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# AXIOM

The Scientific Computation System

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*To my children, Douglas, Daniel, and Susan,  
for their love, support, and understanding over the years.  
R.D.J.*

*To Judith and Kate,  
to whom my debt is beyond computation.  
R.S.S.*



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# Foreword

You are holding in your hands an unusual book. Winston Churchill once said that the empires of the future will be empires of the mind. This book might hold an electronic key to such an empire.

When computers were young and slow, the emerging computer science developed dreams of Artificial Intelligence and Automatic Theorem Proving in which theorems can be proved by machines instead of mathematicians. Now, when computer hardware has matured and become cheaper and faster, there is not too much talk of putting the burden of formulating and proving theorems on the computer's shoulders. Moreover, even in those cases when computer programs do prove theorems, or establish counter-examples (for example, the solution of the four color problem, the non-existence of projective planes of order 10, the disproof of the Mertens conjecture), humans carry most of the burden in the form of programming and verification.

It is the language of computer programming that has turned out to be the crucial instrument of productivity in the evolution of scientific computing. The original Artificial Intelligence efforts gave birth to the first symbolic manipulation systems based on LISP. The first complete symbolic manipulation or, as they are called now, computer algebra packages tried to imbed the development programming and execution of mathematical problems into a framework of familiar symbolic notations, operations and conventions. In the third decade of symbolic computations, a couple of these early systems—REDUCE and MACSYMA—still hold their own among faithful users.

AXIOM was born in the mid-70's as a system called Scratchpad developed by IBM researchers. Scratchpad/AXIOM was born big—its original

platform was an IBM mainframe 3081, and later a 3090. The system was growing and learning during the decade of the 80's, and its development and progress influenced the field of computer algebra. During this period, the first commercially available computer algebra packages for mini and microcomputers made their debut. By now, our readers are aware of Mathematica, Maple, Derive, and Macsyma. These systems (as well as a few special purpose computer algebra packages in academia) emphasize ease of operation and standard scientific conventions, and come with a prepared set of mathematical solutions for typical tasks confronting an applied scientist or an engineer. These features brought a recognition of the enormous benefits of computer algebra to the widest circles of scientists and engineers.

The Scratchpad system took its time to blossom into the beautiful AXIOM product. There is no rival to this powerful environment in its scope and, most importantly, in its structure and organization. AXIOM contains the basis for any comprehensive and elaborate mathematical development. It gives the user all Foundation and Algebra instruments necessary to develop a computer realization of sophisticated mathematical objects in exactly the way a mathematician would do it. AXIOM is also the basis of a complete scientific cyberspace—it provides an environment for mathematical objects used in scientific computation, and the means of controlling and communicating between these objects. Knowledge of only a few AXIOM language features and operating principles is all that is required to make impressive progress in a given domain of interest. The system is powerful. It is not an interactive interpretive environment operating only in response to one line commands—it is a complete language with rich syntax and a full compiler. Mathematics can be developed and explored with ease by the user of AXIOM. In fact, during AXIOM's growth cycle, many detailed mathematical domains were constructed. Some of them are a part of AXIOM's core and are described in this book. For a bird's eye view of the algebra hierarchy of AXIOM, glance inside the book cover.

The crucial strength of AXIOM lies in its excellent structural features and unlimited expandability—it is open, modular system designed to support an ever growing number of facilities with minimal increase in structural complexity. Its design also supports the integration of other computation tools such as numerical software libraries written in Fortran and C. While AXIOM is already a very powerful system, the prospect of scientists using the system to develop their own fields of Science is truly exciting—the day is still young for AXIOM.

Over the last several years Scratchpad/AXIOM has scored many successes in theoretical mathematics, mathematical physics, combinatorics, digital signal processing, cryptography and parallel processing. We have to con-

fess that we enjoyed using Scratchpad/AXIOM. It provided us with an excellent environment for our research, and allowed us to solve problems intractable on other systems. We were able to prove new diophantine results for  $\pi$ ; establish the Grothendieck conjecture for certain classes of linear differential equations; study the arithmetic properties of the uniformization of hyperelliptic and other algebraic curves; construct new factorization algorithms based on formal groups; within Scratchpad/AXIOM we were able to obtain new identities needed for quantum field theory (elliptic genus formula and double scaling limit for quantum gravity), and classify period relations for CM varieties in terms of hypergeometric series.

The AXIOM system is now supported and distributed by NAG, the group that is well known for its high quality software products for numerical and statistical computations. The development of AXIOM in IBM was conducted at IBM T.J. Watson Research Center at Yorktown, New York by a symbolic computation group headed by Richard D. Jenks. Shmuel Winograd of IBM was instrumental in the progress of symbolic research at IBM.

This book opens the wonderful world of AXIOM, guiding the reader and user through AXIOM's definitions, rules, applications and interfaces. A variety of fully developed areas of mathematics are presented as packages, and the user is well advised to take advantage of the sophisticated realization of familiar mathematics. The AXIOM book is easy to read and the AXIOM system is easy to use. It possesses all the features required of a modern computer environment (for example, windowing, integration of operating system features, and interactive graphics). AXIOM comes with a detailed hypertext interface (HyperDoc), an elaborate browser, and complete on-line documentation. The HyperDoc allows novices to solve their problems in a straightforward way, by providing menus for step-by-step interactive entry.

The appearance of AXIOM in the scientific market moves symbolic computing into a higher plane, where scientists can formulate their statements in their own language and receive computer assistance in their proofs. AXIOM's performance on workstations is truly impressive, and users of AXIOM will get more from them than we, the early users, got from mainframes. AXIOM provides a powerful scientific environment for easy construction of mathematical tools and algorithms; it is a symbolic manipulation system, and a high performance numerical system, with full graphics capabilities. We expect every (computer) power hungry scientist will want to take full advantage of AXIOM.

David V. Chudnovsky

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# Contributors

The design and development of AXIOM was led by the Symbolic Computation Group of the Mathematical Sciences Department, IBM Thomas J. Watson Research Center, Yorktown Heights, New York. The current implementation of AXIOM is the product of many people. The primary contributors are:

**Richard D. Jenks** (IBM, Yorktown) received a Ph.D. from the University of Illinois and was a principal architect of the **Scratchpad** computer algebra system (1971). In 1977, Jenks initiated the AXIOM effort with the design of MODLISP, inspired by earlier work with Rüdiger Loos (Tübingen), James Griesmer (IBM, Yorktown), and David Y. Y. Yun (Hawaii). Joint work with David R. Barton (Berkeley, California) and James Davenport led to the design and implementation of prototypes and the concept of categories (1980). More recently, Jenks led the effort on user interface software for AXIOM.

**Barry M. Trager** (IBM, Yorktown) received a Ph.D. from MIT while working in the **MACSYMA** computer algebra group. Trager's thesis laid the groundwork for a complete theory for closed-form integration of elementary functions and its implementation in AXIOM. Trager and Richard Jenks are responsible for the original abstract datatype design and implementation of the programming language with its current MODLISP-based compiler and run-time system. Trager is also responsible for the overall design of the current AXIOM library and for the implementation of many of its components.

**Stephen M. Watt** (IBM, Yorktown) received a Ph.D. from the University of Waterloo and is one of the original authors of the **Maple** computer algebra system. Since joining IBM in 1984, he has made central contributions to the AXIOM language and system design, as well as numerous contributions to the library. He is the principal architect of the new AXIOM compiler, planned for Release 2.

**Robert S. Sutor** (IBM, Yorktown) received a Ph.D. in mathematics

from Princeton University and has been involved with the design and implementation of the system interpreter, system commands, and documentation since 1984. Sutor's contributions to the AXIOM library include factored objects, partial fractions, and the original implementation of finite field extensions. Recently, he has devised technology for producing automatic hard-copy and on-line documentation from single source files.

**Scott C. Morrison** (IBM, Yorktown) received an M.S. from the University of California, Berkeley, and is a principal person responsible for the design and implementation of the AXIOM interface, including the interpreter, HyperDoc, and applications of the computer graphics system.

**Manuel Bronstein** (ETH, Zürich) received a Ph.D. in mathematics from the University of California, Berkeley, completing the theoretical work on closed-form integration by Barry Trager. Bronstein designed and implemented the algebraic structures and algorithms in the AXIOM library for integration, closed form solution of differential equations, operator algebras, and manipulation of top-level mathematical expressions. He also designed (with Richard Jenks) and implemented the current pattern match facility for AXIOM.

**William H. Burge** (IBM, Yorktown) received a Ph.D. from Cambridge University, implemented the AXIOM parser, designed (with Stephen Watt) and implemented the stream and power series structures, and numerous algebraic facilities including those for data structures, power series, and combinatorics.

**Timothy P. Daly** (IBM, Yorktown) is pursuing a Ph.D. in computer science at Brooklyn Polytechnic Institute and is responsible for porting, testing, performance, and system support work for AXIOM.

**James Davenport** (Bath) received a Ph.D. from Cambridge University, is the author of several computer algebra textbooks, and has long recognized the need for AXIOM's generality for computer algebra. He was involved with the early prototype design of system internals and the original category hierarchy for AXIOM (with David R. Barton). More recently, Davenport and Barry Trager designed the algebraic category hierarchy currently used in AXIOM. Davenport is Hebron and Medlock Professor of Information Technology at Bath University.

**Michael Dewar** (Bath) received a Ph.D. from the University of Bath for his work on the IRENA system (an interface between the **REDUCE** computer algebra system and the NAG Library of numerical subprograms), and work on interfacing algebraic and numerical systems in general. He has contributed code to produce FORTRAN output from AXIOM, and is currently developing a comprehensive foreign language interface and a link to the NAG Library for release 2 of AXIOM.

**Albrecht Fortenbacher** (IBM Scientific Center, Heidelberg) received a doctorate from the University of Karlsruhe and is a designer and

implementer of the type-inferencing code in the AXIOM interpreter. The result of research by Fortenbacher on type coercion by rewrite rules will soon be incorporated into AXIOM.

**Patrizia Gianni** (Pisa) received a Laurea in mathematics from the University of Pisa and is the prime author of the polynomial and rational function component of the AXIOM library. Her contributions include algorithms for greatest common divisors, factorization, ideals, Gröbner bases, solutions of polynomial systems, and linear algebra. She is currently Associate Professor of Mathematics at the University of Pisa.

**Johannes Grabmeier** (IBM Scientific Center, Heidelberg) received a Ph.D. from University Bayreuth (Bavaria) and is responsible for many AXIOM packages, including those for representation theory (with Holger Gollan (Essen)), permutation groups (with Gerhard Schneider (Essen)), finite fields (with Alfred Scheerhorn), and non-associative algebra (with Robert Wisbauer (Düsseldorf)).

**Larry Lambe** received a Ph.D. from the University of Illinois (Chicago) and has been using AXIOM for research in homological algebra. Lambe contributed facilities for Lie ring and exterior algebra calculations and has worked with Scott Morrison on various graphics applications.

**Michael Monagan** (ETH, Zürich) received a Ph.D. from the University of Waterloo and is a principal contributor to the **Maple** computer algebra system. He designed and implemented the category hierarchy and domains for data structures (with Stephen Watt), multi-precision floating point arithmetic, code for polynomials modulo a prime, and also worked on the new compiler.

**William Sit** (CCNY) received a Ph.D. from Columbia University. He has been using AXIOM for research in differential algebra, and contributed operations for differential polynomials (with Manuel Bronstein).

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**Jim Wen**, a graduate student in computer graphics at Brown University, designed and implemented the original computer graphics system for AXIOM with pop-up control panels for interactive manipulation of graphic objects.

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Other group members, visitors and contributors to AXIOM include Richard Anderson, George Andrews, David R. Barton, Alexandre Bouyer, Martin Brock, Florian Bundschuh, Cheekai Chin, David V. Chudnovsky, Gregory V. Chudnovsky, Josh Cohen, Gary Cornell, Jean Della Dora, Claire DiCrescendo, Dominique Duval, Lars Erickson, Timothy Freeman, Marc Gaetano, Vladimir A. Grinberg, Florian Bundschuh, Oswald Gschnitzer, Klaus Kusche, Bernhard Kutzler, Mohammed Mobarak, Julian A. Padgett, Michael Rothstein, Alfred Scheerhorn, William F. Schelter, Martin Schönert, Fritz Schwarz, Christine J. Sundaresan, Moss E. Sweedler, Themos T. Tsikas, Bernhard Wall, Robert Wisbauer, and Knut Wolf.

This book has contributions from several people in addition to its principal authors. Scott Morrison is responsible for the computer graphics gallery and the programs in Appendix F. Jonathan Steinbach wrote the original version of Chapter 7. Michael Dewar contributed material on the FORTRAN interface in Chapter 4. Manuel Bronstein, Clifton Williamson, Patricia Gianni, Johannes Grabmeier, and Barry Trager, and Stephen Watt contributed to Chapters 8 and 9 and Appendix E. William Burge, Timothy Daly, Larry Lambe, and William Sit contributed material to Chapter 9.

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# Introduction to AXIOM

---

Welcome to the world of AXIOM. We call AXIOM a scientific computation system: a self-contained toolbox designed to meet your scientific programming needs, from symbolics, to numerics, to graphics.

This introduction is a quick overview of what AXIOM offers.

## **Symbolic computation**

---

Integrate  $\frac{1}{(x^3 (a+bx)^{1/3})}$  with  
respect to  $x$ .

```
integrate(1/(x**3 * (a+b*x)**(1/3)),x)
```

$$\frac{\left( \begin{aligned} &-2 b^2 x^2 \sqrt{3} \log \left( \sqrt[3]{a} \sqrt[3]{b x + a}^2 + \sqrt[3]{a}^2 \sqrt[3]{b x + a} + a \right) + \\ &4 b^2 x^2 \sqrt{3} \log \left( \sqrt[3]{a}^2 \sqrt[3]{b x + a} - a \right) + \\ &12 b^2 x^2 \arctan \left( \frac{2 \sqrt{3} \sqrt[3]{a}^2 \sqrt[3]{b x + a} + a \sqrt{3}}{3 a} \right) + \\ &(12 b x - 9 a) \sqrt{3} \sqrt[3]{a} \sqrt[3]{b x + a}^2 \end{aligned} \right)}{18 a^2 x^2 \sqrt{3} \sqrt[3]{a}} \quad (1)$$

Type: Union(Expression Integer, ...)

AXIOM provides state-of-the-art algebraic machinery to handle your most advanced symbolic problems. For example, AXIOM's integrator gives you the answer when an answer exists. If one does not, it provides a proof that there is no answer. Integration is just one of a multitude of symbolic

AXIOM has a numerical library that includes operations for linear algebra, solution of equations, and special functions. For many of these operations, you can select any number of floating point digits to be carried out in the computation.

Solve  $x^{49} - 49x^4 + 9$  to 49 digits of accuracy.

Type: List Equation Polynomial Float

The output of a computation can be converted to FORTRAN to be used in a later numerical computation. Besides floating point numbers, AXIOM provides literally dozens of kinds of numbers to compute with. These range from various kinds of integers, to fractions, complex numbers, quaternions, continued fractions, and to numbers represented with an arbitrary base.

What is 10 to the 100<sup>th</sup> power in base 32?

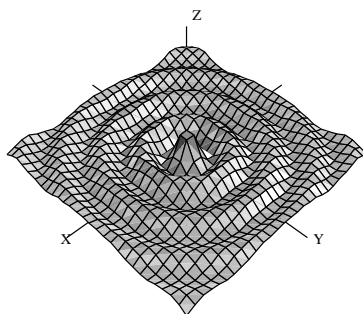
Type: RadixExpansion 32

You may often want to visualize a symbolic formula or draw a graph from a set of numerical values. To do this, you can call upon the AXIOM graphics capability.

Draw  $J_0(\sqrt{x^2 + y^2})$  for  $-20 \leq x, y \leq 20$ .

```
draw(5*besselJ(0,sqrt(x**2+y**2)), x=-20..20, y=-20..20)
Compiling function %J with type (DoubleFloat,
    DoubleFloat) -> DoubleFloat
Transmitting data...
ThreeDimensionalViewport: "5*besselJ(0,(y*y+x*x)**(1/2))" (4)
Type: ThreeDimensionalViewport
```





Graphs in AXIOM are interactive objects you can manipulate with your mouse. Just click on the graph, and a control panel pops up. Using this mouse and the control panel, you can translate, rotate, zoom, change the coloring, lighting, shading, and perspective on the picture. You can also generate a PostScript copy of your graph to produce hard-copy output.

## HyperDoc

---

HyperDoc presents you windows on the world of AXIOM, offering on-line help, examples, tutorials, a browser, and reference material. HyperDoc gives you on-line access to this book in a “hypertext” format. Words that appear in a different font (for example, Matrix, **factor**, and *category*) are generally mouse-active; if you click on one with your mouse, HyperDoc shows you a new window for that word.

As another example of a HyperDoc facility, suppose that you want to compute the roots of  $x^{49} - 49x^4 + 9$  to 49 digits (as in our previous example) and you don’t know how to tell AXIOM to do this. The “basic command” facility of HyperDoc leads the way. Through the series of HyperDoc windows shown in Figure 1 and the specified mouse clicks, you and HyperDoc generate the correct command to issue to compute the answer.

## Interactive Programming

---

AXIOM’s interactive programming language lets you define your own functions. A simple example of a user-defined function is one that computes the successive Legendre polynomials. AXIOM lets you define these polynomials in a piece-wise way.

The first Legendre polynomial.

$p(0) == 1$

Type: Void

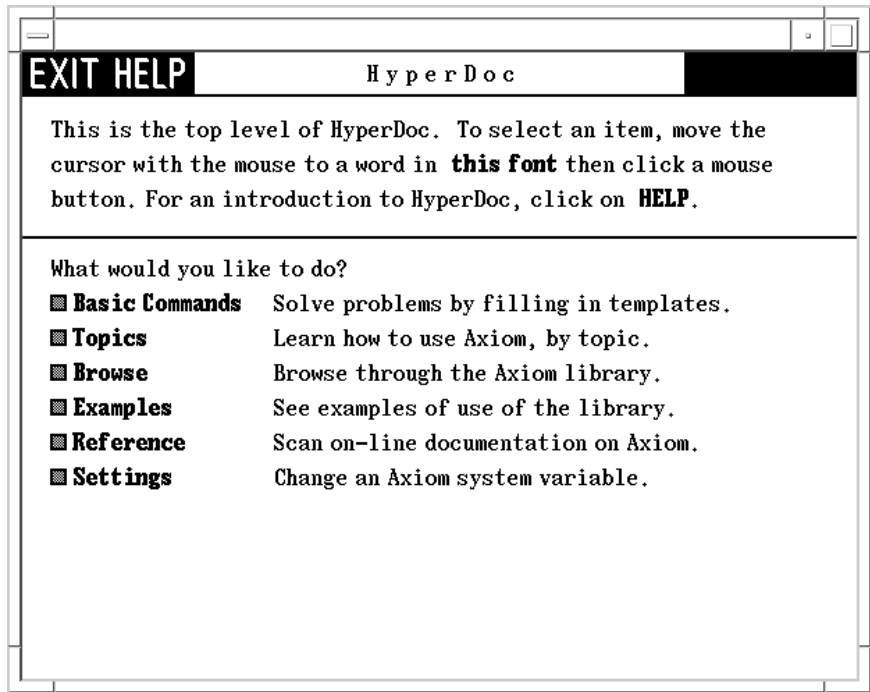


Figure 1: Computing the roots of  $x^{49} - 49x^4 + 9$ .

The second Legendre polynomial.

`p(1) == x`

Type: Void

The  $n^{\text{th}}$  Legendre polynomial for  $(n > 1)$ .

`p(n) == ((2*n-1)*x*p(n-1) - (n-1) * p(n-2))/n`

Type: Void

In addition to letting you define simple functions like this, the interactive language can be used to create entire application packages. All the graphs in the AXIOM Images section in the center of the book, for example, were created by programs written in the interactive language.

The above definitions for `p` do no computation—they simply tell AXIOM how to compute `p(k)` for some positive integer `k`. To actually get a value of a Legendre polynomial, you ask for it.

What is the tenth Legendre polynomial?

`p(10)`

Compiling function `p` with type `Integer -> Polynomial Fraction Integer`

Compiling function `p` as a recurrence relation.

$$\frac{46189}{256} x^{10} - \frac{109395}{256} x^8 + \frac{45045}{128} x^6 - \frac{15015}{128} x^4 + \frac{3465}{256} x^2 - \frac{63}{256} \quad (8)$$

Type: Polynomial Fraction Integer

AXIOM applies the above pieces for `p` to obtain the value of `p(10)`. But it does more: it creates an optimized, compiled function for `p`. The function is formed by putting the pieces together into a single piece of code. By *compiled*, we mean that the function is translated into basic machine-code. By *optimized*, we mean that certain transformations are performed on that code to make it run faster. For `p`, AXIOM actually translates the original definition that is recursive (one that calls itself) to one that is iterative (one that consists of a simple loop).

What is the coefficient of  $x^{90}$  in `p(90)`?

`coefficient(p(90), x, 90)`

$$\frac{5688265542052017822223458237426581853561497449095175}{77371252455336267181195264} \quad (9)$$

Type: Polynomial Fraction Integer

In general, a user function is type-analyzed and compiled on first use. Later, if you use it with a different kind of object, the function is recompiled if necessary.

## Data Structures

A variety of data structures are available for interactive use. These include strings, lists, vectors, sets, multisets, and hash tables. A particularly useful structure for interactive use is the infinite stream:

Create the infinite stream of derivatives of Legendre polynomials

`[D(p(i), x) for i in 1..]`

$$\left[ 1, 3x, \frac{15}{2}x^2 - \frac{3}{2}, \frac{35}{2}x^3 - \frac{15}{2}x, \frac{315}{8}x^4 - \frac{105}{4}x^2 + \frac{15}{8}, \frac{693}{8}x^5 - \frac{315}{4}x^3 + \frac{105}{8}x, \frac{3003}{16}x^6 - \frac{3465}{16}x^4 + \frac{945}{16}x^2 - \frac{35}{16}, \dots \right] \quad (10)$$

Type: Stream Polynomial Fraction Integer

Streams display only a few of their initial elements. Otherwise, they are “lazy”: they only compute elements when you ask for them.

Data structures are an important component for building application software. Advanced users can represent data for applications in optimal fashion. In all, AXIOM offers over forty kinds of aggregate data structures, ranging from mutable structures (such as cyclic lists and flexible arrays) to storage efficient structures (such as bit vectors). As an example, streams

are used as the internal data structure for power series.

What is the series expansion of  $\log(\cot(x))$  about  $x = \pi/2$ ?

```
series(log(cot(x)), x = %pi/2)
```

$$\log\left(\frac{-2x + \pi}{2}\right) + \frac{1}{3}\left(x - \frac{\pi}{2}\right)^2 + \frac{7}{90}\left(x - \frac{\pi}{2}\right)^4 + \frac{62}{2835}\left(x - \frac{\pi}{2}\right)^6 + O\left(\left(x - \frac{\pi}{2}\right)^8\right) \quad (11)$$

Type: GeneralUnivariatePowerSeries(Expression Integer, x, pi/2)

Series and streams make no attempt to compute *all* their elements! Rather, they stand ready to deliver elements on demand.

What is the coefficient of the 50<sup>th</sup> term of this series?

```
coefficient(%, 50)
```

$$\frac{44590788901016030052447242300856550965644}{7131469286438669111584090881309360354581359130859375} \quad (12)$$

Type: Expression Integer

## Mathematical Structures

AXIOM also has many kinds of mathematical structures. These range from simple ones (like polynomials and matrices) to more esoteric ones (like ideals and Clifford algebras). Most structures allow the construction of arbitrarily complicated “types.”

Even a simple input expression can result in a type with several levels.

```
matrix [[x + %i, 0], [1, -2]]
```

$$\begin{bmatrix} x + i & 0 \\ 1 & -2 \end{bmatrix} \quad (13)$$

Type: Matrix Polynomial Complex Integer

The AXIOM interpreter builds types in response to user input. Often, the type of the result is changed in order to be applicable to an operation.

The inverse operation requires that elements of the above matrices are fractions.

```
inverse(%)
```

$$\begin{bmatrix} \frac{1}{x+i} & 0 \\ \frac{1}{2x+2i} & -\frac{1}{2} \end{bmatrix} \quad (14)$$

Type: Union(Matrix Fraction Polynomial Complex Integer, ...)

## Pattern Matching

A convenient facility for symbolic computation is “pattern matching.” Suppose you have a trigonometric expression and you want to transform it to some equivalent form. Use a `rule` command to describe the transformation rules you need. Then give the rules a name and apply that name as a function to your trigonometric expression.

Introduce two rewrite rules.

```
sinCosExpandRules := rule
  sin(x+y) == sin(x)*cos(y) + sin(y)*cos(x)
  cos(x+y) == cos(x)*cos(y) - sin(x)*sin(y)
  sin(2*x) == 2*sin(x)*cos(x)
  cos(2*x) == cos(x)**2 - sin(x)**2

{sin(y+x)==cos(x) sin(y) + cos(y) sin(x),
 cos(y+x)==-sin(x) sin(y) + cos(x) cos(y),
 sin(2 x)==2 cos(x) sin(x), cos(2 x)==-sin(x)**2 + cos(x)**2}

Type: Ruleset(Integer, Integer, Expression Integer)
```

Apply the rules to a simple trigonometric expression.

```
sinCosExpandRules(sin(a+2*b+c))

(-cos(a) sin(b)**2 - 2 cos(b) sin(a) sin(b) + cos(a) cos(b)**2) sin(c) -
cos(c) sin(a) sin(b)**2 + 2 cos(a) cos(b) cos(c) sin(b) +
cos(b)**2 cos(c) sin(a)
```

(16)

Type: Expression Integer

Using input files, you can create your own library of transformation rules relevant to your applications, then selectively apply the rules you need.

## Polymorphic Algorithms

All components of the AXIOM algebra library are written in the AXIOM library language. This language is similar to the interactive language except for protocols that authors are obliged to follow. The library language permits you to write “polymorphic algorithms,” algorithms defined to work in their most natural settings and over a variety of types.

Define a system of polynomial equations  $S$ .

```
S := [3*x**3 + y + 1 = 0, y**2 = 4]
```

$$\left[ y + 3x^3 + 1 = 0, y^2 = 4 \right] \quad (17)$$

Type: List Equation Polynomial Integer

Solve the system  $S$  using rational number arithmetic and 30 digits of accuracy.

```
solve(S, 1/10**30)

[[[y = -2, x = 1757879671211184245283070414507 /
2535301200456458802993406410752],
 [y = 2, x = -1]]]
```

(18)

Type: List List Equation Polynomial Fraction Integer

Solve **S** with the solutions expressed in radicals.

$$\begin{aligned} &\text{radicalSolve(S)} \\ &\left[ \left[ y = 2, x = -1 \right], \left[ y = 2, x = \frac{-\sqrt{-3} + 1}{2} \right], \right. \\ &\left. \left[ y = 2, x = \frac{\sqrt{-3} + 1}{2} \right], \left[ y = -2, x = \frac{1}{\sqrt[3]{3}} \right], \right. \\ &\left. \left[ y = -2, x = \frac{\sqrt{-1} \sqrt{3} - 1}{2 \sqrt[3]{3}} \right], \left[ y = -2, x = \frac{-\sqrt{-1} \sqrt{3} - 1}{2 \sqrt[3]{3}} \right] \right] \end{aligned} \tag{19}$$

Type: List List Equation Expression Integer

While these solutions look very different, the results were produced by the same internal algorithm! The internal algorithm actually works with equations over any “field.” Examples of fields are the rational numbers, floating point numbers, rational functions, power series, and general expressions involving radicals.

## Extensibility

---

Users and system developers alike can augment the AXIOM library, all using one common language. Library code, like interpreter code, is compiled into machine binary code for run-time efficiency.

Using this language, you can create new computational types and new algorithmic packages. All library code is polymorphic, described in terms of a database of algebraic properties. By following the language protocols, there is an automatic, guaranteed interaction between your code and that of colleagues and system implementers.

---

# A Technical Introduction to AXIOM

AXIOM has both an *interactive language* for user interactions and a *programming language* for building library modules. Like Modula 2, PASCAL, FORTRAN, and Ada, the programming language emphasizes strict type-checking. Unlike these languages, types in AXIOM are dynamic objects: they are created at run-time in response to user commands.

Here is the idea of the AXIOM programming language in a nutshell. AXIOM types range from algebraic ones (like polynomials, matrices, and power series) to data structures (like lists, dictionaries, and input files). Types combine in any meaningful way. You can build polynomials of matrices, matrices of polynomials of power series, hash tables with symbolic keys and rational function entries, and so on.

*Categories* define algebraic properties to ensure mathematical correctness. They ensure, for example, that matrices of polynomials are OK, but matrices of input files are not. Through categories, programs can discover that polynomials of continued fractions have a commutative multiplication whereas polynomials of matrices do not.

Categories allow algorithms to be defined in their most natural setting. For example, an algorithm can be defined to solve polynomial equations over *any* field. Likewise a greatest common divisor can compute the “gcd” of two elements from *any* Euclidean domain. Categories foil attempts to

compute meaningless “gcds”, for example, of two hashtables. Categories also enable algorithms to be compiled into machine code that can be run with arbitrary types.

The AXIOM interactive language is oriented towards ease-of-use. The AXIOM interpreter uses type-inferencing to deduce the type of an object from user input. Type declarations can generally be omitted for common types in the interactive language.

So much for the nutshell. Here are these basic ideas described by ten design principles:

## Types are Defined by Abstract Datatype Programs

---

Basic types are called *domains of computation*, or, simply, *domains*. Domains are defined by AXIOM programs of the form:

```
Name(...): Exports == Implementation
```

Each domain has a capitalized **Name** that is used to refer to the class of its members. For example, `Integer` denotes “the class of integers,” `Float`, “the class of floating point numbers,” and `String`, “the class of strings.”

The “...” part following **Name** lists zero or more parameters to the constructor. Some basic ones like `Integer` take no parameters. Others, like `Matrix`, `Polynomial` and `List`, take a single parameter that again must be a domain. For example, `Matrix(Integer)` denotes “matrices over the integers,” `Polynomial(Float)` denotes “polynomial with floating point coefficients,” and `List (Matrix (Polynomial (Integer)))` denotes “lists of matrices of polynomials over the integers.” There is no restriction on the number or type of parameters of a domain constructor.

The **Exports** part specifies operations for creating and manipulating objects of the domain. For example, type `Integer` exports constants `0` and `1`, and operations “+”, “-”, and “\*”. While these operations are common, others such as **odd?** and **bit?** are not.

The **Implementation** part defines functions that implement the exported operations of the domain. These functions are frequently described in terms of another lower-level domain used to represent the objects of the domain.



## **The Type of Basic Objects is a Domain or Subdomain**

---

Every AXIOM object belongs to a *unique* domain. The domain of an object is also called its *type*. Thus the integer 7 has type `Integer` and the string "daniel" has type `String`.

The type of an object, however, is not unique. The type of integer 7 is not only `Integer` but `NonNegativeInteger`, `PositiveInteger`, and possibly, in general, any other “subdomain” of the domain `Integer`. A *subdomain* is a domain with a “membership predicate”. `PositiveInteger` is a subdomain of `Integer` with the predicate “is the integer > 0?”.

Subdomains with names are defined by abstract datatype programs similar to those for domains. The *Export* part of a subdomain, however, must list a subset of the exports of the domain. The *Implementation* part optionally gives special definitions for subdomain objects.

## **Domains Have Types Called Categories**

---

Domain and subdomains in AXIOM are themselves objects that have types. The type of a domain or subdomain is called a *category*. Categories are described by programs of the form:

```
Name(...): Category == Exports
```

The type of every category is the distinguished symbol `Category`. The category `Name` is used to designate the class of domains of that type. For example, category `Ring` designates the class of all rings. Like domains, categories can take zero or more parameters as indicated by the “...” part following `Name`. Two examples are `Module(R)` and `MatrixCategory(R,Row,Col)`.

The *Exports* part defines a set of operations. For example, `Ring` exports the operations “0”, “1”, “+”, “-”, and “\*”. Many algebraic domains such as `Integer` and `Polynomial(Float)` are rings. `String` and `List(R)` (for any domain `R`) are not.

Categories serve to ensure the type-correctness. The definition of matrices states `Matrix(R: Ring)` requiring its single parameter `R` to be a ring. Thus a “matrix of polynomials” is allowed, but “matrix of lists” is not.

## **Operations Can Refer To Abstract Types**

---

All operations have prescribed source and target types. Types can be denoted by symbols that stand for domains, called “symbolic domains.” The following lines of AXIOM code use a symbolic domain `R`:

```
R: Ring
power: (R, NonNegativeInteger): R -> R
power(x, n) == x ** n
```

Line 1 declares the symbol `R` to be a ring. Line 2 declares the type of

`power` in terms of `R`. From the definition on line 3, `power(3,2)` produces 9 for `x = 3` and `R = Integer`. Also, `power(3.0,2)` produces 9.0 for `x = 3.0` and `R = Float`. `power("oxford",2)` however fails since "oxford" has type `String` which is not a ring.

Using symbolic domains, algorithms can be defined in their most natural or general setting.

## Categories Form Hierarchies

Categories form hierarchies (technically, directed-acyclic graphs). A simplified hierarchical world of algebraic categories is shown below in Figure 2. At the top of this world is `SetCategory`, the class of algebraic sets. The notions of parents, ancestors, and descendants is clear. Thus ordered sets (domains of category `OrderedSet`) and rings are also algebraic sets. Likewise, fields and integral domains are rings and algebraic sets. However fields and integral domains are not ordered sets.

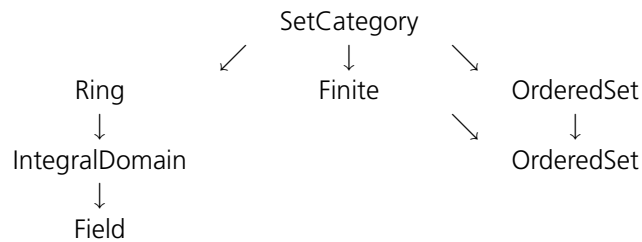


Figure 2: A simplified category hierarchy.

## Domains Belong to Categories by Assertion

A category designates a class of domains. Which domains? You might think that `Ring` designates the class of all domains that export 0, 1, “+”, “-”, and “\*”. But this is not so. Each domain must *assert* which categories it belongs to.

The `Export` part of the definition for `Integer` reads, for example:

```
Join(OrderedSet, IntegralDomain, ...) with ...
```

This definition asserts that `Integer` is both an ordered set and an integral domain. In fact, `Integer` does not explicitly export constants 0 and 1 and operations “+”, “-” and “\*” at all: it inherits them all from `Ring`! Since `IntegralDomain` is a descendant of `Ring`, `Integer` is therefore also a ring.

Assertions can be conditional. For example, `Complex(R)` defines its exports by:

Ring with ... if R has Field then Field ...

Thus `Complex(Float)` is a field but `Complex(Integer)` is not since `Integer` is not a field.

You may wonder: “Why not simply let the set of operations determine whether a domain belongs to a given category?”. AXIOM allows operation names (for example, **norm**) to have very different meanings in different contexts. The meaning of an operation in AXIOM is determined by context. By associating operations with categories, operation names can be reused whenever appropriate or convenient to do so. As a simple example, the operation “<” might be used to denote lexicographic-comparison in an algorithm. However, it is wrong to use the same “<” with this definition of absolute-value: `abs(x) == if x < 0 then -x else x`. Such a definition for **abs** in AXIOM is protected by context: argument `x` is required to be a member of a domain of category `OrderedSet`.

## Packages Are Clusters of Polymorphic Operations

---

In AXIOM, facilities for symbolic integration, solution of equations, and the like are placed in “packages”. A *package* is a special kind of domain: one whose exported operations depend solely on the parameters of the constructor and/or explicit domains.

If you want to use AXIOM, for example, to define some algorithms for solving equations of polynomials over an arbitrary field `F`, you can do so with a package of the form:

```
MySolve(F: Field): Exports == Implementation
```

where **Exports** specifies the **solve** operations you wish to export and **Implementation** defines functions for implementing your algorithms. Once AXIOM has compiled your package, your algorithms can then be used for any `F`: floating-point numbers, rational numbers, complex rational functions, and power series, to name a few.

## The Interpreter Builds Domains Dynamically

---

The AXIOM interpreter reads user input then builds whatever types it needs to perform the indicated computations. For example, to create the matrix

$$M = \begin{pmatrix} x^2 + 1 & 0 \\ 0 & x/2 \end{pmatrix}$$

the interpreter first loads the modules `Matrix`, `Polynomial`, `Fraction`, and `Integer` from the library, then builds the *domain tower* “matrices of polynomials of rational numbers (fractions of integers)”.

Once a domain tower is built, computation proceeds by calling operations down the tower. For example, suppose that the user asks to square the above matrix. To do this, the function “\*” from Matrix is passed M to compute  $M * M$ . The function is also passed an environment containing R that, in this case, is Polynomial (Fraction (Integer)). This results in the successive calling of the “\*” operations from Polynomial, then from Fraction, and then finally from Integer before a result is passed back up the tower.

Categories play a policing role in the building of domains. Because the argument of Matrix is required to be a ring, AXIOM will not build non-sensical types such as “matrices of input files”.

## **AXIOM Code is Compiled**

---

AXIOM programs are statically compiled to machine code, then placed into library modules. Categories provide an important role in obtaining efficient object code by enabling:

- static type-checking at compile time;
- fast linkage to operations in domain-valued parameters;
- optimization techniques to be used for partially specified types (operations for “vectors of R”, for instance, can be open-coded even though R is unknown).

## **AXIOM is Extensible**

---

Users and system implementers alike use the AXIOM language to add facilities to the AXIOM library. The entire AXIOM library is in fact written in the AXIOM source code and available for user modification and/or extension.

AXIOM’s use of abstract datatypes clearly separates the exports of a domain (what operations are defined) from its implementation (how the objects are represented and operations are defined). Users of a domain can thus only create and manipulate objects through these exported operations. This allows implementers to “remove and replace” parts of the library safely by newly upgraded (and, we hope, correct) implementations without consequence to its users.

Categories protect names by context, making the same names available for use in other contexts. Categories also provide for code-economy. Algorithms can be parameterized categorically to characterize their correct and most general context. Once compiled, the same machine code is applicable in all such contexts.

Finally, AXIOM provides an automatic, guaranteed interaction between new and old code. For example:

- if you write a new algorithm that requires a parameter to be a field, then your algorithm will work automatically with every field defined in the system; past, present, or future.
- if you introduce a new domain constructor that produces a field, then the objects of that domain can be used as parameters to any algorithm using field objects defined in the system; past, present, or future.

These are the key ideas. For further information, we particularly recommend your reading chapters 11, 12, and 13, where these ideas are explained in greater detail.



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## PART I

---

# What's new at Release 2.0





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## CHAPTER 1

---

# What's New in AXIOM Version 2.0

Many things have changed in this new version of AXIOM and we describe many of the more important topics here.

### **1.1 Important Things to Read First**

---

If you have any private `.spad` files (that is, library files which were not shipped with AXIOM) you will need to recompile them. For example, if you wrote the file `regress.spad` then you should issue `)compile regress.spad` before trying to use it.

The internal representation of Union has changed. This means that AXIOM data saved with Release 1.x may not be readable by this Release. If you cannot recreate the saved data by recomputing in Release 2.0, please contact NAG for assistance.

### **1.2 The New AXIOM Library Compiler**

---

A new compiler is now available for AXIOM. The programming language is referred to as the AXIOM Extension Language (or Aldor for short), and improves upon the old AXIOM language in many ways. The `)compile` command has been upgraded to be able to invoke the new or old compilers. The language and the compiler are described in the hard-copy documentation which came with your AXIOM system.

To ease the chore of upgrading your `.spad` files (old compiler) to `.as` files

(new compiler), the `)compile` command has been given a `)translate` option. This invokes a special version of the old compiler which parses and analyzes your old code and produces augmented code using the new syntax. Please be aware that the translation is not necessarily one hundred percent complete or correct. You should attempt to compile the output with the Aldor compiler and make any necessary corrections.

## 1.3 The NAG Library Link

---

The NAG Foundation Library link allows you to call NAG Fortran routines from within AXIOM, passing AXIOM objects as parameters and getting them back as results.

The NAG Foundation Library and, consequently, the link are divided into *chapters*, which cover different areas of numerical analysis. The statistical and sorting *chapters* of the Library, however, are not included in the link and various support and utility routines (mainly the F06 and X *chapters*) have been omitted.

Each *chapter* has a short (at most three-letter) name; for example, the *chapter* devoted to the solution of ordinary differential equations is called D02. When using the link via the **HyperDoc interface**, you will be presented with a complete menu of these *chapters*. The names of individual routines within each *chapter* are formed by adding three letters to the *chapter* name, so for example the routine for solving ODEs by Adams method is called **d02cjf**.

### 1.3.1 Interpreting NAG Documentation

---

Information about using the NAG Foundation Library in general, and about using individual routines in particular, can be accessed via HyperDoc. This documentation refers to the Fortran routines directly; the purpose of this subsection is to explain how this corresponds to the AXIOM routines.

For general information about the NAG Foundation Library users should consult **Essential Introduction to the NAG Foundation Library**. The documentation is in ASCII format, and a description of the conventions used to represent mathematical symbols is given in **Introduction to NAG On-Line Documentation**. Advice about choosing a routine from a particular *chapter* can be found in the **Chapter Documents**.

Correspondence Between  
Fortran and AXIOM types

The NAG documentation refers to the Fortran types of objects; in general, the correspondence to AXIOM types is as follows.

- Fortran INTEGER corresponds to AXIOM Integer.
- Fortran DOUBLE PRECISION corresponds to AXIOM DoubleFloat.

- Fortran COMPLEX corresponds to AXIOM Complex DoubleFloat.
- Fortran LOGICAL corresponds to AXIOM Boolean.
- Fortran CHARACTER\*(\*) corresponds to AXIOM String.

(Exceptionally, for NAG EXTERNAL parameters – ASPs in link parlance – REAL and COMPLEX correspond to MachineFloat and MachineComplex, respectively; see Section 1.3.3 on page 24.)

The correspondence for aggregates is as follows.

- A one-dimensional Fortran array corresponds to an AXIOM Matrix with one column.
- A two-dimensional Fortran ARRAY corresponds to an AXIOM Matrix.
- A three-dimensional Fortran ARRAY corresponds to an AXIOM ThreeDimensionalMatrix.

Higher-dimensional arrays are not currently needed for the NAG Foundation Library.

Arguments which are Fortran FUNCTIONS or SUBROUTINES correspond to special ASP domains in AXIOM. See Section 1.3.3 on page 24.

#### Classification of NAG parameters

NAG parameters are classified as belonging to one (or more) of the following categories: **Input**, **Output**, **Workspace** or **External** procedure. Within **External** procedures a similar classification is used, and parameters may also be **Dummies**, or **User Workspace** (data structures not used by the NAG routine but provided for the convenience of the user).

When calling a NAG routine via the link the user only provides values for **Input** and **External** parameters.

The order of the parameters is, in general, different from the order specified in the NAG Foundation Library documentation. The Browser description for each routine helps in determining the correspondence. As a rule of thumb, **Input** parameters come first followed by **Input/Output** parameters. The **External** parameters are always found at the end.

#### IFAIL

NAG routines often return diagnostic information through a parameter called **ifail**. With a few exceptions, the principle is that on input **ifail** takes one of the values  $-1, 0, 1$ . This determines how the routine behaves when it encounters an error:

- a value of 1 causes the NAG routine to return without printing an error message;
- a value of 0 causes the NAG routine to print an error message and abort;
- a value of -1 causes the NAG routine to return and print an error

message.

The user is **STRONGLY ADVISED** to set `ifail` to `-1` when using the link. If `ifail` has been set to `1` or `-1` on input, then its value on output will determine the possible cause of any error. A value of `0` indicates successful completion, otherwise it provides an index into a table of diagnostics provided as part of the routine documentation (accessible via Browse).

### 1.3.2 Using the Link

---

The easiest way to use the link is via the **HyperDoc interface**. You will be presented with a set of fill-in forms where you can specify the parameters for each call. Initially, the forms contain example values, demonstrating the use of each routine (these, in fact, correspond to the standard NAG example program for the routine in question). For some parameters, these values can provide reasonable defaults; others, of course, represent data. When you change a parameter which controls the size of an array, the data in that array are reset to a “neutral” value – usually zero.

When you are satisfied with the values entered, clicking on the “Continue” button will display the AXIOM command needed to run the chosen NAG routine with these values. Clicking on the “Do It” button will then cause AXIOM to execute this command and return the result in the parent AXIOM session, as described below. Note that, for some routines, multiple HyperDoc “pages” are required, due to the structure of the data. For these, returning to an earlier page causes HyperDoc to reset the later pages (this is a general feature of HyperDoc); in such a case, the simplest way to repeat a call, varying a parameter on an earlier page, is probably to modify the call displayed in the parent session.

An alternative approach is to call NAG routines directly in your normal AXIOM session (that is, using the AXIOM interpreter). Such calls return an object of type Result. As not all parameters in the underlying NAG routine are required in the AXIOM call (and the parameter ordering may be different), before calling a NAG routine you should consult the description of the AXIOM operation in the Browser. (The quickest route to this is to type the routine name, in lower case, into the Browser’s input area, then click on **Operations**.) The parameter names used coincide with NAG’s, although they will appear here in lower case. Of course, it is also possible to become familiar with the AXIOM form of a routine by first using it through the **HyperDoc interface**.

As an example of this mode of working, we can find a zero of a function, lying between 3 and 4, as follows:

```
answer:=c05adf(3.0,4.0,1.0e-5,0.0,-1,sin(X)::ASP1(F))
[ifail:Integer , x:DoubleFloat ]
```

(1)  
Type: Result

By default, Result only displays the type of returned values, since the amount of information returned can be quite large. Individual components can be examined as follows:

```
answer . x
3.14159265545896
```

(2)  
Type: DoubleFloat

```
answer . ifail
0
```

(3)  
Type: Integer

In order to avoid conflict with names defined in the workspace, you can also get the values by using the String type (the interpreter automatically coerces them to Symbol)

```
answer "x"
3.14159265545896
```

(4)  
Type: DoubleFloat

It is possible to have AXIOM display the values of scalar or array results automatically. For more details, see the commands **showScalarValues** and **showArrayValues**.

There is also a **.input** file for each NAG routine, containing AXIOM interpreter commands to set up and run the standard NAG example for that routine.

```
)read c05adf.input
--Copyright The Numerical Algorithms Group Limited
1994.
)clear all

All user variables and function definitions have been
cleared.
showArrayValues true

true
```

(1)  
Type: Boolean

```
showScalarValues true

true
```

(2)  
Type: Boolean

```
f:ASP1(F):=exp(-X)-X
F
```

(3)  
Type: Asp1 F

```

a:SF:=0.0
0.0
Type: DoubleFloat

b:SF:=1.0
1.0
Type: DoubleFloat

eps:SF:=1.0e-5
1.0e-05
Type: DoubleFloat

eta:SF:=0.0
0.0
Type: DoubleFloat

result:= c05adf(a,b,eps,eta,-1,f)
[ifail:0, x:0.567143306604963]
Type: Result

```

### 1.3.3 Providing values for Argument Subprograms

For example `c05adf` requires an object of type `Union(fn: FileName,fp: Asp1 F)`

```

)display operation c05adf
There is one exposed function called c05adf :
[1] (DoubleFloat,DoubleFloat,DoubleFloat,DoubleFloat,
      Integer,Union(fn: FileName,fp: Asp1 F)) ->
Result
      from NagRootFindingPackage

```

The user thus has a choice of providing the name of a file containing Fortran source code, or of somehow generating the ASP within AXIOM. If a filename is specified, it is searched for in the *local* machine, i.e., the machine that AXIOM is running on.

Providing ASPs via  
FortranExpression

The FortranExpression domain is used to represent expressions which can be translated into Fortran under certain circumstances. It is very similar to Expression except that only operators which exist in Fortran can be used, and only certain variables can occur. For example the instantiation FortranExpression([X],[M],MachineFloat) is the domain of expressions containing the scalar X and the array M.

This allows us to create expressions like:

```
f : FortranExpression([X],[M],MachineFloat) :=
  sin(X)+M[3,1]
sin(X) + M3,1
Type: FortranExpression([X], [M], MachineFloat)
(1)
```

but not

```
f : FortranExpression([X],[M],MachineFloat) := sin(M)+Y
Cannot convert right-hand side of assignment
sin(M) + Y
to an object of the type FortranExpression([X],[M],
MachineFloat) of the left-hand side.
```

Those ASPs which represent expressions usually export a **coerce** from an appropriate instantiation of FortranExpression (or perhaps Vector FortranExpression etc.). For convenience there are also retractions from appropriate instantiations of Expression, Polynomial and Fraction Polynomial.

Providing ASPs via FortranCode

FortranCode allows us to build arbitrarily complex ASPs via a kind of pseudo-code. It is described fully in Section 1.3.4 on page 26.

Every ASP exports two **coerce** functions: one from FortranCode and one from List FortranCode. There is also a **coerce** from Record( localSymbols: SymbolTable, code: List FortranCode) which is used for passing extra symbol information about the ASP.

So for example, to integrate the function **abs(x)** we could use the built-in **abs** function. But suppose we want to get back to basics and define it directly, then we could do the following:

```
d01ajf(-1.0, 1.0, 0.0, 1.0e-5, 800, 200, -1, cond(LT(X,0),
  assign(F,-X), assign(F,X))) result
1.0
Type: DoubleFloat
(2)
```

The **cond** operation creates a conditional clause and the **assign** an assignment statement.

Providing ASPs via FileName

Suppose we have created the file “asp.f” as follows:

```
DOUBLE PRECISION FUNCTION F(X)
DOUBLE PRECISION X
F=4.0D0/(X*X+1.0D0)
RETURN
END
```

and wish to pass it to the NAG routine **d01ajf** which performs one-dimensional quadrature. We can do this as follows:

```
d01ajf(0.0 ,1.0, 0.0, 1.0e-5, 800, 200, -1, "asp.f")
```

### 1.3.4 General Fortran-generation utilities in AXIOM

---

#### Template Manipulation

This section describes more advanced facilities which are available to users who wish to generate Fortran code from within AXIOM. There are facilities to manipulate templates, store type information, and generate code fragments or complete programs.

A template is a skeletal program which is “fleshed out” with data when it is processed. It is a sequence of *active* and *passive* parts: active parts are sequences of AXIOM commands which are processed as if they had been typed into the interpreter; passive parts are simply echoed verbatim on the Fortran output stream.

Suppose, for example, that we have the following template, stored in the file “test.tem”:

```
-- A simple template
beginVerbatim
    DOUBLE PRECISION FUNCTION F(X)
    DOUBLE PRECISION X
endVerbatim
outputAsFortran("F",f)
beginVerbatim
    RETURN
    END
endVerbatim
```

The passive parts lie between the two tokens `beginVerbatim` and `endVerbatim`. There are two active statements: one which is simply an AXIOM ( `--` ) comment, and one which produces an assignment to the current value of `f`. We could use it as follows:

(4) `->f := 4.0/(1+X**2)`

$$(4) \quad \frac{4}{X^2 + 1}$$

```
(5) ->processTemplate "test.tem"
    DOUBLE PRECISION FUNCTION F(X)
    DOUBLE PRECISION X
    F=4.0D0/(X*X+1.0D0)
    RETURN
    END
```

(5) "CONSOLE"

(A more reliable method of specifying the filename will be introduced below.) Note that the Fortran assignment `F=4.0D0/(X*X+1.0D0)` automatically converted 4.0 and 1 into DOUBLE PRECISION numbers; in general, the AXIOM Fortran generation facility will convert anything which should be a floating point object into either a Fortran REAL or DOUBLE PRECISION object.



Which alternative is used is determined by the command

```
)set fortran precision
----- The precision Option -----

Description: precision of generated FORTRAN objects

The precision option may be followed by any one
of the following:

single
-> double

The current setting is indicated within the list.
```

It is sometimes useful to end a template before the file itself ends (e.g. to allow the template to be tested incrementally or so that a piece of text describing how the template works can be included). It is of course possible to “comment-out” the remainder of the file. Alternatively, the single token `endInput` as part of an active portion of the template will cause processing to be ended prematurely at that point.

The **processTemplate** command comes in two flavours. In the first case, illustrated above, it takes one argument of domain `FileName`, the name of the template to be processed, and writes its output on the current Fortran output stream. In general, a filename can be generated from *directory*, *name* and *extension* components, using the operation **filename**, as in

```
processTemplate filename("", "test", "tem")
```

There is an alternative version of **processTemplate**, which takes two arguments (both of domain `FileName`). In this case the first argument is the name of the template to be processed, and the second is the file in which to write the results. Both versions return the location of the generated Fortran code as their result (`"CONSOLE"` in the above example).

It is sometimes useful to be able to mix active and passive parts of a line or statement. For example you might want to generate a Fortran Comment describing your data set. For this kind of application we provide three functions as follows:

<b>fortranLiteral</b>	writes a string on the Fortran output stream
<b>fortranCarriageReturn</b>	writes a carriage return on the Fortran output stream
<b>fortranLiteralLine</b>	writes a string followed by a return on the Fortran output stream

So we could create our comment  
as follows:

```
m := matrix [[1,2,3],[4,5,6]]
```

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$
(1)

Type: Matrix Integer

```
fortranLiteralLine concat ["C The Matrix has ",
    nrows(m)::String, " rows and ", ncols(m)::String, "
    columns"]
```

```
C      The Matrix has 2 rows and 3 columns
```

Type: Void

or, alternatively:

```
fortranLiteral "C The Matrix has "
```

```
C      The Matrix has
```

```
Type: Void
```

```
fortranLiteral(nrows(m)::String)
```

```
2
```

```
Type: Void
```

```
fortranLiteral " rows and "
```

```
rows and
```

```
Type: Void
```

```
fortranLiteral(ncols(m)::String)
```

```
3
```

```
Type: Void
```

```
fortranLiteral " columns"
```

```
columns
```

```
Type: Void
```

```
fortranCarriageReturn()
```

Type: Void

We should stress that these functions, together with the **outputAsFortran** function are the *only* sure ways of getting output to appear on the Fortran output stream. Attempts to use AXIOM commands such as **output** or **writeline!** may appear to give the required result when displayed on the console, but will give the wrong result when Fortran and algebraic output are sent to differing locations. On the other hand, these functions can be used to send helpful messages to the user, without interfering with the generated Fortran.

## Manipulating the Fortran Output Stream

Sometimes it is useful to manipulate the Fortran output stream in a program, possibly without being aware of its current value. The main use of this is for gathering type declarations (see “Fortran Types” below) but it can be useful in other contexts as well. Thus we provide a set of commands to manipulate a stack of (open) output streams. Only one stream can be written to at any given time. The stack is never empty—its initial value is the console or the current value of the Fortran output stream, and can be determined using

```
topFortranOutputStack()  
"CONSOLE" (9)  
Type: String
```

(see below). The commands available to manipulate the stack are:

<b>clearFortranOutputStack</b>	resets the stack to the console
<b>pushFortranOutputStack</b>	pushes a FileName onto the stack
<b>popFortranOutputStack</b>	pops the stack
<b>showFortranOutputStack</b>	returns the current stack
<b>topFortranOutputStack</b>	returns the top element of the stack

These commands are all part of FortranOutputStackPackage.

## Fortran Types

When generating code it is important to keep track of the Fortran types of the objects which we are generating. This is useful for a number of reasons, not least to ensure that we are actually generating legal Fortran code. The current type system is built up in several layers, and we shall describe each in turn.

### FortranScalarType

This domain represents the simple Fortran datatypes: REAL, DOUBLE PRECISION, COMPLEX, LOGICAL, INTEGER, and CHARACTER. It is possible to **coerce** a String or Symbol into the domain, test whether two objects are equal, and also apply the predicate functions **real?** etc.

### FortranType

This domain represents “full” types: i.e., datatype plus array dimensions (where appropriate) plus whether or not the parameter is an external subprogram. It is possible to **coerce** an object of FortranScalarType into the domain or **construct** one from an element of FortranScalarType, a list of Polynomial Integers (which can of course be simple integers or symbols) representing its dimensions, and a Boolean declaring whether it is external or not. The list of dimensions must be empty if the Boolean is **true**. The functions **scalarTypeOf**, **dimensionsOf** and **external?** return the appropriate parts, and it is possible to get the various basic Fortran Types via functions like **fortranReal**.

For example:

```
type:=construct(real,[i,10],false)$FortranType
REAL (i,10) (10)
```

Type: FortranType

or

```
type:=[real,[i,10],false]$FortranType
REAL (i,10) (11)
```

Type: FortranType

```
scalarTypeOf type
REAL (12)
```

Type: Union(fst: FortranScalarType, ...)

```
dimensionsOf type
[i,10] (13)
```

Type: List Polynomial Integer

```
external? type
false (14)
```

Type: Boolean

```
fortranLogical()
LOGICAL (15)
```

Type: FortranType

```
construct(integer,[],true)$FortranType
EXTERNAL INTEGER (16)
```

Type: FortranType

SymbolTable

This domain creates and manipulates a symbol table for generated Fortran code. This is used by FortranProgram to represent the types of objects in a subprogram. The commands available are:

<b>empty</b>	creates a new SymbolTable
<b>declare!</b>	creates a new entry in a table
<b>fortranTypeOf</b>	returns the type of an object in a table
<b>parametersOf</b>	returns a list of all the symbols in the table
<b>typeList</b>	returns a list of all objects of a given type
<b>typeLists</b>	returns a list of lists of all objects sorted by type
<b>externalList</b>	returns a list of all EXTERNAL objects
<b>printTypes</b>	produces Fortran type declarations from a table

```
symbols := empty()$SymbolTable
table() (17)
```

Type: SymbolTable

```

declare!(X,fortranReal(),symbols)
REAL
Type: FortranType
declare!(M,construct(real,[i,j],false)$FortranType,symbols)
REAL (i,j)
Type: FortranType
declare!([i,j],fortranInteger(),symbols)
INTEGER
Type: FortranType
symbols
table(X = REAL , M = REAL (i,j), i = INTEGER , j = INTEGER )
Type: SymbolTable
fortranTypeOf(i,symbols)
INTEGER
Type: FortranType
typeList(real,symbols)
[X, [M, i, j]]
Type: List Union(name: Symbol, bounds: List Union(S: Symbol, P: Polynomial Integer))
printTypes symbols
    INTEGER j,i
    DOUBLE PRECISION X,M(i,j)
Type: Void

```

## TheSymbolTable

This domain creates and manipulates one global symbol table to be used, for example, during template processing. It is also used when linking to external Fortran routines. The information stored for each subprogram (and the main program segment, where relevant) is:

- its name;
- its return type;
- its argument list;
- and its argument types.

Initially, any information provided is deemed to be for the main program segment.

Issuing the following command indicates that from now on all information refers to the subprogram F.

`newSubProgram F`

Type: Void

It is possible to return to processing the main program segment by issuing the command:

`endSubProgram()`  
`MAIN`

(26)

Type: Symbol

The following commands exist:

<b><code>returnType!</code></b>	declares the return type of the current subprogram
<b><code>returnTypeOf</code></b>	returns the return type of a subprogram
<b><code>argumentList!</code></b>	declares the argument list of the current subprogram
<b><code>argumentListOf</code></b>	returns the argument list of a subprogram
<b><code>declare!</code></b>	provides type declarations for parameters of the current subprogram
<b><code>symbolTableOf</code></b>	returns the symbol table of a subprogram
<b><code>printHeader</code></b>	produces the Fortran header for the current subprogram

In addition there are versions of these commands which are parameterised by the name of a subprogram, and others parameterised by both the name of a subprogram and by an instance of TheSymbolTable.

`newSubProgram F`

Type: Void

`argumentList!(F,[X])`

Type: Void

`returnType!(F,real)`

Type: Void

`declare!(X,fortranReal(),F)`

`REAL`

(30)

Type: FortranType

	<pre> printHeader F       DOUBLE PRECISION FUNCTION F(X)       DOUBLE PRECISION X </pre>	Type: Void
Advanced Fortran Code Generation	This section describes facilities for representing Fortran statements, and building up complete subprograms from them.	
Switch	<p>This domain is used to represent statements like <math>x &lt; y</math>. Although these can be represented directly in AXIOM, it is a little cumbersome, since AXIOM evaluates the last statement, for example, to <b>true</b> (since <math>x</math> is lexicographically less than <math>y</math>).</p> <p>Instead we have a set of operations, such as <b>LT</b> to represent <math>&lt;</math>, to let us build such statements. The available constructors are:</p>	
	<pre>       LT      &lt;       GT      &gt;       LE      ≤       GE      ≥       EQ      =       AND     and       OR      or       NOT     not </pre>	
So for example:	<pre> LT(x,y) </pre> $x < y \tag{32}$	Type: Switch
FortranCode	This domain represents code segments or operations: currently assignments, conditionals, blocks, comments, gotos, continues, various kinds of loops, and return statements.	
For example we can create quite a complicated conditional statement using assignments, and then turn it into Fortran code:	<pre> c := cond(LT(X,Y),assign(F,X),cond(GT(Y,Z),assign(F,Y),       assign(F,Z))) conditional </pre> $\tag{33}$	Type: FortranCode
	<pre> printCode c       IF(X.LT.Y) THEN         F=X       ELSEIF(Y.GT.Z) THEN         F=Y       ELSE         F=Z       ENDIF </pre>	Type: Void

	The Fortran code is printed on the current Fortran output stream.	
FortranProgram	This domain is used to construct complete Fortran subprograms out of elements of FortranCode. It is parameterised by the name of the target subprogram (a Symbol), its return type (from Union(FortranScalarType, “void”)), its arguments (from List Symbol), and its symbol table (from SymbolTable). One can <b>coerce</b> elements of either FortranCode or Expression into it.	
First of all we create a symbol table:	<pre>symbols := empty()\$SymbolTable table()</pre>	(35) Type: SymbolTable
Now put some type declarations into it:	<pre>declare!([X,Y],fortranReal(),symbols) REAL</pre>	(36) Type: FortranType
Then (for convenience) we set up the particular instantiation of FortranProgram	<pre>FP := FortranProgram(F,real,[X,Y],symbols) FortranProgram (F,REAL,[X,Y],table(...,...))</pre>	(37) Type: Domain
Create an object of type Expression(Integer):	<pre>asp := X*sin(Y) X sin(Y)</pre>	(38) Type: Expression Integer
Now <b>coerce</b> it into FP, and print its Fortran form:	<pre>outputAsFortran(asp::FP) DOUBLE PRECISION FUNCTION F(X,Y) DOUBLE PRECISION Y,X F=X*DSIN(Y) RETURN END</pre>	Type: Void
	We can generate a FortranProgram using FortranCode. For example:	
Augment our symbol table:	<pre>declare!(Z,fortranReal(),symbols) REAL</pre>	(40) Type: FortranType



and transform the conditional expression we prepared earlier:

```
outputAsFortran([c,returns()])::FP)
      DOUBLE PRECISION FUNCTION F(X,Y)
      DOUBLE PRECISION Z,Y,X
      IF(X.LT.Y)THEN
        F=X
      ELSEIF(Y.GT.Z)THEN
        F=Y
      ELSE
        F=Z
      ENDIF
      RETURN
      END
```

Type: Void

### 1.3.5 Some technical information

---

The model adopted for the link is a server-client configuration – AXIOM acting as a client via a local agent (a process called **nagman**). The server side is implemented by the **nagd** daemon process which may run on a different host. The **nagman** local agent is started by default whenever you start AXIOM. The **nagd** server must be started separately. Instructions for installing and running the server are supplied in Section ?? on page ???. Use the **)set naglink host** system command to point your local agent to a server in your network.

On the AXIOM side, one sees a set of *packages* (ask Browse for *Nag\**) for each chapter, each exporting operations with the same name as a routine in the NAG Foundation Library. The arguments and return value of each operation belong to standard AXIOM types.

The **man** pages for the NAG Foundation Library are accessible via the description of each operation in Browse (among other places).

In the implementation of each operation, the set of inputs is passed to the local agent **nagman**, which makes a Remote Procedure Call (RPC) to the remote **nagd** daemon process. The local agent receives the RPC results and forwards them to the AXIOM workspace where they are interpreted appropriately.

How are Fortran subroutines turned into RPC calls? For each Fortran routine in the NAG Foundation Library, a C **main()** routine is supplied. Its job is to assemble the RPC input (numeric) data stream into the appropriate Fortran data structures for the routine, call the Fortran routine from C and serialize the results into an RPC output data stream.

Many NAG Foundation Library routines accept ASPs (Argument Subprogram Parameters). These specify user-supplied Fortran routines (e.g. a routine to supply values of a function is required for numerical integra-

tion). How are they handled? There are new facilities in AXIOM to help. A set of AXIOM domains has been provided to turn values in standard AXIOM types (such as Expression Integer) into the appropriate piece of Fortran for each case (a filename pointing to Fortran source for the ASP can always be supplied instead). Ask Browse for *Asp\** to see these domains. The Fortran fragments are included in the outgoing RPC stream, but `nagd` intercepts them, compiles them, and links them with the `main()` C program before executing the resulting program on the numeric part of the RPC stream.

## 1.4 Interactive Front-end and Language

---

The `leave` keyword has been replaced by the `break` keyword for compatibility with the new AXIOM extension language. See section Section 5.4.3 on page 159 for more information.

Curly braces are no longer used to create sets. Instead, use `set` followed by a bracketed expression. For example,

```
set [1,2,3,4]
{1, 2, 3, 4} (1)
```

Type: Set PositiveInteger

Curly braces are now used to enclose a block (see section Section 5.2 on page 153 for more information). For compatibility, a block can still be enclosed by parentheses as well.

“Free functions” created by the Aldor compiler can now be loaded and used within the AXIOM interpreter. A *free function* is a library function that is implemented outside a domain or category constructor.

New coercions to and from type Expression have been added. For example, it is now possible to map a polynomial represented as an expression to an appropriate polynomial type.

Various messages have been added or rewritten for clarity.

## 1.5 Library

---

The FullPartialFractionExpansion domain has been added. This domain computes factor-free full partial fraction expansions. See section ‘FullPartialFractionExpansion’ on page 435 for examples.

We have implemented the Bertrand/Cantor algorithm for integrals of hyperelliptic functions. This brings a major speedup for some classes of algebraic integrals.

We have implemented a new (direct) algorithm for integrating trigonometric functions. This brings a speedup and an improvement in the answer

quality.

The `SmallFloat` domain has been renamed `DoubleFloat` and `SmallInteger` has been renamed `SingleInteger`. The new abbreviations are `DFLOAT` and `SINT`, respectively. We have defined the macro `SF`, the old abbreviation for `SmallFloat`, to expand to `DoubleFloat` and modified the documentation and input file examples to use the new names and abbreviations. You should do the same in any private AXIOM files you have.

There are many new categories, domains and packages related to the NAG Library Link facility. See the file

`$AXIOM/../../src/algebra/exposed.lsp`

for a list of constructors in the **naglink** AXIOM exposure group.

We have made improvements to the differential equation solvers and there is a new facility for solving systems of first-order linear differential equations. In particular, an important fix was made to the solver for inhomogeneous linear ordinary differential equations that corrected the calculation of particular solutions. We also made improvements to the polynomial and transcendental equation solvers including the ability to solve some classes of systems of transcendental equations.

The efficiency of power series have been improved and left and right expansions of `tan(f(x))` at  $x = a$  pole of  $f(x)$  can now be computed. A number of power series bugs were fixed and the `GeneralSeries` domain was added. The power series variable can appear in the coefficients and when this happens, you cannot differentiate or integrate the series. Differentiation and integration with respect to other variables is supported.

A domain was added for representing asymptotic expansions of a function at an exponential singularity.

For limits, the main new feature is the exponential expansion domain used to treat certain exponential singularities. Previously, such singularities were treated in an *ad hoc* way and only a few cases were covered. Now AXIOM can do things like

```
limit((x+1)**(x+1)/x**x-x**x/(x-1)**(x-1), x=%plusInfinity)
```

in a systematic way. It only does one level of nesting, though. In other words, we can handle `exp( some function with a pole )`, but not `exp(exp( some function with a pole ))`.

The computation of integral bases has been improved through careful use of Hermite row reduction. A P-adic algorithm for function fields of algebraic curves in finite characteristic has also been developed.

Miscellaneous: There is improved conversion of definite and indefinite

integrals to InputForm; binomial coefficients are displayed in a new way; some new simplifications of radicals have been implemented; the operation **complexForm** for converting to rectangular coordinates has been added; symmetric product operations have been added to LinearOrdinaryDifferentialOperator.

## 1.6 HyperDoc

---

The buttons on the titlebar and scrollbar have been replaced with ones which have a 3D effect. You can change the foreground and background colors of these “controls” by including and modifying the following lines in your **.Xdefaults** file.

```
Axiom.hyperdoc.ControlBackground: White  
Axiom.hyperdoc.ControlForeground: Black
```

For various reasons, HyperDoc sometimes displays a secondary window. You can control the size and placement of this window by including and modifying the following line in your **.Xdefaults** file.

```
Axiom.hyperdoc.FormGeometry: =950x450+100+0
```

This setting is a standard X Window System geometry specification: you are requesting a window 950 pixels wide by 450 deep and placed in the upper left corner.

Some key definitions have been changed to conform more closely with the CUA guidelines. Press F9 to see the current definitions.

Input boxes (for example, in the Browser) now accept paste-ins from the X Window System. Use the second button to paste in something you have previously copied or cut. An example of how you can use this is that you can paste the type from an AXIOM computation into the main Browser input box.

## 1.7 Documentation

---

We describe here a few additions to the on-line version of the AXIOM book which you can read with HyperDoc.

A section has been added to the graphics chapter, describing how to build two-dimensional graphs from lists of points. An example is given showing how to read the points from a file. See section Section 7.1.9 on page 256 for details.

A further section has been added to that same chapter, describing how to add a two-dimensional graph to a viewport which already contains other graphs. See section Section ?? on page ??? for details.

Chapter 3 and the on-line HyperDoc help have been unified.

An explanation of operation names ending in “?” and “!” has been added to the first chapter. See the end of the section Section 1.3.6 on page 51 for details.

An expanded explanation of using predicates has been added to the sixth chapter. See the example involving **evenRule** in the middle of the section Section 6.21 on page 228 for details.

Documentation for the `)compile`, `)library` and `)load` commands has been greatly changed. This reflects the ability of the `)compile` to now invoke the Aldor compiler, the impending deletion of the `)load` command and the new `)library` command. The `)library` command replaces `)load` and is compatible with the compiled output from both the old and new compilers.



---

## PART II

---

# Basic Features of AXIOM





---

# An Overview of AXIOM

Welcome to the AXIOM environment for interactive computation and problem solving. Consider this chapter a brief, whirlwind tour of the AXIOM world. We introduce you to AXIOM's graphics and the AXIOM language. Then we give a sampling of the large variety of facilities in the AXIOM system, ranging from the various kinds of numbers, to data types (like lists, arrays, and sets) and mathematical objects (like matrices, integrals, and differential equations). We conclude with the discussion of system commands and an interactive "undo."

Before embarking on the tour, we need to brief those readers working interactively with AXIOM on some details. Others can skip right immediately to Section 1.2 on page 46.

## 1.1 Starting Up and Winding Down

---

You need to know how to start the AXIOM system and how to stop it. We assume that AXIOM has been correctly installed on your machine (as described in another AXIOM document).

To begin using AXIOM, issue the command **axiom** to the operating system shell. There is a brief pause, some start-up messages, and then one or more windows appear.

If you are not running AXIOM under the X Window System, there is only one window (the console). At the lower left of the screen there is a prompt that looks like

```
(1) ->
```

When you want to enter input to AXIOM, you do so on the same line after the prompt. The “1” in “(1)” is the computation step number and is incremented after you enter AXIOM statements. Note, however, that a system command such as **)clear all** may change the step number in other ways. We talk about step numbers more when we discuss system commands and the workspace history facility.

If you are running AXIOM under the X Window System, there may be two windows: the console window (as just described) and the HyperDoc main menu. HyperDoc is a multiple-window hypertext system that lets you view AXIOM documentation and examples on-line, execute AXIOM expressions, and generate graphics. If you are in a graphical windowing environment, it is usually started automatically when AXIOM begins. If it is not running, issue **)hd** to start it. We discuss the basics of HyperDoc in Chapter 3.

To interrupt an AXIOM computation, hold down the **Ctrl** (control) key and press **C**. This brings you back to the AXIOM prompt.

To exit from AXIOM, move to the console window, type **)quit** at the input prompt and press the **Enter** key. You will probably be prompted with the following message:

Please enter **y** or **yes** if you really want to leave the  
interactive environment and return to the operating system

You should respond **yes**, for example, to exit AXIOM.

We are purposely vague in describing exactly what your screen looks like or what messages AXIOM displays. AXIOM runs on a number of different machines, operating systems and window environments, and these differences all affect the physical look of the system. You can also change the

way that AXIOM behaves via *system commands* described later in this chapter and in Appendix A. System commands are special commands, like `)set`, that begin with a closing parenthesis and are used to change your environment. For example, you can set a system variable so that you are not prompted for confirmation when you want to leave AXIOM.

### 1.1.1 Clef

---

If you are using AXIOM under the X Window System, the Clef command line editor is probably available and installed. With this editor you can recall previous lines with the up and down arrow keys (`↑` and `↓`). To move forward and backward on a line, use the right and left arrows (`→` and `←`). You can use the `Insert` key to toggle insert mode on or off. When you are in insert mode, the cursor appears as a large block and if you type anything, the characters are inserted into the line without deleting the previous ones.

If you press the `Home` key, the cursor moves to the beginning of the line and if you press the `End` key, the cursor moves to the end of the line. Pressing `Ctrl-End` deletes all the text from the cursor to the end of the line.

Clef also provides AXIOM operation name completion for a limited set of operations. If you enter a few letters and then press the `Tab` key, Clef tries to use those letters as the prefix of an AXIOM operation name. If a name appears and it is not what you want, press `Tab` again to see another name.

You are ready to begin your journey into the world of AXIOM. Proceed to the first stop.

## 1.2 Typographic Conventions

---

In this book we have followed these typographical conventions:

- Categories, domains and packages are displayed in a sans-serif typeface: `Ring`, `Integer`, `DiophantineSolutionPackage`.
- Prefix operators, infix operators, and punctuation symbols in the AXIOM language are displayed in the text like this: “+”, “\$”, “+→”.
- AXIOM expressions or expression fragments are displayed in a monospace typeface: `inc(x) == x + 1`.
- For clarity of presentation,  $\mathrm{T}_{\mathrm{E}}\mathrm{X}$  is often used to format expressions:  $g(x) = x^2 + 1$ .
- Function names and HyperDoc button names are displayed in the text in a bold typeface: **factor**, **integrate**, **Lighting**.
- Italics are used for emphasis and for words defined in the glossary: *category*.

This book contains over 2500 examples of AXIOM input and output. All examples were run through AXIOM and their output was created in  $\mathrm{T}_{\mathrm{E}}\mathrm{X}$  form for this book by the AXIOM `TexFormat` package. We have deleted system messages from the example output if those messages are not important for the discussions in which the examples appear.

## 1.3 The AXIOM Language

---

The AXIOM language is a rich language for performing interactive computations and for building components of the AXIOM library. Here we present only some basic aspects of the language that you need to know for the rest of this chapter. Our discussion here is intentionally informal, with details unveiled on an “as needed” basis. For more information on a particular construct, we suggest you consult the index at the back of the book.

### 1.3.1 Arithmetic Expressions

---

AXIOM puts implicit parentheses around operations of higher precedence, and groups those of equal precedence from left to right.

The above expression is equivalent to this.

If an expression contains subexpressions enclosed in parentheses, the parenthesized subexpressions are evaluated first (from left to right, from inside out).

For arithmetic expressions, use the “+” and “-” *operators* as in mathematics. Use “\*” for multiplication, and “\*\*” for exponentiation. To create a fraction, use “/”. When an expression contains several operators, those of highest *precedence* are evaluated first. For arithmetic operators, “\*\*” has highest precedence, “\*” and “/” have the next highest precedence, and “+” and “-” have the lowest precedence.

$$1 + 2 - 3 / 4 * 3 ** 2 - 1$$

$$-\frac{19}{4}$$
(1)

Type: Fraction Integer

$$((1 + 2) - ((3 / 4) * (3 ** 2))) - 1$$

$$-\frac{19}{4}$$
(2)

Type: Fraction Integer

$$1 + 2 - 3 / (4 * 3 ** (2 - 1))$$

$$\frac{11}{4}$$
(3)

Type: Fraction Integer

### 1.3.2 Previous Results

---

Use the percent sign (“%”) to refer to the last result. Also, use “%%” to refer to previous results. %%(-1) is equivalent to “%”, %%(-2) returns the next to the last result, and so on. %(1) returns the result from step number 1, %(2) returns the result from step number 2, and so on. %(0) is not defined.

This is ten to the tenth power.

$$10 ** 10$$

$$10000000000$$
(1)

Type: PositiveInteger

This is the last result minus one.	$\% - 1$ 9999999999	(2) Type: PositiveInteger
This is the last result.	$\%\% (-1)$ 9999999999	(3) Type: PositiveInteger
This is the result from step number 1.	$\%\% (1)$ 10000000000	(4) Type: PositiveInteger

### 1.3.3 Some Types

---

Everything in AXIOM has a type. The type determines what operations you can perform on an object and how the object can be used. An entire chapter of this book (Chapter 2) is dedicated to the interactive use of types. Several of the final chapters discuss how types are built and how they are organized in the AXIOM library.

Positive integers are given type PositiveInteger.	8 8	(1) Type: PositiveInteger
Negative ones are given type Integer. This fine distinction is helpful to the AXIOM interpreter.	-8 -8	(2) Type: Integer
Here a positive integer exponent gives a polynomial result.	$x^{**8}$ $x^8$	(3) Type: Polynomial Integer
Here a negative integer exponent produces a fraction.	$x^{**(-8)}$ $\frac{1}{x^8}$	(4) Type: Fraction Polynomial Integer

### 1.3.4 Symbols, Variables, Assignments, and Declarations

---

A *symbol* is a literal used for the input of things like the “variables” in polynomials and power series.

We use the three symbols  $x$ ,  $y$ , and  $z$  in entering this polynomial.

$$(x - y*z)**2$$

$$y^2 z^2 - 2 x y z + x^2 \quad (1)$$

Type: Polynomial Integer

A symbol has a name beginning with an uppercase or lowercase alphabetic character, “%”, or “!”. Successive characters (if any) can be any of the above, digits, or “?”. Case is distinguished: the symbol `points` is different from the symbol `Points`.

A symbol can also be used in AXIOM as a *variable*. A variable refers to a value. To *assign* a value to a variable, the operator “:=” is used.<sup>1</sup> A variable initially has no restrictions on the kinds of values to which it can refer.

This assignment gives the value 4 (an integer) to a variable named  $x$ .

$$x := 4$$

$$4 \quad (2)$$

Type: PositiveInteger

This gives the value  $z + 3/5$  (a polynomial) to  $x$ .

$$x := z + 3/5$$

$$z + \frac{3}{5} \quad (3)$$

Type: Polynomial Fraction Integer

To restrict the types of objects that can be assigned to a variable, use a *declaration*

$$y : \text{Integer}$$

Type: Void

After a variable is declared to be of some type, only values of that type can be assigned to that variable.

$$y := 89$$

$$89 \quad (5)$$

Type: Integer

The declaration for  $y$  forces values assigned to  $y$  to be converted to integer values.

$$y := \sin \%pi$$

$$0 \quad (6)$$

Type: Integer

If no such conversion is possible, AXIOM refuses to assign a value to  $y$ .

$$y := 2/3$$

Cannot convert right-hand side of assignment

$$\frac{2}{3}$$

to an object of the type Integer of the left-hand side.

<sup>1</sup>AXIOM actually has two forms of assignment: *immediate* assignment, as discussed here, and *delayed assignment*. See Section 5.1 on page 150 for details.

```
f : Float := 2/3
0.6666666666666667
```

(7)  
Type: Float

```
f : Float; f := 2/3
0.6666666666666667
Type: Float
```

$$q \quad (9)$$

Type: Variable q

$$\begin{array}{l} [q, r] \\ [q, r] \end{array} \quad \text{Type: List OrderedVariableList [q, r]} \quad (10)$$

f	
0.666666666666666667	(11)
	Type: Float

$$f \quad \text{Type: Variable f} \quad (12)$$

Objects of one type can usually be “converted” to objects of several other types. To *convert* an object to a new type, use the “`::`” infix operator.<sup>2</sup> For example, to display an object, it is necessary to convert the object to type `OutputForm`.

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This produces a polynomial with rational number coefficients.

$$\text{p} := r^{**2} + 2/3$$

$$r^2 + \frac{2}{3}$$
(1)

Type: Polynomial Fraction Integer

Create a quotient of polynomials with integer coefficients by using “::”.

$$\text{p} :: \text{Fraction Polynomial Integer}$$

$$\frac{3 r^2 + 2}{3}$$
(2)

Type: Fraction Polynomial Integer

Some conversions can be performed automatically when AXIOM tries to evaluate your input. Others conversions must be explicitly requested.

### 1.3.6 Calling Functions

As we saw earlier, when you want to add or subtract two values, you place the arithmetic operator “+” or “-” between the two *arguments* denoting the values. To use most other AXIOM operations, however, you use another syntax: write the name of the operation first, then an open parenthesis, then each of the arguments separated by commas, and, finally, a closing parenthesis. If the operation takes only one argument and the argument is a number or a symbol, you can omit the parentheses.

This calls the operation **factor** with the single integer argument 120.

$$\text{factor}(120)$$

$$2^3 3 5$$
(1)

Type: Factored Integer

This is a call to **divide** with the two integer arguments 125 and 7.

$$\text{divide}(125, 7)$$

$$[\text{quotient} = 17, \text{remainder} = 6]$$
(2)

Type: Record(quotient: Integer, remainder: Integer)

This calls **quatern** with four floating-point arguments.

$$\text{quatern}(3.4, 5.6, 2.9, 0.1)$$

$$3.4 + 5.6 i + 2.9 j + 0.1 k$$
(3)

Type: Quaternion Float

This is the same as **factorial**(10).

$$\text{factorial } 10$$

$$3628800$$
(4)

Type: PositiveInteger

An operations that returns a Boolean value (that is, **true** or **false**) frequently has a name suffixed with a question mark (“?”). For example, the **even?** operation returns **true** if its integer argument is an even number, **false** otherwise.

An operation that can be destructive on one or more arguments usually has a name ending in a exclamation point (“!”). This actually means that

it is *allowed* to update its arguments but it is not *required* to do so. For example, the underlying representation of a collection type may not allow the very last element to be removed and so an empty object may be returned instead. Therefore, it is important that you use the object returned by the operation and not rely on a physical change having occurred within the object. Usually, destructive operations are provided for efficiency reasons.

### 1.3.7 Some Predefined Macros

---

AXIOM provides several *macros* for your convenience.<sup>3</sup> Macros are names (or forms) that expand to larger expressions for commonly used values.

<code>%i</code>	The square root of -1.
<code>%e</code>	The base of the natural logarithm.
<code>%pi</code>	$\pi$ .
<code>%infinity</code>	$\infty$ .
<code>%plusInfinity</code>	$+\infty$ .
<code>%minusInfinity</code>	$-\infty$ .

### 1.3.8 Long Lines

---

When you enter AXIOM expressions from your keyboard, there will be times when they are too long to fit on one line. AXIOM does not care how long your lines are, so you can let them continue from the right margin to the left side of the next line.

Alternatively, you may want to enter several shorter lines and have AXIOM glue them together. To get this glue, put an underscore (`_`) at the end of each line you wish to continue.

```
2_  
+_  
3
```

is the same as if you had entered

```
2+3
```

If you are putting your AXIOM statements in an input file (see Section 4.1 on page 139), you can use indentation to indicate the structure of your program. (see Section 5.2 on page 153).

### 1.3.9 Comments

---

Comment statements begin with two consecutive hyphens or two consecutive plus signs and continue until the end of the line.

---

<sup>3</sup>See Section 6.2 on page 179 for a discussion on how to write your own macros.

The comment beginning with -- 2 + 3 -- this is rather simple, no?  
is ignored by AXIOM.

5

(1)

Type: PositiveInteger

There is no way to write long multi-line comments other than starting  
each line with "--" or "++".

## 1.4 Graphics

This is an example of AXIOM's two-dimensional plotting. From the 2D Control Panel you can rescale the plot, turn axes and units on and off and save the image, among other things. This PostScript image was produced by clicking on the **PS** 2D Control Panel button.

AXIOM has a two- and three-dimensional drawing and rendering package that allows you to draw, shade, color, rotate, translate, map, clip, scale and combine graphic output of AXIOM computations. The graphics interface is capable of plotting functions of one or more variables and plotting parametric surfaces. Once the graphics figure appears in a window, move your mouse to the window and click. A control panel appears immediately and allows you to interactively transform the object.

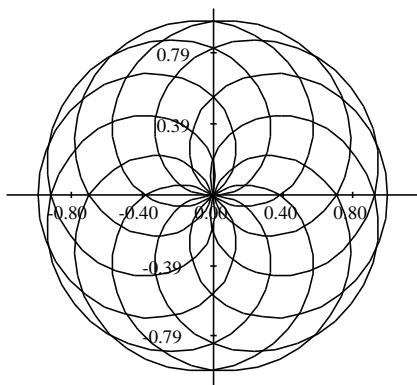
```
draw(cos(5*t/8), t=0..16*%pi, coordinates==polar)
```

```
Compiling function %B with type DoubleFloat ->
DoubleFloat
Graph data being transmitted to the viewport
manager...
AXIOM2D data being transmitted to the viewport
manager...
```

TwoDimensionalViewport: "cos (5\*t)/8"

(1)

Type: TwoDimensionalViewport



This is an example of AXIOM's three-dimensional plotting. It is a monochrome graph of the complex arctangent function. The image displayed was rotated and had the "shade" and "outline" display options set from the 3D Control Panel. The PostScript output was produced by clicking on the **save** 3D Control Panel button and then clicking on the **PS** button. See Section 8.1 on page 264 for more details and examples of AXIOM's numeric and graphics capabilities.

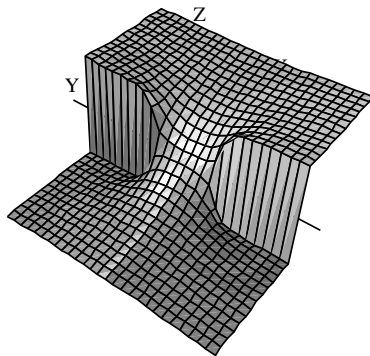
```
draw((x,y) +-> real atan complex(x,y), -%pi..%pi, -
      %pi..%pi, colorFunction == (x,y) +-> argument atan
      complex(x,y))
```

Transmitting data...

ThreeDimensionalViewport: "AXIOM3D"

(2)

Type: ThreeDimensionalViewport



An exhibit of AXIOM Images is given in the center section of this book. For a description of the commands and programs that produced these figures, see Appendix F. PostScript output is available so that AXIOM images can be printed.<sup>4</sup> See Chapter 7 for more examples and details about using AXIOM's graphics facilities.

---

<sup>4</sup>PostScript is a trademark of Adobe Systems Incorporated, registered in the United States.

## 1.5 Numbers

AXIOM distinguishes very carefully between different kinds of numbers, how they are represented and what their properties are. Here are a sampling of some of these kinds of numbers and some things you can do with them.

Integer arithmetic is always exact.

```
11**13 * 13**11 * 17**7 - 19**5 * 23**3
25387751112538918594666224484237298
```

(1)  
Type: PositiveInteger

Integers can be represented in factored form.

```
factor 643238070748569023720594412551704344145570763243
1113 1311 177 195 233 292
```

(2)  
Type: Factored Integer

Results stay factored when you do arithmetic. Note that the 12 is automatically factored for you.

```
% * 12
22 3 1113 1311 177 195 233 292
```

(3)  
Type: Factored Integer

Integers can also be displayed to bases other than 10. This is an integer in base 11.

```
radix(25937424601,11)
10000000000
```

(4)  
Type: RadixExpansion 11

Roman numerals are also available for those special occasions.

```
roman(1992)
MCMXCII
```

(5)  
Type: RomanNumeral

Rational number arithmetic is also exact.

```
r := 10 + 9/2 + 8/3 + 7/4 + 6/5 + 5/6 + 4/7 + 3/8 + 2/9
55739
-----
2520
```

(6)  
Type: Fraction Integer

To factor fractions, you have to map **factor** onto the numerator and denominator.

```
map(factor,r)
139 401
-----
23 32 5 7
```

(7)  
Type: Fraction Factored Integer

Type `SingleInteger` refers to machine word-length integers. In English, this expression means “11 as a small integer”.

```
11@SingleInteger
11
```

(8)  
Type: SingleInteger

Machine double-precision floating-point numbers are also available for numeric and graphical applications.

```
123.21@DoubleFloat
123.21000000000001
```

(9)  
Type: DoubleFloat

The normal floating-point type in AXIOM, `Float`, is a software implementation of floating-point numbers in which the exponent and the man-

tissa may have any number of digits.<sup>5</sup> The types `Complex(Float)` and `Complex(DoubleFloat)` are the corresponding software implementations of complex floating-point numbers.

This is a floating-point approximation to about twenty digits. The “`::`” is used here to change from one kind of object (here, a rational number) to another (a floating-point number).

```
r :: Float
22.118650793650793651
```

(10)  
Type: Float

Use **digits** to change the number of digits in the representation. This operation returns the previous value so you can reset it later.

```
digits(22)
20
```

(11)  
Type: PositiveInteger

To 22 digits of precision, the number  $e^{\pi\sqrt{163.0}}$  appears to be an integer.

```
exp(%pi * sqrt 163.0)
262537412640768744.0
```

(12)  
Type: Float

Increase the precision to forty digits and try again.

```
digits(40); exp(%pi * sqrt 163.0)
262537412640768743.9999999999992500725976
```

(13)  
Type: Float

Here are complex numbers with rational numbers as real and imaginary parts.

```
(2/3 + %i)**3
-46/27 + 1/3 i
```

(14)  
Type: Complex Fraction Integer

The standard operations on complex numbers are available.

```
conjugate %
-46/27 - 1/3 i
```

(15)  
Type: Complex Fraction Integer

You can factor complex integers.

```
factor(89 - 23 * %i)
-(1 + i) (2 + i)^2 (3 + 2 i)^2
```

(16)  
Type: Factored Complex Integer

Complex numbers with floating point parts are also available.

```
exp(%pi/4.0 * %i)
0.7071067811865475244008443621048490392849+
0.7071067811865475244008443621048490392848 i
```

(17)  
Type: Complex Float

---

<sup>5</sup>See ‘Float’ on page 427 and ‘DoubleFloat’ on page 404 for additional information on floating-point types.

Every rational number has an exact representation as a repeating decimal expansion (see ‘DecimalExpansion’ on page 401).

$$\text{decimal}(1/352)$$

$$0.00284\overline{09} \quad (18)$$

Type: DecimalExpansion

A rational number can also be expressed as a continued fraction (see ‘ContinuedFraction’ on page 385).

$$\text{continuedFraction}(6543/210)$$

$$31 + \frac{1}{6} + \frac{1}{2} + \frac{1}{1} + \frac{1}{3} \quad (19)$$

Type: ContinuedFraction Integer

Also, partial fractions can be used and can be displayed in a compact ...

$$\text{partialFraction}(1, \text{factorial}(10))$$

$$\frac{159}{2^8} - \frac{23}{3^4} - \frac{12}{5^2} + \frac{1}{7} \quad (20)$$

Type: PartialFraction Integer

or expanded format (see ‘PartialFraction’ on page 525).

$$\text{padicFraction}(\%)$$

$$\frac{1}{2} + \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6} + \frac{1}{2^7} + \frac{1}{2^8} - \frac{2}{3^2} - \frac{1}{3^3} - \frac{2}{3^4} - \frac{2}{5} - \frac{2}{5^2} + \frac{1}{7} \quad (21)$$

Type: PartialFraction Integer

Like integers, bases (radices) other than ten can be used for rational numbers (see ‘RadixExpansion’ on page 537). Here we use base eight.

$$\text{radix}(4/7, 8)$$

$$0.\overline{4} \quad (22)$$

Type: RadixExpansion 8

Of course, there are complex versions of these as well. AXIOM decides to make the result a complex rational number.

$$\% + 2/3*\%i$$

$$\frac{4}{7} + \frac{2}{3}i \quad (23)$$

Type: Complex Fraction Integer

You can also use AXIOM to manipulate fractional powers.

$$(5 + \text{sqrt } 63 + \text{sqrt } 847)**(1/3)$$

$$\sqrt[3]{14 \sqrt{7} + 5} \quad (24)$$

Type: AlgebraicNumber

You can also compute with integers modulo a prime.

$$\mathbf{x} : \text{PrimeField } 7 := 5$$

$$5 \quad (25)$$

Type: PrimeField 7

Arithmetic is then done modulo 7.

$$\mathbf{x}**3$$

$$6 \quad (26)$$

Type: PrimeField 7

Since 7 is prime, you can invert nonzero values.

$$1/\mathbf{x}$$

$$3 \quad (27)$$

Type: PrimeField 7



You can also compute modulo an integer that is not a prime.

```
y : IntegerMod 6 := 5
5
(28)
Type: IntegerMod 6
```

All of the usual arithmetic operations are available.

```
y**3
5
(29)
Type: IntegerMod 6
```

Inversion is not available if the modulus is not a prime number. Modular arithmetic and prime fields are discussed in Section 8.11.1 on page 316.

```
1/y
There are 11 exposed and 12 unexposed library
operations named / having 2 argument(s) but none
was determined to be applicable. Use HyperDoc
Browse, or issue
)display op /
to learn more about the available operations.
Perhaps package-calling the operation or using
coercions on the arguments will allow you to apply
the operation.

Cannot find a definition or applicable library
operation named / with argument type(s)
PositiveInteger
IntegerMod 6

Perhaps you should use "@" to indicate the
required return type, or "$" to specify which
version of the function you need.
```

This defines **a** to be an algebraic number, that is, a root of a polynomial equation.

```
a := rootOf(a**5 + a**3 + a**2 + 3,a)
a
(30)
Type: Expression Integer
```

Computations with **a** are reduced according to the polynomial equation.

```
(a + 1)**10
-85 a^4 - 264 a^3 - 378 a^2 - 458 a - 287
(31)
Type: Expression Integer
```

Define **b** to be an algebraic number involving **a**.

```
b := rootOf(b**4 + a,b)
b
(32)
Type: Expression Integer
```

Do some arithmetic.

```
2/(b - 1)
2
b - 1
(33)
Type: Expression Integer
```

To expand and simplify this,  
call **ratDenom** to rationalize  
the denominator.

$$\begin{aligned} & \text{ratDenom}(\%) \\ & \left( a^4 - a^3 + 2 a^2 - a + 1 \right) b^3 + \left( a^4 - a^3 + 2 a^2 - a + 1 \right) b^2 + \\ & \left( a^4 - a^3 + 2 a^2 - a + 1 \right) b + a^4 - a^3 + 2 a^2 - a + 1 \end{aligned} \quad (34)$$

Type: Expression Integer

If we do this, we should get **b**.

$$\begin{aligned} & 2/\%+1 \\ & \left( \left( a^4 - a^3 + 2 a^2 - a + 1 \right) b^3 + \left( a^4 - a^3 + 2 a^2 - a + 1 \right) b^2 + \right. \\ & \left. \left( a^4 - a^3 + 2 a^2 - a + 1 \right) b + a^4 - a^3 + 2 a^2 - a + 3 \right) \\ & \frac{\left( \left( a^4 - a^3 + 2 a^2 - a + 1 \right) b^3 + \left( a^4 - a^3 + 2 a^2 - a + 1 \right) b^2 + \right.}{\left( a^4 - a^3 + 2 a^2 - a + 1 \right) b + a^4 - a^3 + 2 a^2 - a + 1} \end{aligned} \quad (35)$$

Type: Expression Integer

But we need to rationalize the  
denominator again.

$$\begin{aligned} & \text{ratDenom}(\%) \\ & b \end{aligned} \quad (36)$$

Type: Expression Integer

Types Quaternion and Octonion  
are also available.  
Multiplication of quaternions is  
non-commutative, as expected.

$$\begin{aligned} & \mathbf{q} := \text{quaternion}(1, 2, 3, 4) * \text{quaternion}(5, 6, 7, 8) - \\ & \text{quaternion}(5, 6, 7, 8) * \text{quaternion}(1, 2, 3, 4) \\ & -8 i + 16 j - 8 k \end{aligned} \quad (37)$$

Type: Quaternion Integer

## 1.6 Data Structures

AXIOM has a large variety of data structures available. Many data structures are particularly useful for interactive computation and others are useful for building applications. The data structures of AXIOM are organized into *category hierarchies* as shown on the inside back cover.

A *list* is the most commonly used data structure in AXIOM for holding objects all of the same type.<sup>6</sup> The name *list* is short for “linked-list of nodes.” Each node consists of a value (**first**) and a link (**rest**) that *points* to the next node, or to a distinguished value denoting the empty list. To get to, say, the third element, AXIOM starts at the front of the list, then traverses across two links to the third node.

Write a list of elements using square brackets with commas separating the elements.

```
u := [1, -7, 11]
[1, -7, 11] (1)
```

Type: List Integer

This is the value at the third node. Alternatively, you can say `u.3`.

```
first rest rest u
11 (2)
```

Type: PositiveInteger

Many operations are defined on lists, such as: **empty?**, to test that a list has no elements; **cons**(*x*, *l*), to create a new list with **first** element *x* and **rest** *l*; **reverse**, to create a new list with elements in reverse order; and **sort**, to arrange elements in order.

An important point about lists is that they are “mutable”: their constituent elements and links can be changed “in place.” To do this, use any of the operations whose names end with the character “!”.

The operation **concat!**(*u*, *v*) replaces the last link of the list *u* to point to some other list *v*. Since *u* refers to the original list, this change is seen by *u*.

```
concat!(u, [9, 1, 3, -4]); u
[1, -7, 11, 9, 1, 3, -4] (3)
```

Type: List Integer

A *cyclic list* is a list with a “cycle”: a link pointing back to an earlier node of the list. To create a cycle, first get a node somewhere down the list.

```
lastnode := rest(u, 3)
[9, 1, 3, -4] (4)
```

Type: List Integer

Use **setrest!** to change the link emanating from that node to point back to an earlier part of the list.

```
setrest!(lastnode, rest(u, 2)); u
[1, -7, 11, 9] (5)
```

Type: List Integer

A *stream* is a structure that (potentially) has an infinite number of distinct

---

<sup>6</sup>Lists are discussed in ‘List’ on page 489 and in Section 5.5 on page 171.

elements.<sup>7</sup> Think of a stream as an “infinite list” where elements are computed successively.

Create an infinite stream of factored integers. Only a certain number of initial elements are computed and displayed.

```
[factor(i) for i in 2.. by 2]
```

$$\left[2, 2^2, 2 \cdot 3, 2^3, 2 \cdot 5, 2^2 \cdot 3, 2 \cdot 7, \dots\right] \quad (6)$$

Type: Stream Factored Integer

AXIOM represents streams by a collection of already-computed elements together with a function to compute the next element “on demand.” Asking for the  $n^{\text{th}}$  element causes elements 1 through  $n$  to be evaluated.

```
%.36
```

$$2^3 \cdot 3^2 \quad (7)$$

Type: Factored Integer

Streams can also be finite or cyclic. They are implemented by a linked list structure similar to lists and have many of the same operations. For example, **first** and **rest** are used to access elements and successive nodes of a stream.

A *one-dimensional array* is another data structure used to hold objects of the same type.<sup>8</sup> Unlike lists, one-dimensional arrays are inflexible—they are implemented using a fixed block of storage. Their advantage is that they give quick and equal access time to any element.

A simple way to create a one-dimensional array is to apply the operation **oneDimensionalArray** to a list of elements.

```
a := oneDimensionalArray [1, -7, 3, 3/2]
```

$$\left[1, -7, 3, \frac{3}{2}\right] \quad (8)$$

Type: OneDimensionalArray Fraction Integer

One-dimensional arrays are also mutable: you can change their constituent elements “in place.”

```
a.3 := 11; a
```

$$\left[1, -7, 11, \frac{3}{2}\right] \quad (9)$$

Type: OneDimensionalArray Fraction Integer

<sup>7</sup>Streams are discussed in ‘Stream’ on page 575 and in Section 5.5 on page 171.

<sup>8</sup>See ‘OneDimensionalArray’ on page 514 for details.

However, one-dimensional arrays are not flexible structures. You cannot destructively **concat!** them together.

```
concat!(a,oneDimensionalArray [1,-2])
```

There are 5 exposed and 0 unexposed library operations named concat! having 2 argument(s) but none was determined to be applicable. Use HyperDoc Browse, or issue  
`)display op concat!`  
 to learn more about the available operations. Perhaps package-calling the operation or using coercions on the arguments will allow you to apply the operation.

```
Cannot find a definition or applicable library
operation named concat! with argument type(s)
    OneDimensionalArray Fraction Integer
    OneDimensionalArray Integer
```

Perhaps you should use "@" to indicate the required return type, or "\$" to specify which version of the function you need.

Examples of datatypes similar to OneDimensionalArray are: Vector (vectors are mathematical structures implemented by one-dimensional arrays), String (arrays of “characters,” represented by byte vectors), and Bits (represented by “bit vectors”).

A vector of 32 bits, each representing the Boolean value **true**.

```
bits(32,true)
```

```
"11111111111111111111111111111111" (10)
```

Type: Bits

A *flexible array* is a cross between a list and a one-dimensional array.<sup>9</sup> Like a one-dimensional array, a flexible array occupies a fixed block of storage. Its block of storage, however, has room to expand! When it gets full, it grows (a new, larger block of storage is allocated); when it has too much room, it contracts.

Create a flexible array of three elements.

```
f := flexibleArray [2, 7, -5]
```

```
[2, 7, -5] (11)
```

Type: FlexibleArray Integer

Insert some elements between the second and third elements.

```
insert!(flexibleArray [11, -3],f,2)
```

```
[2, 11, -3, 7, -5] (12)
```

Type: FlexibleArray Integer

Flexible arrays are used to implement “heaps.” A *heap* is an example of a data structure called a *priority queue*, where elements are ordered

<sup>9</sup>See ‘FlexibleArray’ on page 425 for details.

with respect to one another.<sup>10</sup> A heap is organized so as to optimize insertion and extraction of maximum elements. The **extract!** operation returns the maximum element of the heap, after destructively removing that element and reorganizing the heap so that the next maximum element is ready to be delivered.

An easy way to create a heap is to apply the operation **heap** to a list of values.

```
h := heap [-4, 7, 11, 3, 4, -7]
[11, 4, 7, -4, 3, -7]
```

(13)

Type: Heap Integer

This loop extracts elements one-at-a-time from **h** until the heap is exhausted, returning the elements as a list in the order they were extracted.

```
[extract!(h) while not empty?(h)]
[11, 7, 4, 3, -4, -7]
```

(14)

Type: List Integer

A *binary tree* is a “tree” with at most two branches per node: it is either empty, or else is a node consisting of a value, and a left and right subtree (again, binary trees).<sup>11</sup>

A *binary search tree* is a binary tree such that, for each node, the value of the node is greater than all values (if any) in the left subtree, and less than or equal all values (if any) in the right subtree.

```
binarySearchTree [5, 3, 2, 9, 4, 7, 11]
[[2, 3, 4], 5, [7, 9, 11]]
```

(15)

Type: BinarySearchTree PositiveInteger

A *balanced binary tree* is useful for doing modular computations. Given a list **lm** of moduli, **modTree(a, lm)** produces a balanced binary tree with the values  $a \bmod m$  at its leaves.

```
modTree(8, [2, 3, 5, 7])
[0, 2, 3, 1]
```

(16)

Type: List Integer

A *set* is a collection of elements where duplication and order is irrelevant.<sup>12</sup> Sets are always finite and have no corresponding structure like streams for infinite collections.

```
fs := set [1/3, 4/5, -1/3, 4/5]
{ -1/3, 1/3, 4/5 }
```

(17)

Type: Set Fraction Integer

---

<sup>10</sup>See ‘Heap’ on page 443 for more details. Heaps are also examples of data structures called *bags*. Other bag data structures are Stack, Queue, and Dequeue.

<sup>11</sup>Example of binary tree types are BinarySearchTree (see ‘BinarySearchTree’ on page 361, PendantTree, TournamentTree, and BalancedBinaryTree (see ‘BalancedBinaryTree’ on page 354).

<sup>12</sup>See ‘Set’ on page 563 for more details.

For all the primes  $p$  between 2 and 1000, find the distribution of  $p \bmod 5$ .

A *multiset* is a set that keeps track of the number of duplicate values.<sup>13</sup>

```
multiset [x rem 5 for x in primes(2,1000)]
```

{0, 42:3, 40:1, 38:4, 47:2} (18)

Type: Multiset Integer

A *table* is conceptually a set of “key–value” pairs and is a generalization of a multiset.<sup>14</sup> The domain `Table(Key, Entry)` provides a general-purpose type for tables with *values* of type `Entry` indexed by *keys* of type `Key`.

Compute the above distribution of primes using tables. First, let  $t$  denote an empty table of keys and values, each of type `Integer`.

```
t : Table(Integer,Integer) := empty()
table()
```

(19)

Type: Table(Integer, Integer)

We define a function **howMany** to return the number of values of a given modulus  $k$  seen so far. It calls **search(k,t)** which returns the number of values stored under the key  $k$  in table  $t$ , or “failed” if no such value is yet stored in  $t$  under  $k$ .

In English, this says “Define **howMany(k)** as follows. First, let  $n$  be the value of **search(k,t)**. Then, if  $n$  has the value “failed”, return the value 1; otherwise return  $n + 1$ .”

```
howMany(k) == (n:=search(k,t); n case "failed" => 1; n+1)
```

Type: Void

Run through the primes to create the table, then print the table. The expression `t.m := howMany(m)` updates the value in table  $t$  stored under key  $m$ .

```
for p in primes(2,1000) repeat (m:= p rem 5; t.m:=
  howMany(m)); t
```

```
Compiling function howMany with type Integer ->
Integer
```

```
table (2 = 47, 4 = 38, 1 = 40, 3 = 42, 0 = 1)
```

(21)

Type: Table(Integer, Integer)

A *record* is an example of an inhomogeneous collection of objects.<sup>15</sup> A record consists of a set of named *selectors* that can be used to access its components.

Declare that **daniel** can only be assigned a record with two prescribed fields.

```
daniel : Record(age : Integer, salary : Float)
```

Type: Void

<sup>13</sup>See ‘MultiSet’ on page 506 for details.

<sup>14</sup>For examples of tables, see `AssociationList` (‘AssociationList’ on page 352), `HashTable`, `KeyedAccessFile` (‘KeyedAccessFile’ on page 460), `Library` (‘Library’ on page 474), `SparseTable` (‘SparseTable’ on page 568), `StringTable` (‘StringTable’ on page 581), and `Table` (‘Table’ on page 585).

<sup>15</sup>See Section 2.4 on page 105 for details.

Give **daniel** a value, using square brackets to enclose the values of the fields.

```
daniel := [28, 32005.12]
[age = 28, salary = 32005.12]
Type: Record(age: Integer, salary: Float)
```

(23)

Give **daniel** a raise.

```
daniel.salary := 35000; daniel
[age = 28, salary = 35000.0]
Type: Record(age: Integer, salary: Float)
```

(24)

A *union* is a data structure used when objects have multiple types.<sup>16</sup>

Let **dog** be either an integer or a string value.

```
dog: Union(licenseNumber: Integer, name: String)
```

Type: Void

Give **dog** a name.

```
dog := "Whisper"
"Whisper"
Type: Union(name: String, ...)
```

(26)

All told, there are over forty different data structures in AXIOM. Using the domain constructors described in Chapter 13, you can add your own data structure or extend an existing one. Choosing the right data structure for your application may be the key to obtaining good performance.

---

<sup>16</sup>See Section 2.5 on page 108 for details.



## 1.7 Expanding to Higher Dimensions

To get higher dimensional aggregates, you can create one-dimensional aggregates with elements that are themselves aggregates, for example, lists of lists, one-dimensional arrays of lists of multisets, and so on. For applications requiring two-dimensional homogeneous aggregates, you will likely find *two-dimensional arrays* and *matrices* most useful.

The entries in TwoDimensionalArray and Matrix objects are all the same type, except that those for Matrix must belong to a Ring. You create and access elements in roughly the same way. Since matrices have an understood algebraic structure, certain algebraic operations are available for matrices but not for arrays. Because of this, we limit our discussion here to Matrix, that can be regarded as an extension of TwoDimensionalArray.<sup>17</sup>

You can create a matrix from a list of lists, where each of the inner lists represents a row of the matrix.

```
m := matrix([[1,2], [3,4]])
```

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad (1)$$

Type: Matrix Integer

The “collections” construct (see Section 5.5 on page 171) is useful for creating matrices whose entries are given by formulas.

```
matrix([[1/(i + j - x) for i in 1..4] for j in 1..4])
```

$$\begin{bmatrix} -\frac{1}{x-2} & -\frac{1}{x-3} & -\frac{1}{x-4} & -\frac{1}{x-5} \\ -\frac{1}{x-3} & -\frac{1}{x-4} & -\frac{1}{x-5} & -\frac{1}{x-6} \\ -\frac{1}{x-4} & -\frac{1}{x-5} & -\frac{1}{x-6} & -\frac{1}{x-7} \\ -\frac{1}{x-5} & -\frac{1}{x-6} & -\frac{1}{x-7} & -\frac{1}{x-8} \end{bmatrix} \quad (2)$$

Type: Matrix Fraction Polynomial Integer

Let *vm* denote the three by three Vandermonde matrix.

```
vm := matrix [[1,1,1], [x,y,z], [x*x,y*y,z*z]]
```

$$\begin{bmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{bmatrix} \quad (3)$$

Type: Matrix Polynomial Integer

Use this syntax to extract an entry in the matrix.

```
vm(3,3)
```

$$z^2 \quad (4)$$

Type: Polynomial Integer

<sup>17</sup>See ‘TwoDimensionalArray’ on page 590 for more information about arrays. For more information about AXIOM’s linear algebra facilities, see ‘Matrix’ on page 500, ‘Permanent’ on page 528, ‘SquareMatrix’ on page 569, ‘Vector’ on page 601, Section 8.4 on page 280 (computation of eigenvalues and eigenvectors), and Section 8.5 on page 283 (solution of linear and polynomial equations).

You can also pull out a **row** or a **column**.

$$\text{column}(\text{vm}, 2) \quad [1, y, y^2] \quad (5)$$

Type: Vector Polynomial Integer

You can do arithmetic.

$$\text{vm} * \text{vm} \quad \begin{bmatrix} x^2 + x + 1 & y^2 + y + 1 & z^2 + z + 1 \\ x^2 z + x y + x & y^2 z + y^2 + x & z^3 + y z + x \\ x^2 z^2 + x y^2 + x^2 & y^2 z^2 + y^3 + x^2 & z^4 + y^2 z + x^2 \end{bmatrix} \quad (6)$$

Type: Matrix Polynomial Integer

You can perform operations such as **transpose**, **trace**, and **determinant**.

$$\text{factor determinant vm} \quad (y - x) (z - y) (z - x) \quad (7)$$

Type: Factored Polynomial Integer

## 1.8 Writing Your Own Functions

AXIOM provides you with a very large library of predefined operations and objects to compute with. You can use the AXIOM library of constructors to create new objects dynamically of quite arbitrary complexity. For example, you can make lists of matrices of fractions of polynomials with complex floating point numbers as coefficients. Moreover, the library provides a wealth of operations that allow you to create and manipulate these objects.

For many applications, you need to interact with the interpreter and write some AXIOM programs to tackle your application. AXIOM allows you to write functions interactively, thereby effectively extending the system library. Here we give a few simple examples, leaving the details to Chapter 6.

We begin by looking at several ways that you can define the “factorial” function in AXIOM. The first way is to give a piece-wise definition of the function. This method is best for a general recurrence relation since the pieces are gathered together and compiled into an efficient iterative function. Furthermore, enough previously computed values are automatically saved so that a subsequent call to the function can pick up from where it left off.

Define the value of **fact** at 0.

```
fact(0) == 1
```

Type: Void

Define the value of **fact(n)** for general **n**.

$$\text{fact}(n) == n * \text{fact}(n-1)$$

Type: Void

Ask for the value at 50. The resulting function created by AXIOM computes the value by iteration.

```
fact(50)
```

```
Compiling function fact with type Integer -> Integer
Compiling function fact as a recurrence relation.
```

$\begin{matrix} 30414093201713378043612608166064768844377641568960512 \\ \textcircled{\hspace{1cm}}\end{matrix}$

Type: PositiveInteger

A second definition uses an `if-then-else` and recursion.

```
fac(n) == if n < 3 then n else n * fac(n - 1)
```

Type: Void

```
fac(50)
Compiling function fac with type Integer -> Integer
30414093201713378043612608166064768844377641568960512
000000000000                                (5)
```

```
fa(n) == (a := 1; for i in 2..n repeat a := a*i; a)
```

Type: Void

```
fa(50)
Compiling function fa with type PositiveInteger ->
    PositiveInteger
30414093201713378043612608166064768844377641568960512
00000000000000                                (7)
```

```
f(n) == reduce(*,[i for i in 2..n])
```

Type: Void

```
f(50)
Compiling function f with type PositiveInteger ->
    PositiveInteger
30414093201713378043612608166064768844377641568960512
00000000000000                                         (9)
```

```
factorial(50)
30414093201713378043612608166064768844377641568960512      (10)
0000000000000
Type: PositiveInteger
```

You are not limited to one-line functions in AXIOM. If you place your function definitions in **.input** files (see Section 4.1 on page 139), you can have multi-line functions that use indentation for grouping.

Given `n` elements, `diagonalMatrix` creates an `n` by `n` matrix with those elements down the diagonal. This function uses a permutation matrix that interchanges the `i`th and `j`th rows of a matrix by which it is right-multiplied.

This function definition shows a style of definition that can be used in **.input** files. Indentation is used to create *blocks*: sequences of expressions that are evaluated in sequence except as modified by control statements such as **if-then-else** and **return**.

```
permMat(n, i, j) ==
  m := diagonalMatrix
    [(if i = k or j = k then 0 else 1)
     for k in 1..n]
  m(i,j) := 1
  m(j,i) := 1
  m
```

Type: Void

This creates a four by four matrix that interchanges the second and third rows.

```
p := permMat(4,2,3)

Compiling function permMat with type (PositiveInteger
,PositiveInteger,PositiveInteger) -> Matrix
Integer
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (12)$$

Type: Matrix Integer

Create an example matrix to permute.

```
m := matrix [[4*i + j for j in 1..4] for i in 0..3]
```

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} \quad (13)$$

Type: Matrix Integer

Interchange the second and third rows of m.

```
permMat(4,2,3) * m
```

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 9 & 10 & 11 & 12 \\ 5 & 6 & 7 & 8 \\ 13 & 14 & 15 & 16 \end{bmatrix} \quad (14)$$

Type: Matrix Integer

A function can also be passed as an argument to another function, which then applies the function or passes it off to some other function that does. You often have to declare the type of a function that has functional arguments.

This declares **t** to be a two-argument function that returns a **Float**. The first argument is a function that takes one **Float** argument and returns a **Float**.

```
t : (Float -> Float, Float) -> Float
```

Type: Void

This is the definition of `t`.

```
t(fun, x) == fun(x)**2 + sin(x)**2
```

Type: Void

We have not defined a `cos` in the workspace. The one from the AXIOM library will do.

```
t(cos, 5.2058)
Compiling function t with type ((Float -> Float),
Float) -> Float
```

1.0 (17)

Type: Float

Here we define our own (user-defined) function.

```
cosinv(y) == cos(1/y)
```

Type: Void

Pass this function as an argument to `t`.

```
t(cosinv, 5.2058)
Compiling function cosinv with type Float -> Float
```

1.739223724180051649254147684772932520785 (19)

Type: Float

AXIOM also has pattern matching capabilities for simplification of expressions and for defining new functions by rules. For example, suppose that you want to apply regularly a transformation that groups together products of radicals:

$$\sqrt{a} \sqrt{b} \mapsto \sqrt{ab}, \quad (\forall a)(\forall b)$$

Note that such a transformation is not generally correct. AXIOM never uses it automatically.

Give this rule the name **groupSqrt**.

```
groupSqrt := rule(sqrt(a) * sqrt(b) == sqrt(a*b))
```

%E  $\sqrt{a} \sqrt{b} = \%E \sqrt{a b}$  (20)

Type: RewriteRule(Integer, Integer, Expression Integer)

Here is a test expression.

```
a := (sqrt(x) + sqrt(y) + sqrt(z))**4
```

$$\begin{aligned} & ((4z + 4y + 12x) \sqrt{y} + (4z + 12y + 4x) \sqrt{x}) \sqrt{z} + \\ & (12z + 4y + 4x) \sqrt{x} \sqrt{y} + z^2 + (6y + 6x)z + y^2 + 6xy + x^2 \end{aligned} \quad (21)$$

Type: Expression Integer

The rule **groupSqrt** successfully simplifies the expression.

```
groupSqrt a
```

$$\begin{aligned} & (4z + 4y + 12x) \sqrt{yz} + (4z + 12y + 4x) \sqrt{xz} + \\ & (12z + 4y + 4x) \sqrt{xy} + z^2 + (6y + 6x)z + y^2 + 6xy + x^2 \end{aligned} \quad (22)$$

Type: Expression Integer

## 1.9 Polynomials

Polynomials are the commonly used algebraic types in symbolic computation. Interactive users of AXIOM generally only see one type of polynomial: `Polynomial(R)`. This type represents polynomials in any number of unspecified variables over a particular coefficient domain `R`. This type represents its coefficients *sparsely*: only terms with non-zero coefficients are represented.

In building applications, many other kinds of polynomial representations are useful. Polynomials may have one variable or multiple variables, the variables can be named or unnamed, the coefficients can be stored sparsely or densely. So-called “distributed multivariate polynomials” store polynomials as coefficients paired with vectors of exponents. This type is particularly efficient for use in algorithms for solving systems of non-linear polynomial equations.

The polynomial constructor most familiar to the interactive user is `Polynomial`.

$$(x^{**2} - x*y^{**3} + 3*y)^{**2}$$

$$x^2 y^6 - 6 x y^4 - 2 x^3 y^3 + 9 y^2 + 6 x^2 y + x^4 \quad (1)$$

Type: Polynomial Integer

If you wish to restrict the variables used, `UnivariatePolynomial` provides polynomials in one variable.

$$p: UP(x, INT) := (3*x-1)^{**2} * (2*x + 8)$$

$$18 x^3 + 60 x^2 - 46 x + 8 \quad (2)$$

Type: UnivariatePolynomial(x, Integer)

The constructor `MultivariatePolynomial` provides polynomials in one or more specified variables.

$$m: MPOLY([x,y], INT) := (x^{**2} - x*y^{**3} + 3*y)^{**2}$$

$$x^4 - 2 y^3 x^3 + (y^6 + 6 y) x^2 - 6 y^4 x + 9 y^2 \quad (3)$$

Type: MultivariatePolynomial([x, y], Integer)

You can change the way the polynomial appears by modifying the variable ordering in the explicit list.

$$m :: MPOLY([y,x], INT)$$

$$x^2 y^6 - 6 x y^4 - 2 x^3 y^3 + 9 y^2 + 6 x^2 y + x^4 \quad (4)$$

Type: MultivariatePolynomial([y, x], Integer)

The constructor `DistributedMultivariatePolynomial` provides polynomials in one or more specified variables with the monomials ordered lexicographically.

$$m :: DMP([y,x], INT)$$

$$y^6 x^2 - 6 y^4 x - 2 y^3 x^3 + 9 y^2 + 6 y x^2 + x^4 \quad (5)$$

Type: DistributedMultivariatePolynomial([y, x], Integer)

The constructor `HomogeneousDistributedMultivariatePolynomial` is similar except that the monomials are ordered by total order refined by reverse lexicographic order.

$$m :: HDMP([y,x], INT)$$

$$y^6 x^2 - 2 y^3 x^3 - 6 y^4 x + x^4 + 6 y x^2 + 9 y^2 \quad (6)$$

Type: HomogeneousDistributedMultivariatePolynomial([y, x], Integer)

More generally, the domain constructor `GeneralDistributedMultivariatePolynomial` allows the user to provide an arbitrary predicate to define his own term ordering. These last three constructors are typically used in

Gröbner basis applications and when a flat (that is, non-recursive) display is wanted and the term ordering is critical for controlling the computation.



## 1.10 Limits

AXIOM's **limit** function is usually used to evaluate limits of quotients where the numerator and denominator both tend to zero or both tend to infinity. To find the limit of an expression **f** as a real variable **x** tends to a limit value **a**, enter **limit(f, x=a)**. Use **complexLimit** if the variable is complex. Additional information and examples of limits are in Section 8.6 on page 288.

You can take limits of functions with parameters.

$$g := \csc(a \cdot x) / \operatorname{csch}(b \cdot x)$$

$$\frac{\csc(ax)}{\operatorname{csch}(bx)}$$
(1)

Type: Expression Integer

As you can see, the limit is expressed in terms of the parameters.

$$\operatorname{limit}(g, x=0)$$

$$\frac{b}{a}$$
(2)

Type: Union(OrderedCompletion Expression Integer, ...)

A variable may also approach plus or minus infinity:

$$h := (1 + k/x)^{**x}$$

$$\frac{x + k^x}{x}$$
(3)

Type: Expression Integer

Use **%plusInfinity** and **%minusInfinity** to denote  $\infty$  and  $-\infty$ .

$$\operatorname{limit}(h, x=\%plusInfinity)$$

$$e^k$$
(4)

Type: Union(OrderedCompletion Expression Integer, ...)

A function can be defined on both sides of a particular value, but may tend to different limits as its variable approaches that value from the left and from the right.

$$\operatorname{limit}(\sqrt{y^{**2}}/y, y = 0)$$

$$[leftHandLimit = -1, rightHandLimit = 1]$$
(5)

Type: Union(Record(leftHandLimit: Union(OrderedCompletion Expression Integer, "failed"), rightHandLimit: Union(OrderedCompletion Expression Integer, "failed")), ...)

As **x** approaches 0 along the real axis, **exp(-1/x\*\*2)** tends to 0.

$$\operatorname{limit}(\exp(-1/x^{**2}), x = 0)$$

$$0$$
(6)

Type: Union(OrderedCompletion Expression Integer, ...)

However, if **x** is allowed to approach 0 along any path in the complex plane, the limiting value of **exp(-1/x\*\*2)** depends on the path taken because the function has an essential singularity at **x=0**. This is reflected in the error message returned by the function.

$$\operatorname{complexLimit}(\exp(-1/x^{**2}), x = 0)$$

$$\text{"failed"}$$
(7)

Type: Union("failed", ...)

## 1.11 Series

You can convert a functional expression to a power series by using the operation **series**. In this example, **sin(a\*x)** is expanded in powers of  $(x - 0)$ , that is, in powers of  $x$ .

This expression expands **sin(a\*x)** in powers of  $(x - \pi/4)$ .

AXIOM provides *Puiseux series*: series with rational number exponents. The first argument to **series** is an in-place function that computes the  $n^{\text{th}}$  coefficient. (Recall that the “+>” is an infix operator meaning “maps to.”)

Once you have created a power series, you can perform arithmetic operations on that series. We compute the Taylor expansion of  $1/(1-x)$ .

Compute the square of the series.

AXIOM also provides power series. By default, AXIOM tries to compute and display the first ten elements of a series. Use **)set streams calculate** to change the default value to something else. For the purposes of this book, we have used this system command to display fewer than ten terms. For more information about working with series, see Section 8.9 on page 295.

```
series(sin(a*x), x = 0)
```

$$a x - \frac{a^3}{6} x^3 + \frac{a^5}{120} x^5 - \frac{a^7}{5040} x^7 + O(x^9) \quad (1)$$

Type: UnivariatePuiseuxSeries(Expression Integer, x, 0)

```
series(sin(a*x), x = %pi/4)
```

$$\begin{aligned} &\sin\left(\frac{a\pi}{4}\right) + a \cos\left(\frac{a\pi}{4}\right) \left(x - \frac{\pi}{4}\right) - \frac{a^2 \sin\left(\frac{a\pi}{4}\right)}{2} \left(x - \frac{\pi}{4}\right)^2 - \\ &\frac{a^3 \cos\left(\frac{a\pi}{4}\right)}{6} \left(x - \frac{\pi}{4}\right)^3 + \frac{a^4 \sin\left(\frac{a\pi}{4}\right)}{24} \left(x - \frac{\pi}{4}\right)^4 + \frac{a^5 \cos\left(\frac{a\pi}{4}\right)}{120} \times \\ &\left(x - \frac{\pi}{4}\right)^5 - \frac{a^6 \sin\left(\frac{a\pi}{4}\right)}{720} \left(x - \frac{\pi}{4}\right)^6 - \frac{a^7 \cos\left(\frac{a\pi}{4}\right)}{5040} \left(x - \frac{\pi}{4}\right)^7 + \\ &O\left(\left(x - \frac{\pi}{4}\right)^8\right) \end{aligned} \quad (2)$$

Type: UnivariatePuiseuxSeries(Expression Integer, x, pi/4)

```
series(n +> (-1)**((3*n - 4)/6)/factorial(n - 1/3), x = 0, 4/3.., 2)
```

$$x^{\frac{4}{3}} - \frac{1}{6} x^{\frac{10}{3}} + O(x^4) \quad (3)$$

Type: UnivariatePuiseuxSeries(Expression Integer, x, 0)

```
f := series(1/(1-x), x = 0)
```

$$1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + O(x^8) \quad (4)$$

Type: UnivariatePuiseuxSeries(Expression Integer, x, 0)

```
f ** 2
```

$$1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + 7x^6 + 8x^7 + O(x^8) \quad (5)$$

Type: UnivariatePuiseuxSeries(Expression Integer, x, 0)

```
f := series(1/(1-x), x = 0)
1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + O(x^8)
Type: UnivariatePuisseuxSeries(Expression Integer, x, 0)
```

$$g := \log(f)$$

$$x + \frac{1}{2} x^2 + \frac{1}{3} x^3 + \frac{1}{4} x^4 + \frac{1}{5} x^5 + \frac{1}{6} x^6 + \frac{1}{7} x^7 + \frac{1}{8} x^8 + O(x^9) \quad (7)$$

Type: UnivariatePuisseuxSeries(Expression Integer, x, 0)

$$\exp(g) = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + O(x^8) \quad (8)$$

Type: UnivariatePuisseuxSeries(Expression Integer, x, 0)

$$1 + x + \frac{1}{2} x^2 + \frac{1}{6} x^3 + \frac{1}{24} x^4 + \frac{1}{120} x^5 + \frac{1}{720} x^6 + \frac{1}{5040} x^7 + O(x^8) \quad (9)$$

Type: UnivariateTaylorSeries(Expression Integer, x, 0)

$$\begin{aligned} & \text{eval}(f, 1.0) \\ & [1.0, 2.0, 2.5, \\ & 2.6667, \\ & 2.708333333333333333333333333333333333333333, \\ & 2.716666666666666666666666666666666666666667, \\ & 2.71805555555555555555555555555555555555556, \dots] \end{aligned} \tag{10}$$

Type: Stream Expression Float

## 1.12 Derivatives

To find the derivative of an expression **f** with respect to a variable **x**, enter **D(f, x)**.

Use the AXIOM function **D** to differentiate an expression.

$$\text{f} := \exp \exp \text{x} \\ e^{e^x} \quad (1)$$

Type: Expression Integer

$$\text{D}(\text{f}, \text{x}) \\ e^x e^{e^x} \quad (2)$$

Type: Expression Integer

An optional third argument **n** in **D** asks AXIOM for the **n**<sup>th</sup> derivative of **f**. This finds the fourth derivative of **f** with respect to **x**.

$$\text{D}(\text{f}, \text{x}, 4) \\ (e^{x^4} + 6 e^{x^3} + 7 e^{x^2} + e^x) e^{e^x} \quad (3)$$

Type: Expression Integer

You can also compute partial derivatives by specifying the order of differentiation.

$$\text{g} := \sin(\text{x}^2 + \text{y}) \\ \sin(y + x^2) \quad (4)$$

Type: Expression Integer

$$\text{D}(\text{g}, \text{y}) \\ \cos(y + x^2) \quad (5)$$

Type: Expression Integer

$$\text{D}(\text{g}, [\text{y}, \text{y}, \text{x}, \text{x}]) \\ 4 x^2 \sin(y + x^2) - 2 \cos(y + x^2) \quad (6)$$

Type: Expression Integer

AXIOM can manipulate the derivatives (partial and iterated) of expressions involving formal operators. All the dependencies must be explicit.

This returns 0 since **F** (so far) does not explicitly depend on **x**.

$$\text{D}(\text{F}, \text{x}) \\ 0 \quad (7)$$

Type: Polynomial Integer

Suppose that we have **F** a function of **x**, **y**, and **z**, where **x** and **y** are themselves functions of **z**.

Start by declaring that **F**, **x**, and **y** are operators.

$$\text{F} := \text{operator 'F}; \text{x} := \text{operator 'x}; \text{y} := \text{operator 'y} \\ y \quad (8)$$

Type: BasicOperator

You can use  $F$ ,  $x$ , and  $y$  in expressions.

$$a := F(x(z), y(z), z^2) + x(z)y(z+1) \\ x(y(z+1)) + F(x(z), y(z), z^2) \quad (9)$$

Type: Expression Integer

Differentiate formally with respect to  $z$ . The formal derivatives appearing in  $dadz$  are not just formal symbols, but do represent the derivatives of  $x$ ,  $y$ , and  $F$ .

$$dadz := D(a, z) \\ 2z F_{,3}(x(z), y(z), z^2) + y'(z) F_{,2}(x(z), y(z), z^2) + \\ x'(z) F_{,1}(x(z), y(z), z^2) + x'(y(z+1)) y'(z+1) \quad (10)$$

Type: Expression Integer

You can evaluate the above for particular functional values of  $F$ ,  $x$ , and  $y$ . If  $x(z)$  is  $\exp(z)$  and  $y(z)$  is  $\log(z+1)$ , then this evaluates  $dadz$ .

$$\text{eval}(\text{eval}(dadz, 'x, z \mapsto \exp z), 'y, z \mapsto \log(z+1)) \\ \left( \frac{\left( (2z^2 + 2z) F_{,3}(e^z, \log(z+1), z^2) + F_{,2}(e^z, \log(z+1), z^2) \right) \right. \\ \left. + (z+1) e^z F_{,1}(e^z, \log(z+1), z^2) + z + 1 \right)}{z+1} \quad (11)$$

Type: Expression Integer

You obtain the same result by first evaluating  $a$  and then differentiating.

$$\text{eval}(\text{eval}(a, 'x, z \mapsto \exp z), 'y, z \mapsto \log(z+1)) \\ F(e^z, \log(z+1), z^2) + z + 2 \quad (12)$$

Type: Expression Integer

$$D(\%, z) \\ \left( \frac{\left( (2z^2 + 2z) F_{,3}(e^z, \log(z+1), z^2) + F_{,2}(e^z, \log(z+1), z^2) \right) \right. \\ \left. + (z+1) e^z F_{,1}(e^z, \log(z+1), z^2) + z + 1 \right)}{z+1} \quad (13)$$

Type: Expression Integer

## 1.13 Integration

We use a factorization-free algorithm.

AXIOM has extensive library facilities for integration.

The first example is the integration of a fraction with denominator that factors into a quadratic and a quartic irreducible polynomial. The usual partial fraction approach used by most other computer algebra systems either fails or introduces expensive unneeded algebraic numbers.

$$\text{integrate}((x^2 + 2x + 1) / ((x + 1)^6 + 1), x)$$

$$\frac{\arctan(x^3 + 3x^2 + 3x + 1)}{3} \quad (1)$$

Type: Union(Expression Integer, ...)

When real parameters are present, the form of the integral can depend on the signs of some expressions.

Rather than query the user or make sign assumptions, AXIOM returns all possible answers.

$$\text{integrate}(1/(x^2 + a), x)$$

$$\left[ \frac{\log\left(\frac{(x^2 - a)\sqrt{-a} + 2ax}{x^2 + a}\right)}{2\sqrt{-a}}, \frac{\arctan\left(\frac{x\sqrt{a}}{a}\right)}{\sqrt{a}} \right] \quad (2)$$

Type: Union(List Expression Integer, ...)

The **integrate** operation generally assumes that all parameters are real. The only exception is when the integrand has complex valued quantities.

If the parameter is complex instead of real, then the notion of sign is undefined and there is a unique answer. You can request this answer by “prepending” the word “complex” to the command name:

$$\text{complexIntegrate}(1/(x^2 + a), x)$$

$$\frac{\log\left(\frac{x\sqrt{-a} + a}{\sqrt{-a}}\right) - \log\left(\frac{x\sqrt{-a} - a}{\sqrt{-a}}\right)}{2\sqrt{-a}} \quad (3)$$

Type: Expression Integer

The following two examples illustrate the limitations of table-based approaches. The two integrands are very similar, but the answer to one of them requires the addition of two new algebraic numbers.

This one is the easy one. The next one looks very similar but the answer is much more complicated.

$$\text{integrate}(x^3 / (a + bx)^{1/3}, x)$$

$$\frac{(120b^3x^3 - 135ab^2x^2 + 162a^2bx - 243a^3)\sqrt[3]{bx+a}^2}{440b^4} \quad (4)$$

Type: Union(Expression Integer, ...)

Only an algorithmic approach is guaranteed to find what new constants must be added in order to find a solution.

$$\text{integrate}(1 / (x^{**3} * (a+b*x)**(1/3)), x)$$

$$\frac{\left( \begin{aligned} &-2 b^2 x^2 \sqrt{3} \log \left( \sqrt[3]{a} \sqrt[3]{b x + a}^2 + \sqrt[3]{a}^2 \sqrt[3]{b x + a} + a \right) + \\ &4 b^2 x^2 \sqrt{3} \log \left( \sqrt[3]{a}^2 \sqrt[3]{b x + a} - a \right) + \\ &12 b^2 x^2 \arctan \left( \frac{2 \sqrt{3} \sqrt[3]{a}^2 \sqrt[3]{b x + a} + a \sqrt{3}}{3 a} \right) + \\ &(12 b x - 9 a) \sqrt{3} \sqrt[3]{a} \sqrt[3]{b x + a}^2 \end{aligned} \right)}{18 a^2 x^2 \sqrt{3} \sqrt[3]{a}} \quad (5)$$

Type: Union(Expression Integer, ...)

Some computer algebra systems use heuristics or table-driven approaches to integration. When these systems cannot determine the answer to an integration problem, they reply “I don’t know.” AXIOM uses a algorithm for integration. that conclusively proves that an integral cannot be expressed in terms of elementary functions.

When AXIOM returns an integral sign, it has proved that no answer exists as an elementary function.

$$\text{integrate}(\log(1 + \text{sqrt}(a*x + b)) / x, x)$$

$$\int x \frac{\log(\sqrt{b + \%V} a + 1)}{\%V} d \%V \quad (6)$$

Type: Union(Expression Integer, ...)

AXIOM can handle complicated mixed functions much beyond what you can find in tables.

Whenever possible, AXIOM tries to express the answer using the functions present in the integrand.

$$\text{integrate}((\sinh(1+\text{sqrt}(x+b))+2*\text{sqrt}(x+b)) / (\text{sqrt}(x+b) * (x + \cosh(1+\text{sqrt}(x + b))))), x)$$

$$2 \log \left( \frac{-2 \cosh(\sqrt{x+b} + 1) - 2 x}{\sinh(\sqrt{x+b} + 1) - \cosh(\sqrt{x+b} + 1)} \right) - 2 \sqrt{x+b} \quad (7)$$

Type: Union(Expression Integer, ...)

A strong structure-checking algorithm in AXIOM finds hidden algebraic relationships between functions.

$$\text{integrate}(\tan(\text{atan}(x)/3), x)$$

$$\frac{8 \log \left( 3 \tan \left( \frac{\text{arctan}(x)}{3} \right)^2 - 1 \right) - 3 \tan \left( \frac{\text{arctan}(x)}{3} \right)^2 + 18 x \tan \left( \frac{\text{arctan}(x)}{3} \right)}{18} \quad (8)$$

Type: Union(Expression Integer, ...)

The discovery of this algebraic relationship is necessary for correct integration of this function. Here are the details:

1. If  $x = \tan t$  and  $g = \tan(t/3)$  then the following algebraic relation

is true:

$$g^3 - 3xg^2 - 3g + x = 0$$

2. Integrate  $g$  using this algebraic relation; this produces:

$$\frac{(24g^2 - 8)\log(3g^2 - 1) + (81x^2 + 24)g^2 + 72xg - 27x^2 - 16}{54g^2 - 18}$$

3. Rationalize the denominator, producing:

$$\frac{8\log(3g^2 - 1) - 3g^2 + 18xg + 16}{18}$$

Replace  $g$  by the initial definition  $g = \tan(\arctan(x)/3)$  to produce the final result.

This is an example of a mixed function where the algebraic layer is over the transcendental one.

`integrate((x + 1) / (x*(x + log x) ** (3/2)), x)`

$$-\frac{2\sqrt{\log(x) + x}}{\log(x) + x} \quad (9)$$

Type: Union(Expression Integer, ...)

While incomplete for non-elementary functions, AXIOM can handle some of them.

`integrate(exp(-x**2) * erf(x) / (erf(x)**3 - erf(x)**2 - erf(x) + 1), x)`

$$\frac{(\operatorname{erf}(x) - 1)\sqrt{\pi}\log\left(\frac{\operatorname{erf}(x) - 1}{\operatorname{erf}(x) + 1}\right) - 2\sqrt{\pi}}{8\operatorname{erf}(x) - 8} \quad (10)$$

Type: Union(Expression Integer, ...)

More examples of AXIOM's integration capabilities are discussed in Section 8.8 on page 292.



## 1.14 Differential Equations

Let's solve some differential equations. Let  $y$  be the unknown function in terms of  $x$ .

Here we solve a third order equation with polynomial coefficients.

Here we find all the algebraic function solutions of the equation.

This example has solutions whose logarithmic derivative is an algebraic function of degree two.

The general approach used in integration also carries over to the solution of linear differential equations.

```
y := operator 'y
y
```

(1)

Type: BasicOperator

```
deq := x**3 * D(y x, x, 3) + x**2 * D(y x, x, 2) - 2 * x *
      D(y x, x) + 2 * y x = 2 * x**4
```

$$x^3 y'''(x) + x^2 y''(x) - 2x y'(x) + 2y(x) = 2x^4 \quad (2)$$

Type: Equation Expression Integer

```
solve(deq, y, x)
```

$$\left[ \begin{aligned} & \text{particular} = \frac{x^5 - 10x^3 + 20x^2 + 4}{15x}, \\ & \text{basis} = \left[ \frac{2x^3 - 3x^2 + 1}{x}, \frac{x^3 - 1}{x}, \frac{x^3 - 3x^2 - 1}{x} \right] \end{aligned} \right] \quad (3)$$

Type: Union(Record(particular: Expression Integer, basis: List Expression Integer), ...)

```
deq := (x**2 + 1) * D(y x, x, 2) + 3 * x * D(y x, x) + y x
      = 0
```

$$(x^2 + 1) y''(x) + 3x y'(x) + y(x) = 0 \quad (4)$$

Type: Equation Expression Integer

```
solve(deq, y, x)
```

$$\left[ \text{particular} = 0, \text{basis} = \left[ \frac{1}{\sqrt{x^2 + 1}}, \frac{\log(\sqrt{x^2 + 1} - x)}{\sqrt{x^2 + 1}} \right] \right] \quad (5)$$

Type: Union(Record(particular: Expression Integer, basis: List Expression Integer), ...)

Coefficients of differential equations can come from arbitrary constant fields. For example, coefficients can contain algebraic numbers.

```
eq := 2*x**3 * D(y x,x,2) + 3*x**2 * D(y x,x) - 2 * y x
2 x^3 y''(x) + 3 x^2 y'(x) - 2 y(x)
```

(6)

Type: Expression Integer

`solve(eq,y,x).basis`

$$\left[ e^{\left(-\frac{2}{\sqrt{x}}\right)}, e^{\frac{2}{\sqrt{x}}} \right] \quad (7)$$

Type: List Expression Integer

Here's another differential equation to solve.

`deq := D(y x, x) = y(x) / (x + y(x) * log y x)`

$$y'(x) = \frac{y(x)}{y(x) \log(y(x)) + x} \quad (8)$$

Type: Equation Expression Integer

`solve(deq, y, x)`

$$\frac{y(x) \log(y(x))^2 - 2 x}{2 y(x)} \quad (9)$$

Type: Union(Expression Integer, ...)

Rather than attempting to get a closed form solution of a differential equation, you instead might want to find an approximate solution in the form of a series.

Let's solve a system of nonlinear first order equations and get a solution in power series. Tell AXIOM that `x` is also an operator.

`x := operator 'x`

`x` (10)

Type: BasicOperator

Here are the two equations forming our system.

`eq1 := D(x(t), t) = 1 + x(t)**2`

$$x'(t) = x(t)^2 + 1 \quad (11)$$

Type: Equation Expression Integer

`eq2 := D(y(t), t) = x(t) * y(t)`

$$y'(t) = x(t) y(t) \quad (12)$$

Type: Equation Expression Integer

We can solve the system around  $t = 0$  with the initial conditions  $x(0) = 0$  and  $y(0) = 1$ . Notice that since we give the unknowns in the order  $[x, y]$ , the answer is a list of two series in the order `[series for x(t), series for y(t)]`.

```
seriesSolve([eq2, eq1], [x, y], t = 0, [y(0) = 1, x(0) = 0])
```

```
Compiling function %BT with type List
  UnivariateTaylorSeries(Expression Integer,t,0) ->
  UnivariateTaylorSeries(Expression Integer,t,0)
Compiling function %BU with type List
  UnivariateTaylorSeries(Expression Integer,t,0) ->
  UnivariateTaylorSeries(Expression Integer,t,0)
```

$$\left[ t + \frac{1}{3} t^3 + \frac{2}{15} t^5 + \frac{17}{315} t^7 + O(t^8), 1 + \frac{1}{2} t^2 + \frac{5}{24} t^4 + \frac{61}{720} t^6 + O(t^8) \right] \quad (13)$$

Type: List UnivariateTaylorSeries(Expression Integer, t, 0)

## 1.15 Solution of Equations

A system of two equations involving a symbolic parameter  $t$ .

AXIOM also has state-of-the-art algorithms for the solution of systems of polynomial equations. When the number of equations and unknowns is the same, and you have no symbolic coefficients, you can use **solve** for real roots and **complexSolve** for complex roots. In each case, you tell AXIOM how accurate you want your result to be. All operations in the **solve** family return answers in the form of a list of solution sets, where each solution set is a list of equations.

$S(t) == [x^2 - 2y^2 - t, x^2y - y^2 - 5x + 5]$

Type: Void

Find the real roots of  $S(19)$  with rational arithmetic, correct to within  $1/10^{20}$ .

`solve(S(19), 1/10**20)`

Compiling function S with type PositiveInteger ->  
List Polynomial Integer

$$\left[ \left[ y = 5, x = -\frac{2451682632253093442511}{295147905179352825856} \right], \right. \\ \left. \left[ y = 5, x = \frac{2451682632253093442511}{295147905179352825856} \right] \right] \quad (2)$$

Type: List List Equation Polynomial Fraction Integer

Find the complex roots of  $S(19)$  with floating point coefficients to 20 digits accuracy in the mantissa.

`complexSolve(S(19), 10.e-20)`

$$\left[ \left[ y = 5.0, x = 8.306623862918074852561669055295290320373 \right], \right. \\ \left[ y = 5.0, x = -8.306623862918074852561669055295290320373 \right], \quad (3) \\ \left[ y = -3.0i, x = 1.0 \right], \left[ y = 3.0i, x = 1.0 \right] \right]$$

Type: List List Equation Polynomial Complex Float

If a system of equations has symbolic coefficients and you want a solution in radicals, try **radicalSolve**.

`radicalSolve(S(a), [x,y])`

Compiling function S with type Variable a -> List  
Polynomial Integer

$$\left[ \left[ x = -\sqrt{a+50}, y = 5 \right], \left[ x = \sqrt{a+50}, y = 5 \right], \right. \\ \left. \left[ x = 1, y = \sqrt{\frac{-a+1}{2}} \right], \left[ x = 1, y = -\sqrt{\frac{-a+1}{2}} \right] \right] \quad (4)$$

Type: List List Equation Expression Integer

For systems of equations with symbolic coefficients, you can apply **solve**, listing the variables that you want AXIOM to solve for. For polynomial equations, a solution cannot usually be expressed solely in terms of the other variables. Instead, the solution is presented as a “triangular” system of equations, where each polynomial has coefficients involving only the succeeding variables. This is analogous to converting a linear system of

equations to “triangular form”.

A system of three equations in five variables.

$$\begin{aligned} \text{eqns} &:= [\mathbf{x}^{**2} - \mathbf{y} + \mathbf{z}, \mathbf{x}^{**2} * \mathbf{z} + \mathbf{x}^{**4} - \mathbf{b} * \mathbf{y}, \mathbf{y}^{**2} * \mathbf{z} - \mathbf{a} - \\ &\quad \mathbf{b} * \mathbf{x}] \\ &[z - y + x^2, x^2 z - b y + x^4, y^2 z - b x - a] \end{aligned} \tag{5}$$

Type: List Polynomial Integer

Solve the system for unknowns  $[x, y, z]$ , reducing the solution to triangular form.

$$\begin{aligned} &\text{solve}(\text{eqns}, [\mathbf{x}, \mathbf{y}, \mathbf{z}]) \\ &\left[ \left[ x = -\frac{a}{b}, y = 0, z = -\frac{a^2}{b^2} \right], \left[ x = \frac{z^3 + 2 b z^2 + b^2 z - a}{b}, \right. \right. \\ &y = z + b, z^6 + 4 b z^5 + 6 b^2 z^4 + (4 b^3 - 2 a) z^3 + \\ &\left. \left. (b^4 - 4 a b) z^2 - 2 a b^2 z - b^3 + a^2 = 0 \right] \right] \end{aligned} \tag{6}$$

Type: List List Equation Fraction Polynomial Integer

## 1.16 System Commands

---

We conclude our tour of AXIOM with a brief discussion of *system commands*. System commands are special statements that start with a closing parenthesis (“)”). They are used to control or display your AXIOM environment, start the HyperDoc system, issue operating system commands and leave AXIOM. For example, `)system` is used to issue commands to the operating system from AXIOM. Here is a brief description of some of these commands. For more information on specific commands, see Appendix A.

Perhaps the most important user command is the `)clear all` command that initializes your environment. Every section and subsection in this book has an invisible `)clear all` that is read prior to the examples given in the section. `)clear all` gives you a fresh, empty environment with no user variables defined and the step number reset to 1. The `)clear` command can also be used to selectively clear values and properties of system variables.

Another useful system command is `)read`. A preferred way to develop an application in AXIOM is to put your interactive commands into a file, say `my.input` file. To get AXIOM to read this file, you use the system command `)read my.input`. If you need to make changes to your approach or definitions, go into your favorite editor, change `my.input`, then `)read my.input` again.

Other system commands include: `)history`, to display previous input and/or output lines; `)display`, to display properties and values of workspace variables; and `)what`.

Issue `)what` to get a list of AXIOM objects that contain a given substring in their name.

```
)what operations integrate
```

```
Operations whose names satisfy the above pattern(s):
```

HermiteIntegrate	algintegrate
complexIntegrate	expintegrate
extendedIntegrate	fintegrate
infieldIntegrate	integrate
internalIntegrate	internalIntegrate0
lazyGintegrate	lazyIntegrate
lfintegrate	limitedIntegrate
monomialIntegrate	nagPolygonIntegrate
palgintegrate	pmComplexintegrate
pmintegrate	primintegrate
tanintegrate	

```
To get more information about an operation such as
HermiteIntegrate , issue the command )display op
HermiteIntegrate
```

A useful system command is `)undo`. Sometimes while computing interac-

tively with AXIOM, you make a mistake and enter an incorrect definition or assignment. Or perhaps you need to try one of several alternative approaches, one after another, to find the best way to approach an application. For this, you will find the *undo* facility of AXIOM helpful.

System command `)undo n` means “undo back to step *n*”; it restores the values of user variables to those that existed immediately after input expression *n* was evaluated. Similarly, `)undo -n` undoes changes caused by the last *n* input expressions. Once you have done an `)undo`, you can continue on from there, or make a change and **redo** all your input expressions from the point of the `)undo` forward. The `)undo` is completely general: it changes the environment like any user expression. Thus you can `)undo` any previous undo.

Here is a sample dialogue between user and AXIOM.

“Let me define two mutually dependent functions *f* and *g* piece-wise.”

`f(0) == 1; g(0) == 1`

Type: Void

“Here is the general term for *f*.”

`f(n) == e/2*f(n-1) - x*g(n-1)`

Type: Void

“And here is the general term for *g*.”

`g(n) == -x*f(n-1) + d/3*g(n-1)`

Type: Void

“What is value of *f*(3)?”

`f(3)`

Compiling function g with type Integer -> Polynomial  
 Fraction Integer  
 Compiling function g as a recurrence relation.  
 Compiling function g with type Integer -> Polynomial  
 Fraction Integer  
 Compiling function g as a recurrence relation.

+++ |\*1;g;1;initial;AUX| redefined

+++ |\*1;g;1;initial| redefined

Compiling function f with type Integer -> Polynomial  
 Fraction Integer  
 Compiling function f as a recurrence relation.

+++ |\*1;f;1;initial| redefined

$$-x^3 + \left(e + \frac{1}{3}d\right)x^2 + \left(-\frac{1}{4}e^2 - \frac{1}{6}de - \frac{1}{9}d^2\right)x + \frac{1}{8}e^3 \quad (4)$$

Type: Polynomial Fraction Integer

```

“Hmm, I think I want to define f differently. Undo to the
environment right after I defined f.”      )undo 2

“Here is how I think I want f to be defined instead.”
f(n) == d/3*f(n-1) - x*g(n-1)
1 old definition(s) deleted for function or rule f
Type: Void

Redo the computation from expression 3 forward.      )undo )redo

“I want my old definition of f after all. Undo the undo and
restore the environment to that immediately after (4).”      )undo 4

“Check that the value of f(3) is restored.”      f(3)
Compiling function g with type Integer -> Polynomial
Fraction Integer
Compiling function g as a recurrence relation.
+++ |*1;g;1;initial;AUX| redefined
+++ |*1;g;1;initial| redefined
Compiling function g with type Integer -> Polynomial
Fraction Integer
Compiling function g as a recurrence relation.
+++ |*1;g;1;initial;AUX| redefined
+++ |*1;g;1;initial| redefined
Compiling function f with type Integer -> Polynomial
Fraction Integer
Compiling function f as a recurrence relation.
+++ |*1;f;1;initial;AUX| redefined
+++ |*1;f;1;initial| redefined

$$-x^3 + \left(e + \frac{1}{3}d\right)x^2 + \left(-\frac{1}{4}e^2 - \frac{1}{6}de - \frac{1}{9}d^2\right)x + \frac{1}{8}e^3 \quad (6)$$

Type: Polynomial Fraction Integer

```

After you have gone off on several tangents, then backtracked to previous points in your conversation using `)undo`, you might want to save all the “correct” input commands you issued, disregarding those undone. The system command `)history )write mynew.input` writes a clean straight-line program onto the file **mynew.input** on your disk.

This concludes your tour of AXIOM. To disembark, issue the system command `)quit` to leave AXIOM and return to the operating system.



---

# Using Types and Modes

In this chapter we look at the key notion of *type* and its generalization *mode*. We show that every AXIOM object has a type that determines what you can do with the object. In particular, we explain how to use types to call specific functions from particular parts of the library and how types and modes can be used to create new objects from old. We also look at Record and Union types and the special type Any. Finally, we give you an idea of how AXIOM manipulates types and modes internally to resolve ambiguities.

## 2.1 The Basic Idea

The AXIOM world deals with many kinds of objects. There are mathematical objects such as numbers and polynomials, data structure objects such as lists and arrays, and graphics objects such as points and graphic images. Functions are objects too.

AXIOM organizes objects using the notion of *domain of computation*, or simply *domain*. Each domain denotes a class of objects. The class of objects it denotes is usually given by the name of the domain: Integer for the integers, Float for floating-point numbers, and so on. The convention is that the first letter of a domain name is capitalized. Similarly, the domain Polynomial(Integer) denotes “polynomials with integer coefficients.” Also, Matrix(Float) denotes “matrices with floating-point entries.”

Every basic AXIOM object belongs to a unique domain. The integer 3 belongs to the domain Integer and the polynomial  $x + 3$  belongs to the domain Polynomial(Integer). The domain of an object is also called its *type*. Thus we speak of “the type Integer” and “the type Polynomial(Integer).”

After an AXIOM computation, the type is displayed toward the right-hand side of the page (or screen).

- 3  
-3  
(1)  
Type: Integer

Here we create a rational number but it looks like the last result. The type however tells you it is different. You cannot identify the type of an object by how AXIOM displays the object.

- 3 / 1  
-3  
(2)  
Type: Fraction Integer

When a computation produces a result of a simpler type, AXIOM leaves the type unsimplified. Thus no information is lost.

$x + 3 - x$   
3  
(3)  
Type: Polynomial Integer

This seldom matters since AXIOM retracts the answer to the simpler type if it is necessary.

factorial(%)  
6  
(4)  
Type: Expression Integer

When you issue a positive number, the type PositiveInteger is printed. Surely, 3 also has type Integer! The curious reader may now have two questions. First, is the type of an object not unique? Second, how is PositiveInteger related to Integer? Read on!

3  
3  
(5)  
Type: PositiveInteger

Any domain can be refined to a *subdomain* by a membership *predicate*.<sup>1</sup> For example, the domain Integer can be refined to the subdomain PositiveInteger, the set of integers  $x$  such that  $x > 0$ , by giving the AXIOM predicate  $x \mapsto x > 0$ . Similarly, AXIOM can define subdomains such as “the subdomain of diagonal matrices,” “the subdomain of lists of length two,” “the subdomain of monic irreducible polynomials in  $x$ ,” and so on. Trivially, any domain is a subdomain of itself.

While an object belongs to a unique domain, it can belong to any number of subdomains. Any subdomain of the domain of an object can be used as the *type* of that object. The type of 3 is indeed both Integer and PositiveInteger as well as any other subdomain of integer whose predicate is satisfied, such as “the prime integers,” “the odd positive integers between 3 and 17,” and so on.

### 2.1.1 Domain Constructors

To ask for “the factorial of 7” you enter this expression to AXIOM. This applies the function `factorial` to the value 7 to compute the result.

```
factorial(7)
```

```
5040
```

(1)

Type: PositiveInteger

Enter the type Polynomial(Integer) as an expression to AXIOM. This looks much like a function call as well. It is! The result is appropriately stated to be of type Domain, which according to our usual convention, denotes the class of all domains.

```
Polynomial(Integer)
```

```
Polynomial Integer
```

(2)

Type: Domain

The most basic operation involving domains is that of building a new domain from a given one. To create the domain of “polynomials over the integers,” AXIOM applies the function Polynomial to the domain Integer. A function like Polynomial is called a *domain constructor* or, more simply, a *constructor*. A domain constructor is a function that creates a domain. An argument to a domain constructor can be another domain or, in general, an arbitrary kind of object. Polynomial takes a single domain argument while SquareMatrix takes a positive integer as an argument to give its dimension and a domain argument to give the type of its components.

---

<sup>1</sup>A predicate is a function that, when applied to an object of the domain, returns either `true` or `false`.

What kinds of domains can you use as the argument to Polynomial or SquareMatrix or List? Well, the first two are mathematical in nature. You want to be able to perform algebraic operations like “+” and “\*” on polynomials and square matrices, and operations such as **determinant** on square matrices. So you want to allow polynomials of integers *and* polynomials of square matrices with complex number coefficients and, in general, anything that “makes sense.” At the same time, you don’t want AXIOM to be able to build nonsense domains such as “polynomials of strings!”

In contrast to algebraic structures, data structures can hold any kind of object. Operations on lists such as **insert**, **delete**, and **concat** just manipulate the list itself without changing or operating on its elements. Thus you can build List over almost any datatype, including itself.

Create a complicated algebraic domain.

```
List (List (Matrix (Polynomial (Complex (Fraction
(Integer))))))
```

```
List List Matrix Polynomial Complex Fraction Integer (3)
```

Type: Domain

Try to create a meaningless domain.

```
Polynomial(String)
```

```
Polynomial String is not a valid type.
```

Evidently from our last example, AXIOM has some mechanism that tells what a constructor can use as an argument. This brings us to the notion of *category*. As domains are objects, they too have a domain. The domain of a domain is a category. A category is simply a type whose members are domains.

A common algebraic category is Ring, the class of all domains that are “rings.” A ring is an algebraic structure with constants 0 and 1 and operations “+”, “-”, and “\*”. These operations are assumed “closed” with respect to the domain, meaning that they take two objects of the domain and produce a result object also in the domain. The operations are understood to satisfy certain “axioms,” certain mathematical principles providing the algebraic foundation for rings. For example, the *additive inverse axiom* for rings states:

Every element  $x$  has an additive inverse  $y$  such that  $x + y = 0$ .

The prototypical example of a domain that is a ring is the integers. Keep them in mind whenever we mention Ring.

Many algebraic domain constructors such as Complex, Polynomial, Fraction, take rings as arguments and return rings as values. You can use the infix

operator “has” to ask a domain if it belongs to a particular category.

All numerical types are rings.  
Domain constructor Polynomial builds “the ring of polynomials over any other ring.”

```
Polynomial(Integer) has Ring
true
```

(4)  
Type: Boolean

Constructor List never produces a ring.

```
List(Integer) has Ring
false
```

(5)  
Type: Boolean

The constructor Matrix(R) builds “the domain of all matrices over the ring R.” This domain is never a ring since the operations “+”, “-”, and “\*” on matrices of arbitrary shapes are undefined.

```
Matrix(Integer) has Ring
false
```

(6)  
Type: Boolean

Thus you can never build polynomials over matrices.

```
Polynomial(Matrix(Integer))
Polynomial Matrix Integer is not a valid type.
```

Use SquareMatrix(n,R) instead. For any positive integer n, it builds “the ring of n by n matrices over R.”

```
Polynomial(SquareMatrix(7,Complex(Integer)))
Polynomial SquareMatrix (7, Complex Integer )
```

(7)  
Type: Domain

Another common category is Field, the class of all fields. A field is a ring with additional operations. For example, a field has commutative multiplication and a closed operation “/” for the division of two elements. Integer is not a field since, for example, 3/2 does not have an integer result. The prototypical example of a field is the rational numbers, that is, the domain Fraction(Integer). In general, the constructor Fraction takes a ring as an argument and returns a field.<sup>2</sup> Other domain constructors, such as Complex, build fields only if their argument domain is a field.

The complex integers (often called the “Gaussian integers”) do not form a field.

```
Complex(Integer) has Field
false
```

(8)  
Type: Boolean

But fractions of complex integers do.

```
Fraction(Complex(Integer)) has Field
true
```

(9)  
Type: Boolean

---

<sup>2</sup>Actually, the argument domain must have some additional properties so as to belong to category IntegralDomain.

The algebraically equivalent domain of complex rational numbers is a field since domain constructor `Complex` produces a field whenever its argument is a field.

```
Complex(Fraction(Integer)) has Field
true
```

(10)  
Type: Boolean

The most basic category is `Type`. It denotes the class of all domains and subdomains.<sup>3</sup> Domain constructor `List` is able to build “lists of elements from domain `D`” for arbitrary `D` simply by requiring that `D` belong to category `Type`.

Now, you may ask, what exactly is a category? Like domains, categories can be defined in the AXIOM language. A category is defined by three components:

1. a name (for example, `Ring`), used to refer to the class of domains that the category represents;
2. a set of operations, used to refer to the operations that the domains of this class support (for example, “+”, “-”, and “\*” for rings); and
3. an optional list of other categories that this category extends.

This last component is a new idea. And it is key to the design of AXIOM! Because categories can extend one another, they form hierarchies. Detailed charts showing the category hierarchies in AXIOM are displayed in the endpages of this book. There you see that all categories are extensions of `Type` and that `Field` is an extension of `Ring`.

The operations supported by the domains of a category are called the *exports* of that category because these are the operations made available for system-wide use. The exports of a domain of a given category are not only the ones explicitly mentioned by the category. Since a category extends other categories, the operations of these other categories—and all categories these other categories extend—are also exported by the domains.

For example, polynomial domains belong to `PolynomialCategory`. This category explicitly mentions some twenty-nine operations on polynomials, but it extends eleven other categories (including `Ring`). As a result, the current system has over one hundred operations on polynomials.

If a domain belongs to a category that extends, say, `Ring`, it is convenient to say that the domain exports `Ring`. The name of the category thus provides a convenient shorthand for the list of operations exported by the category. Rather than listing operations such as “+” and “\*” of `Ring` each time they are needed, the definition of a type simply asserts that it

---

<sup>3</sup>`Type` does not denote the class of all types. The type of all categories is `Category`. The type of `Type` itself is undefined.

exports category Ring.

The category name, however, is more than a shorthand. The name Ring, in fact, implies that the operations exported by rings are required to satisfy a set of “axioms” associated with the name Ring.<sup>4</sup>

Why is it not correct to assume that some type is a ring if it exports all of the operations of Ring? Here is why. Some languages such as **APL** denote the Boolean constants **true** and **false** by the integers 1 and 0 respectively, then use “+” and “\*” to denote the logical operators **or** and **and**. But with these definitions Boolean is not a ring since the additive inverse axiom is violated.<sup>5</sup> This alternative definition of Boolean can be easily and correctly implemented in AXIOM, since Boolean simply does not assert that it is of category Ring. This prevents the system from building meaningless domains such as Polynomial(Boolean) and then wrongfully applying algorithms that presume that the ring axioms hold.

Enough on categories. To learn more about them, see Chapter 12. We now return to our discussion of domains.

Domains *export* a set of operations to make them available for system-wide use. Integer, for example, exports the operations “+” and “=” given by the *signatures* “+”: (Integer,Integer)  $\rightarrow$  Integer and “=”: (Integer,Integer)  $\rightarrow$  Boolean, respectively. Each of these operations takes two Integer arguments. The “+” operation also returns an Integer but “=” returns a Boolean: **true** or **false**. The operations exported by a domain usually manipulate objects of the domain—but not always.

The operations of a domain may actually take as arguments, and return as values, objects from any domain. For example, Fraction(Integer) exports the operations “/”: (Integer,Integer)  $\rightarrow$  Fraction(Integer) and **characteristic**:  $\rightarrow$  NonNegativeInteger.

Suppose all operations of a domain take as arguments and return as values, only objects from *other* domains. This kind of domain is what AXIOM calls a *package*.

A package does not designate a class of objects at all. Rather, a package is just a collection of operations. Actually the bulk of the AXIOM library of algorithms consists of packages. The facilities for factorization; integration; solution of linear, polynomial, and differential equations; computation of limits; and so on, are all defined in packages. Domains needed by algorithms can be passed to a package as arguments or used by name if

---

<sup>4</sup>This subtle but important feature distinguishes AXIOM from other abstract datatype designs.

<sup>5</sup>There is no inverse element **a** such that  $1 + \mathbf{a} = 0$ , or, in the usual terms: **true** or **a** = **false**.

they are not “variable.” Packages are useful for defining operations that convert objects of one type to another, particularly when these types have different parameterizations. As an example, the package `PolynomialFunction2(R,S)` defines operations that convert polynomials over a domain `R` to polynomials over `S`. To convert an object from `Polynomial(Integer)` to `Polynomial(Float)`, AXIOM builds the package `PolynomialFunctions2(Integer,Float)` in order to create the required conversion function. (This happens “behind the scenes” for you: see Section 2.7 on page 113 for details on how to convert objects.)

AXIOM categories, domains and packages and all their contained functions are written in the AXIOM programming language and have been compiled into machine code. This is what comprises the AXIOM *library*. In the rest of this book we show you how to use these domains and their functions and how to write your own functions.



## 2.2 Writing Types and Modes

---

When might you need to write a type or mode? You need to do so when you declare variables.

We have already seen in the last section several examples of types. Most of these examples had either no arguments (for example, `Integer`) or one argument (for example, `Polynomial(Integer)`). In this section we give details about writing arbitrary types. We then define *modes* and discuss how to write them. We conclude the section with a discussion on constructor abbreviations.

`a : PositiveInteger`

Type: Void

You need to do so when you declare functions (Section 2.3 on page 103),

`f : Integer -> String`

Type: Void

You need to do so when you convert an object from one type to another (Section 2.7 on page 113).

`factor(2 :: Complex(Integer))`

$-i(1+i)^2$  (3)

Type: Factored Complex Integer

`(2 = 3)$Integer`

`false` (4)

Type: Boolean

You need to do so when you give computation target type information (Section 2.9 on page 119).

`(2 = 3)@Boolean`

`false` (5)

Type: Boolean

### 2.2.1 Types with No Arguments

---

A constructor with no arguments can be written either with or without trailing opening and closing parentheses (“`()`”).

`Boolean()` is the same as `Boolean`

`String()` is the same as `String`

`Integer()` is the same as `Integer`

`Void()` is the same as `Void`

It is customary to omit the parentheses.

### 2.2.2 Types with One Argument

---

A constructor with one argument can frequently be written with no parentheses. Types nest from right to left so that `Complex Fraction Polynomial Integer` is the same as `Complex(Fraction(Polynomial(Integer)))`. You need to use parentheses to force the application of a constructor to the correct argument, but you need not use any more than is necessary to remove ambiguities.

Here are some guidelines for using parentheses (they are possibly slightly more restrictive than they need to be).

If the argument is an expression like `2 + 3` then you must enclose the argument in parentheses.

```
e : PrimeField(2 + 3)
```

Type: Void

If the type is to be used with package calling then you must enclose the argument in parentheses.

```
content(2)$Polynomial(Integer)
2
```

(2)

Type: Integer

Alternatively, you can write the type without parentheses then enclose the whole type expression with parentheses.

```
content(2)$ (Polynomial Complex Fraction Integer)
2
```

(3)

Type: Complex Fraction Integer

If you supply computation target type information (Section 2.9 on page 119) then you should enclose the argument in parentheses.

```
(2/3)@Fraction(Polynomial(Integer))
2
3
```

(4)

Type: Fraction Polynomial Integer

If the type itself has parentheses around it and we are not in the case of the first example above, then the parentheses can usually be omitted.

```
(2/3)@Fraction(Polynomial Integer)
2
3
```

(5)

Type: Fraction Polynomial Integer

If the type is used in a declaration and the argument is a single-word type, integer or symbol, then the parentheses can usually be omitted.

```
(d,f,g) : Complex Polynomial Integer
```

Type: Void

### 2.2.3 Types with More Than One Argument

If a constructor has more than one argument, you must use parentheses. Some examples are

```
UnivariatePolynomial(x, Float)
MultivariatePolynomial([z,w,r], Complex Float)
SquareMatrix(3, Integer)
FactoredFunctions2(Integer,Fraction Integer)
```

### 2.2.4 Modes

A *mode* is a type that possibly is a question mark (“?”) or contains one in an argument position. For example, the following are all modes.

```

?                               Polynomial ?
Matrix Polynomial ?             SquareMatrix(3,?)
Integer                         OneDimensionalArray(Float)
```

As is evident from these examples, a mode is a type with a part that is not specified (indicated by a question mark). Only one “?” is allowed per mode and it must appear in the most deeply nested argument that is a type. Thus `?(Integer)`, `Matrix?(Polynomial)`, `SquareMatrix?(Integer)` and `SquareMatrix(?, ?)` are all invalid. The question mark must take the place of a domain, not data (for example, the integer that is the dimension of a square matrix). This rules out, for example, the two `SquareMatrix` expressions.

Modes can be used for declarations (Section 2.3 on page 103) and conversions (Section 2.7 on page 113). However, you cannot use a mode for package calling or giving target type information.

## 2.2.5 Abbreviations

Every constructor has an abbreviation that you can freely substitute for the constructor name. In some cases, the abbreviation is nothing more than the capitalized version of the constructor name.

Aside from allowing types to be written more concisely, abbreviations are used by AXIOM to name various system files for constructors (such as library filenames, test input files and example files). Here are some common abbreviations.

COMPLEX abbreviates Complex	DFLOAT abbreviates DoubleFloat
EXPR abbreviates Expression	FLOAT abbreviates Float
FRAC abbreviates Fraction	INT abbreviates Integer
MATRIX abbreviates Matrix	NNI abbreviates NonNegativeInteger
PI abbreviates PositiveInteger	POLY abbreviates Polynomial
STRING abbreviates String	UP abbreviates UnivariatePolynomial

You can combine both full constructor names and abbreviations in a type expression. Here are some types using abbreviations.

POLY INT is the same as Polynomial(INT)  
 POLY(Integer) is the same as Polynomial(Integer)  
 POLY(Integer) is the same as Polynomial(INT)  
 FRAC(COMPLEX(INT)) is the same as Fraction Complex Integer  
 FRAC(COMPLEX(INT)) is the same as FRAC(Complex Integer)

There are several ways of finding the names of constructors and their abbreviations. For a specific constructor, use `)abbreviation query`. You can also use the `)what` system command to see the names and abbreviations of constructors. For more information about `)what`, see Section

A.28 on page 748.

)abbreviation query can be abbreviated (no pun intended) to )abb q.

```
)abb q Integer
```

```
INT abbreviates domain Integer
```

The )abbreviation query command lists the constructor name if you give the abbreviation. Issue )abb q if you want to see the names and abbreviations of all AXIOM constructors.

```
)abb q DMP
```

```
DMP abbreviates domain
  DistributedMultivariatePolynomial
```

Issue this to see all packages whose names contain the string "ode".

```
)what packages ode
```

```
----- Packages -----
```

```
Packages with names matching patterns:
ode
```

```
EXPRODE  ExpressionSpaceODESolver
FCPAK1    FortranCodePackage1  GRAY      GrayCode
LODEEF    ElementaryFunctionLODESolver
NODE1     NonLinearFirstOrderODESolver
ODECONST  ConstantLODE
ODEEF     ElementaryFunctionODESolver
ODEINT    ODEIntegration        ODEPAL
PureAlgebraicLODE
ODERAT    RationalLODE          ODERED    ReduceLODE
ODESYS    SystemODESolver        ODETOOLS  ODETools
UTSODE    UnivariateTaylorSeriesODESolver
UTSODETL  UTSodetools
```

## 2.3 Declarations

A *declaration* is an expression used to restrict the type of values that can be assigned to variables. A colon (":") is always used after a variable or list of variables to be declared.

For a single variable, the syntax for declaration is

*variableName* : *typeOrMode*

For multiple variables, the syntax is

*(variableName<sub>1</sub>, variableName<sub>2</sub>, ... variableName<sub>N</sub>)* : *typeOrMode*

You can always combine a declaration with an assignment. When you do, it is equivalent to first giving a declaration statement, then giving an assignment. For more information on assignment, see Section 1.3.4 on page 48 and Section 5.1 on page 150. To see how to declare your own functions, see Section 6.4 on page 183.

This declares one variable to have a type.

`a : Integer`

Type: Void

This declares several variables to have a type.

`(b,c) : Integer`

Type: Void

`a`, `b` and `c` can only hold integer values.

`a := 45`  
`45`

(3)

Type: Integer

If a value cannot be converted to a declared type, an error message is displayed.

`b := 4/5`  
Cannot convert right-hand side of assignment  
`4`  
`-`  
`5`  
  
to an object of the type Integer of the left-hand side.

This declares a variable with a mode.

`n : Complex ?`

Type: Void

This declares several variables with a mode.

`(p,q,r) : Matrix Polynomial ?`

Type: Void

This complex object has integer real and imaginary parts.

`n := -36 + 9 * %i`  
 $-36 + 9 i$

(6)

Type: Complex Integer

This complex object has fractional symbolic real and imaginary parts.

`n := complex(4/(x + y),y/x)`  
 $\frac{4}{y + x} + \frac{y}{x} i$

(7)

Type: Complex Fraction Polynomial Integer

This matrix has entries that are polynomials with integer coefficients.

`p := [[1,2],[3,4],[5,6]]`  

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

(8)

Type: Matrix Polynomial Integer

This matrix has a single entry that is a polynomial with rational number coefficients.

`q := [[x - 2/3]]`  

$$\begin{bmatrix} x - \frac{2}{3} \end{bmatrix}$$

(9)

Type: Matrix Polynomial Fraction Integer

This matrix has entries that are polynomials with complex integer coefficients.

`r := [[1-%i*x,7*y+4*i]]`  

$$\begin{bmatrix} -i x + 1 & 7 y + 4 i \end{bmatrix}$$

(10)

Type: Matrix Polynomial Complex Integer

Note the difference between this and the next example. This is a complex object with polynomial real and imaginary parts.

`f : COMPLEX POLY ? := (x + y*i)**2`  
 $-y^2 + x^2 + 2 x y i$

(11)

Type: Complex Polynomial Integer

This is a polynomial with complex integer coefficients. The objects are convertible from one to the other. See Section 2.7 on page 113 for more information.

`g : POLY COMPLEX ? := (x + y*i)**2`  
 $-y^2 + 2 i x y + x^2$

(12)

Type: Polynomial Complex Integer

## 2.4 Records

A Record is an object composed of one or more other objects, each of which is referenced with a *selector*. Components can all belong to the same type or each can have a different type.

The syntax for writing a Record type is

```
Record(selector1:type1, selector2:type2, ..., selectorN:typeN)
```

You must be careful if a selector has the same name as a variable in the workspace. If this occurs, precede the selector name by a single quote.

Record components are implicitly ordered. All the components of a record can be set at once by assigning the record a bracketed *tuple* of values of the proper length (for example, `r : Record(a: Integer, b: String) := [1, "two"]`). To access a component of a record `r`, write the name `r`, followed by a period, followed by a selector.

The object returned by this computation is a record with two components: a **quotient** part and a **remainder** part.

```
u := divide(5,2)
[quotient = 2, remainder = 1]
Type: Record(quotient: Integer, remainder: Integer)
```

This is the quotient part.

```
u.quotient
2
Type: PositiveInteger
```

This is the remainder part.

```
u.remainder
1
Type: PositiveInteger
```

You can use selector expressions on the left-hand side of an assignment to change destructively the components of a record.

```
u.quotient := 8978
8978
Type: PositiveInteger
```

The selected component **quotient** has the value 8978, which is what is returned by the assignment. Check that the value of `u` was modified.

```
u
[quotient = 8978, remainder = 1]
Type: Record(quotient: Integer, remainder: Integer)
```

Selectors are evaluated. Thus you can use variables that evaluate to selectors instead of the selectors themselves.

```
s := 'quotient
quotient
Type: Variable quotient
```

Be careful! A selector could have the same name as a variable in the workspace. If this occurs, precede the selector name by a single quote, as in <code>u.'quotient</code> .	<code>divide(5,2).s</code> <code>2</code>	(7) Type: PositiveInteger
Here we declare that the value of <code>bd</code> has two components: a string, to be accessed via <code>name</code> , and an integer, to be accessed via <code>birthdayMonth</code> .	<code>bd : Record(name : String, birthdayMonth : Integer)</code>	Type: Void
You must initially set the value of the entire Record at once.	<code>bd := ["Judith", 3]</code> <code>[name = "Judith", birthdayMonth = 3]</code>	(9) Type: Record(name: String, birthdayMonth: Integer)
Once set, you can change any of the individual components.	<code>bd.name := "Katie"</code> <code>"Katie"</code>	(10) Type: String
Records may be nested and the selector names can be shared at different levels.	<code>r : Record(a : Record(b: Integer, c: Integer), b: Integer)</code>	Type: Void
The record <code>r</code> has a <code>b</code> selector at two different levels. Here is an initial value for <code>r</code> .	<code>r := [[1,2],3]</code> <code>[a = [b = 1, c = 2], b = 3]</code>	(12) Type: Record(a: Record(b: Integer, c: Integer), b: Integer)
This extracts the <code>b</code> component from the <code>a</code> component of <code>r</code> .	<code>r.a.b</code> <code>1</code>	(13) Type: PositiveInteger
This extracts the <code>b</code> component from <code>r</code> .	<code>r.b</code> <code>3</code>	(14) Type: PositiveInteger
You can also use spaces or parentheses to refer to Record components. This is the same as <code>r.a</code> .	<code>r(a)</code> <code>[b = 1, c = 2]</code>	(15) Type: Record(b: Integer, c: Integer)
This is the same as <code>r.b</code> .	<code>r b</code> <code>3</code>	(16) Type: PositiveInteger



This is the same as `r.b := 10`.      `r(b) := 10`      (17)

Type: PositiveInteger

Look at `r` to make sure it was modified.      `r`      (18)

`[a = [b = 1, c = 2], b = 10]`

Type: Record(a: Record(b: Integer, c: Integer), b: Integer)

## 2.5 Unions

Type Union is used for objects that can be of any of a specific finite set of types. Two versions of unions are available, one with selectors (like records) and one without.

### 2.5.1 Unions Without Selectors

The declaration `x : Union(Integer, String, Float)` states that `x` can have values that are integers, strings or “big” floats. If, for example, the Union object is an integer, the object is said to belong to the Integer *branch* of the Union.<sup>6</sup>

The syntax for writing a Union type without selectors is

`Union(type1, type2, ..., typeN)`

The types in a union without selectors must be distinct.

It is possible to create unions like `Union(Integer, PositiveInteger)` but they are difficult to work with because of the overlap in the branch types. See below for the rules AXIOM uses for converting something into a union object.

The `case` infix operator returns a Boolean and can be used to determine the branch in which an object lies.

This function displays a message stating in which branch of the Union the object (defined as `x` above) lies.

```
sayBranch(x : Union(Integer,String,Float)) : Void ==
  output
    x case Integer => "Integer branch"
    x case String  => "String branch"
    "Float branch"
```

Function declaration `sayBranch : Union(Integer,String,Float) -> Void` has been added to workspace.

Type: Void

This tries `sayBranch` with an integer.

```
sayBranch 1
Compiling function sayBranch with type Union(Integer,
String,Float) -> Void
Integer branch
```

Type: Void

This tries `sayBranch` with a string.

```
sayBranch "hello"
String branch
```

Type: Void

<sup>6</sup>Note that we are being a bit careless with the language here. Technically, the type of `x` is always `Union(Integer, String, Float)`. If it belongs to the Integer branch, `x` may be converted to an object of type Integer.

This tries **sayBranch** with a floating-point number.

```
sayBranch 2.718281828
```

```
Float branch
```

Type: Void

There are two things of interest about this particular example to which we would like to draw your attention.

1. AXIOM normally converts a result to the target value before passing it to the function. If we left the declaration information out of this function definition then the **sayBranch** call would have been attempted with an Integer rather than a Union, and an error would have resulted.

2. The types in a Union are searched in the order given. So if the type were given as

```
sayBranch(x: Union(String,Integer,Float,Any)): Void
```

then the result would have been “String branch” because there is a conversion from Integer to String.

Sometimes Union types can have extremely long names. AXIOM therefore abbreviates the names of unions by printing the type of the branch first within the Union and then eliding the remaining types with an ellipsis (“...”).

Here the Integer branch is displayed first. Use “::” to create a Union object from an object.

```
78 :: Union(Integer,String)
```

```
78
```

(5)

Type: Union(Integer, ...)

Here the String branch is displayed first.

```
s := "string" :: Union(Integer,String)
```

```
"string"
```

(6)

Type: Union(String, ...)

Use **typeOf** to see the full and actual Union type.

```
typeOf s
```

```
Union (Integer ,String )
```

(7)

Type: Domain

A common operation that returns a union is **exquo** which returns the “exact quotient” if the quotient is exact,...

```
three := exquo(6,2)
```

```
3
```

(8)

Type: Union(Integer, ...)

and “failed” if the quotient is not exact.

```
exquo(5,2)
```

```
"failed"
```

(9)

Type: Union("failed", ...)

A union with a "failed" is frequently used to indicate the failure or lack of applicability of an object. As another example, assign an integer a variable `r` declared to be a rational number.

```
r: FRAC INT := 3
3
Type: Fraction Integer
```

(10)

The operation **retractIfCan** tries to retract the fraction to the underlying domain Integer. It produces a union object. Here it succeeds.

```
retractIfCan(r)
3
Type: Union(Integer, ...)
```

(11)

Assign it a rational number.

```
r := 3/2
3
2
Type: Fraction Integer
```

(12)

Here the retraction fails.

```
retractIfCan(r)
"failed"
Type: Union("failed", ...)
```

(13)

## 2.5.2 Unions With Selectors

Like records (Section 2.4 on page 105), you can write Union types with selectors.

The syntax for writing a Union type with selectors is

```
Union(selector1:type1, selector2:type2, ..., selectorN:typeN)
```

You must be careful if a selector has the same name as a variable in the workspace. If this occurs, precede the selector name by a single quote. It is an error to use a selector that does not correspond to the branch of the Union in which the element actually lies.

Be sure to understand the difference between records and unions with selectors. Records can have more than one component and the selectors are used to refer to the components. Unions always have one component but the type of that one component can vary. An object of type `Record(a: Integer, b: Float, c: String)` contains an integer *and* a float *and* a string. An object of type `Union(a: Integer, b: Float, c: String)` contains an integer *or* a float *or* a string.

Here is a version of the **sayBranch** function (cf. Section 2.5.1 on page 108) that works with a union with selectors. It displays a message stating in which branch of the Union the object lies.

```
sayBranch(x:Union(i:Integer,s:String,f:Float)):Void==
output
  x case i => "Integer branch"
  x case s  => "String branch"
  "Float branch"
```

Note that **case** uses the selector name as its right-hand argument. If you accidentally use the branch type on the right-hand side of **case**, **false** will be returned.

Declare variable **u** to have a union type with selectors.

```
u : Union(i : Integer, s : String)
```

Type: Void

Give an initial value to **u**.

```
u := "good morning"
"good morning"
```

(2)

Type: Union(s: String, ...)

Use **case** to determine in which branch of a Union an object lies.

```
u case i
false
```

(3)

Type: Boolean

```
u case s
true
```

(4)

Type: Boolean

To access the element in a particular branch, use the selector.

```
u.s
"good morning"
```

(5)

Type: String

## 2.6 The “Any” Domain

---

With the exception of objects of type `Record`, all AXIOM data structures are homogenous, that is, they hold objects all of the same type. If you need to get around this, you can use type `Any`. Using `Any`, for example, you can create lists whose elements are integers, rational numbers, strings, and even other lists.

Declare `u` to have type `Any`.

```
u : Any
```

Type: `Void`

Assign a list of mixed type values to `u`

```
u := [1, 7.2, 3/2, x**2, "wally"]
```

$$\left[ 1, 7.2, \frac{3}{2}, x^2, \text{"wally"} \right] \quad (2)$$

Type: `List Any`

When we ask for the elements, AXIOM displays these types.

```
u.1
```

1 (3)

Type: `PositiveInteger`

Actually, these objects belong to `Any` but AXIOM automatically converts them to their natural types for you.

```
u.3
```

$$\frac{3}{2} \quad (4)$$

Type: `Fraction Integer`

Since type `Any` can be anything, it can only belong to type `Type`. Therefore it cannot be used in algebraic domains.

```
v : Matrix(Any)
```

```
Daly Bug
```

```
Matrix Any is not a valid type.
```

Perhaps you are wondering how AXIOM internally represents objects of type `Any`. An object of type `Any` consists not only a data part representing its normal value, but also a type part (a *badge*) giving its type. For example, the value 1 of type `PositiveInteger` as an object of type `Any` internally looks like `[1,PositiveInteger()]`.

## 2.7 Conversion

*Conversion* is the process of changing an object of one type into an object of another type. The syntax for conversion is:

$$\text{object} :: \text{newType}$$

By default, 3 has the type  
PositiveInteger.

3

3

(1)

Type: PositiveInteger

We can change this into an  
object of type Fraction Integer by  
using “::”.

3 :: Fraction Integer

3

(2)

Type: Fraction Integer

A *coercion* is a special kind of conversion that AXIOM is allowed to do automatically when you enter an expression. Coercions are usually somewhat safer than more general conversions. The AXIOM library contains operations called **coerce** and **convert**. Only the **coerce** operations can be used by the interpreter to change an object into an object of another type unless you explicitly use a “::”.

By now you will be quite familiar with what types and modes look like. It is useful to think of a type or mode as a pattern for what you want the result to be.

Let's start with a square matrix  
of polynomials with complex  
rational number coefficients.

m : SquareMatrix(2,POLY COMPLEX FRAC INT)

Type: Void

m := matrix [[x-3/4\*i,z\*y\*\*2+1/2],[3/7\*i\*y\*\*4 - x,12-  
%i\*9/5]]

$$\begin{bmatrix} x - \frac{3}{4}i & y^2 z + \frac{1}{2} \\ \frac{3}{7}i y^4 - x & 12 - \frac{9}{5}i \end{bmatrix} \quad (4)$$

Type: SquareMatrix(2, Polynomial Complex Fraction Integer)

We first want to interchange the  
Complex and Fraction layers. We  
do the conversion by doing the  
interchange in the type  
expression.

m1 := m :: SquareMatrix(2,POLY FRAC COMPLEX INT)

$$\begin{bmatrix} x - \frac{3}{4}i & y^2 z + \frac{1}{2} \\ \frac{3}{7}i y^4 - x & \frac{60-9i}{5} \end{bmatrix} \quad (5)$$

Type: SquareMatrix(2, Polynomial Fraction Complex Integer)

Interchange the Polynomial and the Fraction levels.

$$\begin{aligned} \text{m2} &:= \text{m1} :: \text{SquareMatrix}(2, \text{FRAC POLY COMPLEX INT}) \\ &\left[ \begin{array}{cc} \frac{4x-3i}{3i y^4-7x} & \frac{2y^2 z+1}{60-9i} \end{array} \right] \end{aligned} \quad (6)$$

Type: SquareMatrix(2, Fraction Polynomial Complex Integer)

Interchange the Polynomial and the Complex levels.

$$\begin{aligned} \text{m3} &:= \text{m2} :: \text{SquareMatrix}(2, \text{FRAC COMPLEX POLY INT}) \\ &\left[ \begin{array}{cc} \frac{4x-3i}{-7x+3y^4i} & \frac{2y^2 z+1}{60-9i} \end{array} \right] \end{aligned} \quad (7)$$

Type: SquareMatrix(2, Fraction Complex Polynomial Integer)

All the entries have changed types, although in comparing the last two results only the entry in the lower left corner looks different. We did all the intermediate steps to show you what AXIOM can do.

In fact, we could have combined all these into one conversion.

$$\begin{aligned} \text{m} &:: \text{SquareMatrix}(2, \text{FRAC COMPLEX POLY INT}) \\ &\left[ \begin{array}{cc} \frac{4x-3i}{-7x+3y^4i} & \frac{2y^2 z+1}{60-9i} \end{array} \right] \end{aligned} \quad (8)$$

Type: SquareMatrix(2, Fraction Complex Polynomial Integer)

There are times when AXIOM is not be able to do the conversion in one step. You may need to break up the transformation into several conversions in order to get an object of the desired type.

We cannot move either Fraction or Complex above (or to the left of, depending on how you look at it) SquareMatrix because each of these levels requires that its argument type have commutative multiplication, whereas SquareMatrix does not.<sup>7</sup> The Integer level did not move anywhere because it does not allow any arguments. We also did not move the SquareMatrix part anywhere, but we could have.

Recall that **m** looks like this.

$$\begin{aligned} \text{m} & \\ &\left[ \begin{array}{cc} x - \frac{3}{4}i & y^2 z + \frac{1}{2} \\ \frac{3}{7}i y^4 - x & 12 - \frac{9}{5}i \end{array} \right] \end{aligned} \quad (9)$$

Type: SquareMatrix(2, Polynomial Complex Fraction Integer)

---

<sup>7</sup>Fraction requires that its argument belong to the category IntegralDomain and Complex requires that its argument belong to CommutativeRing. See Section 2.1 on page 92 for a brief discussion of categories.



If we want a polynomial with matrix coefficients rather than a matrix with polynomial entries, we can just do the conversion.

$$\begin{aligned} \text{m} &:: \text{POLY SquareMatrix}(2, \text{COMPLEX FRAC INT}) \\ &\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} y^2 z + \begin{bmatrix} 0 & 0 \\ \frac{3}{7} i & 0 \end{bmatrix} y^4 + \\ &\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} -\frac{3}{4} i & \frac{1}{2} \\ 0 & 12 - \frac{9}{5} i \end{bmatrix} \end{aligned} \quad (10)$$

Type: Polynomial SquareMatrix(2, Complex Fraction Integer)

We have not yet used modes for any conversions. Modes are a great shorthand for indicating the type of the object you want. Instead of using the long type expression in the last example, we could have simply said this.

$$\begin{aligned} \text{m} &:: \text{POLY ?} \\ &\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} y^2 z + \begin{bmatrix} 0 & 0 \\ \frac{3}{7} i & 0 \end{bmatrix} y^4 + \\ &\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} -\frac{3}{4} i & \frac{1}{2} \\ 0 & 12 - \frac{9}{5} i \end{bmatrix} \end{aligned} \quad (11)$$

Type: Polynomial SquareMatrix(2, Complex Fraction Integer)

We can also indicate more structure if we want the entries of the matrices to be fractions.

$$\begin{aligned} \text{m} &:: \text{POLY SquareMatrix}(2, \text{FRAC ?}) \\ &\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} y^2 z + \begin{bmatrix} 0 & 0 \\ \frac{3}{7} i & 0 \end{bmatrix} y^4 + \\ &\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} -\frac{3}{4} i & \frac{1}{2} \\ 0 & \frac{60-9}{5} i \end{bmatrix} \end{aligned} \quad (12)$$

Type: Polynomial SquareMatrix(2, Fraction Complex Integer)

## 2.8 Subdomains Again

---

A *subdomain* *S* of a domain *D* is a domain consisting of

1. those elements of *D* that satisfy some *predicate* (that is, a test that returns **true** or **false**) and
2. a subset of the operations of *D*.

Every domain is a subdomain of itself, trivially satisfying the membership test: **true**.

Currently, there are only two system-defined subdomains in AXIOM that receive substantial use. `PositiveInteger` and `NonNegativeInteger` are subdomains of `Integer`. An element *x* of `NonNegativeInteger` is an integer that is greater than or equal to zero, that is, satisfies  $x \geq 0$ . An element *x* of `PositiveInteger` is a nonnegative integer that is, in fact, greater than zero, that is, satisfies  $x > 0$ . Not all operations from `Integer` are available for these subdomains. For example, negation and subtraction are not provided since the subdomains are not closed under those operations. When you use an integer in an expression, AXIOM assigns to it the type that is the most specific subdomain whose predicate is satisfied.

This is a positive integer.	5	
	5	(1) Type: PositiveInteger
This is a nonnegative integer.	0	
	0	(2) Type: NonNegativeInteger
This is neither of the above.	-5	
	-5	(3) Type: Integer
Furthermore, unless you are assigning an integer to a declared variable or using a conversion, any integer result has as type the most specific subdomain.	(-2) - (-3)	
	1	(4) Type: PositiveInteger
	0 :: Integer	
	0	(5) Type: Integer
	x : NonNegativeInteger := 5	
	5	(6) Type: NonNegativeInteger

When necessary, AXIOM converts an integer object into one belonging to a less specific subdomain. For example, in 3-2, the arguments to “-” are both elements of PositiveInteger, but this type does not provide a subtraction operation. Neither does NonNegativeInteger, so 3 and 2 are viewed as elements of Integer, where their difference can be calculated. The result is 1, which AXIOM then automatically assigns the type PositiveInteger.

Certain operations are very sensitive to the subdomains to which their arguments belong. This is an element of PositiveInteger.

```
2 ** 2
4
Type: PositiveInteger
```

(7)

This is an element of Fraction Integer.

```
2 ** (-2)
1
4
Type: Fraction Integer
```

(8)

It makes sense then that this is a list of elements of PositiveInteger.

```
[10**i for i in 2..5]
[100, 1000, 10000, 100000]
Type: List PositiveInteger
```

(9)

What should the type of [10\*\*(i-1) for i in 2..5] be? On one hand, i-1 is always an integer greater than zero as i ranges from 2 to 5 and so 10\*\*i is also always a positive integer. On the other, i-1 is a very simple function of i. AXIOM does not try to analyze every such function over the index’s range of values to determine whether it is always positive or nowhere negative. For an arbitrary AXIOM function, this analysis is not possible.

So, to be consistent no such analysis is done and we get this.

```
[10**(i-1) for i in 2..5]
[10, 100, 1000, 10000]
Type: List Fraction Integer
```

(10)

To get a list of elements of PositiveInteger instead, you have two choices. You can use a conversion.

```
[10**((i-1) :: PI) for i in 2..5]
Compiling function G82568 with type Integer ->
Boolean
Compiling function G82580 with type
NonNegativeInteger -> Boolean
[10, 100, 1000, 10000]
Type: List PositiveInteger
```

(11)

Or you can use pretend.

```
[10**((i-1) pretend PI) for i in 2..5]
[10, 100, 1000, 10000]
Type: List PositiveInteger
```

(12)

The operation pretend is used to defeat the AXIOM type system. The

expression `object pretend D` means “make a new object (without copying) of type `D` from `object`.” If `object` were an integer and you told AXIOM to pretend it was a list, you would probably see a message about a fatal error being caught and memory possibly being damaged. Lists do not have the same internal representation as integers!

You use `pretend` at your peril.

Use `pretend` with great care!  
AXIOM trusts you that the  
value is of the specified type.

(2/3) pretend Complex Integer

$2 + 3i$

(13)

Type: Complex Integer

## 2.9 Package Calling and Target Types

AXIOM works hard to figure out what you mean by an expression without your having to qualify it with type information. Nevertheless, there are times when you need to help it along by providing hints (or even orders!) to get AXIOM to do what you want.

We saw in Section 2.3 on page 103 that declarations using types and modes control the type of the results produced. For example, we can either produce a complex object with polynomial real and imaginary parts or a polynomial with complex integer coefficients, depending on the declaration.

*Package calling* is how you tell AXIOM to use a particular function from a particular part of the library.

Use the “/” from Fraction Integer to create a fraction of two integers.

$$\frac{2}{3} \quad (1)$$

Type: Fraction Integer

If we wanted a floating point number, we can say “use the “/” in Float.”

$$(2/3)\$Float \quad (2)$$

Type: Float

Perhaps we actually wanted a fraction of complex integers.

$$(2/3)\$Fraction(Complex Integer) \quad (3)$$

Type: Fraction Complex Integer

In each case, AXIOM used the indicated operations, sometimes first needing to convert the two integers into objects of an appropriate type. In these examples, “/” is written as an infix operator.

To use package calling with an infix operator, use the following syntax:

$$(arg_1 \text{ op } arg_1)\$type$$

We used, for example,  $(2/3)\$Float$ . The expression  $2 + 3 + 4$  is equivalent to  $(2+3) + 4$ . Therefore in the expression  $(2 + 3 + 4)\$Float$  the second “+” comes from the Float domain. Can you guess whether the first “+” comes from Integer or Float?<sup>8</sup>

<sup>8</sup>Float, because the package call causes AXIOM to convert  $(2 + 3)$  and 4 to type Float. Before the sum is converted, it is given a target type (see below) of Float by AXIOM and then evaluated. The target type causes the “+” from Float to be used.

For an operator written before its arguments, you must use parentheses around the arguments (even if there is only one), and follow the closing parenthesis by a “\$” and then the type.

$$\text{fun } ( \text{arg}_1, \text{arg}_1, \dots, \text{arg}_N ) \$type$$

For example, to call the “minimum” function from DoubleFloat on two integers, you could write `min(4,89)$DoubleFloat`. Another use of package calling is to tell AXIOM to use a library function rather than a function you defined. We discuss this in Section 6.9 on page 191.

Sometimes rather than specifying where an operation comes from, you just want to say what type the result should be. We say that you provide a *target type* for the expression. Instead of using a “\$”, use a “@” to specify the requested target type. Otherwise, the syntax is the same. Note that giving a target type is not the same as explicitly doing a conversion. The first says “try to pick operations so that the result has such-and-such a type.” The second says “compute the result and then convert to an object of such-and-such a type.”

Sometimes it makes sense, as in this expression, to say “choose the operations in this expression so that the final result is a Float.”

`(2/3)@Float`

`0.66666666666666666667`

(4)

Type: Float

Here we used “@” to say that the target type of the left-hand side was Float. In this simple case, there was no real difference between using “\$” and “@”. You can see the difference if you try the following.

This says to try to choose “+” so that the result is a string. AXIOM cannot do this.

`(2 + 3)@String`

An expression involving @ String actually evaluated to one of type PositiveInteger . Perhaps you should use :: String .

This says to get the “+” from String and apply it to the two integers. AXIOM also cannot do this because there is no “+” exported by String.

`(2 + 3)$String`

The function + is not implemented in String .

(By the way, the operation **concat** or juxtaposition is used to concatenate two strings.)

When we have more than one operation in an expression, the difference is even more evident. The following two expressions show that AXIOM uses the target type to create different objects. The “+”, “\*” and “\*\*” operations are all chosen so that an object of the correct final type is

	created.	
This says that the operations should be chosen so that the result is a Complex object.	$((x + y * \%i)**2)@(\text{Complex Polynomial Integer})$ $-y^2 + x^2 + 2 x y i$	(5) Type: Complex Polynomial Integer
This says that the operations should be chosen so that the result is a Polynomial object.	$((x + y * \%i)**2)@(\text{Polynomial Complex Integer})$ $-y^2 + 2 i x y + x^2$	(6) Type: Polynomial Complex Integer
What do you think might happen if we left off all target type and package call information in this last example?	$(x + y * \%i)**2$ $-y^2 + 2 i x y + x^2$	(7) Type: Polynomial Complex Integer
We can convert it to Complex as an afterthought. But this is more work than just saying making what we want in the first place.	$\% :: \text{Complex ?}$ $-y^2 + x^2 + 2 x y i$	(8) Type: Complex Polynomial Integer
	Finally, another use of package calling is to qualify fully an operation that is passed as an argument to a function.	
Start with a small matrix of integers.	$h := \text{matrix } [[8,6],[-4,9]]$ $\begin{bmatrix} 8 & 6 \\ -4 & 9 \end{bmatrix}$	(9) Type: Matrix Integer
We want to produce a new matrix that has for entries the multiplicative inverses of the entries of <b>h</b> . One way to do this is by calling <b>map</b> with the <b>inv</b> function from <b>Fraction(Integer)</b> .	$\text{map}(\text{inv}\$ \text{Fraction(Integer)}, h)$ $\begin{bmatrix} \frac{1}{8} & \frac{1}{6} \\ -\frac{1}{4} & \frac{1}{9} \end{bmatrix}$	(10) Type: Matrix Fraction Integer
We could have been a bit less verbose and used abbreviations.	$\text{map}(\text{inv}\$ \text{FRAC(INT)}, h)$ $\begin{bmatrix} \frac{1}{8} & \frac{1}{6} \\ -\frac{1}{4} & \frac{1}{9} \end{bmatrix}$	(11) Type: Matrix Fraction Integer
As it turns out, AXIOM is smart enough to know what we mean anyway. We can just say this.	$\text{map}(\text{inv}, h)$ $\begin{bmatrix} \frac{1}{8} & \frac{1}{6} \\ -\frac{1}{4} & \frac{1}{9} \end{bmatrix}$	(12) Type: Matrix Fraction Integer

## 2.10 Resolving Types

In this section we briefly describe an internal process by which AXIOM determines a type to which two objects of possibly different types can be converted. We do this to give you further insight into how AXIOM takes your input, analyzes it, and produces a result.

What happens when you enter  $x + 1$  to AXIOM? Let's look at what you get from the two terms of this expression.

This is a symbolic object whose type indicates the name.

$x$   
 $x$   
(1)  
Type: Variable x

This is a positive integer.

1  
1  
(2)  
Type: PositiveInteger

There are no operations in PositiveInteger that add positive integers to objects of type Variable(x) nor are there any in Variable(x). Before it can add the two parts, AXIOM must come up with a common type to which both  $x$  and 1 can be converted. We say that AXIOM must *resolve* the two types into a common type. In this example, the common type is Polynomial(Integer).

Once this is determined, both parts are converted into polynomials, and the addition operation from Polynomial(Integer) is used to get the answer.

$x + 1$   
 $x + 1$   
(3)  
Type: Polynomial Integer

AXIOM can always resolve two types: if nothing resembling the original types can be found, then Any is used. This is fine and useful in some cases.

`["string", 3.14159]`  
`["string", 3.14159]`  
(4)  
Type: List Any



In other cases objects of type  
Any can't be used by the  
operations you specified.

```
"string" + 3.14159
```

There are 11 exposed and 5 unexposed library  
operations named + having 2 argument(s) but none  
was determined to be applicable. Use HyperDoc  
Browse, or issue  
                                  )display op +  
to learn more about the available operations.  
Perhaps package-calling the operation or using  
coercions on the arguments will allow you to apply  
the operation.

Daly Bug

Cannot find a definition or applicable library  
operation named + with argument type(s)  
                                  String  
                                  Float

Perhaps you should use "@" to indicate the  
required return type, or "\$" to specify which  
version of the function you need.

Although this example was contrived, your expressions may need to be  
qualified slightly to help AXIOM resolve the types involved. You may  
need to declare a few variables, do some package calling, provide some  
target type information or do some explicit conversions.

We suggest that you just enter the expression you want evaluated and  
see what AXIOM does. We think you will be impressed with its ability  
to “do what I mean.” If AXIOM is still being obtuse, give it some hints.  
As you work with AXIOM, you will learn where it needs a little help to  
analyze quickly and perform your computations.

## 2.11 Exposing Domains and Packages

---

In this section we discuss how AXIOM makes some operations available to you while hiding others that are meant to be used by developers or only in rare cases. If you are a new user of AXIOM, it is likely that everything you need is available by default and you may want to skip over this section on first reading.

Every domain and package in the AXIOM library is either *exposed* (meaning that you can use its operations without doing anything special) or it is *hidden* (meaning you have to either package call (see Section 2.9 on page 119) the operations it contains or explicitly expose it to use the operations). The initial exposure status for a constructor is set in the file **exposed.lsp** (see the *Installer's Note* for AXIOM if you need to know the location of this file). Constructors are collected together in *exposure groups*. Categories are all in the exposure group “categories” and the bulk of the basic set of packages and domains that are exposed are in the exposure group “basic.” Here is an abbreviated sample of the file (without the Lisp parentheses):

```
basic
    AlgebraicNumber          AN
    AlgebraGivenByStructuralConstants  ALGSC
    Any                      ANY
    AnyFunctions1            ANY1
    BinaryExpansion          BINARY
    Boolean                   BOOLEAN
    CardinalNumber           CARD
    CartesianTensor          CARTEN
    Character                 CHAR
    CharacterClass            CCLASS
    CliffordAlgebra          CLIF
    Color                     COLOR
    Complex                   COMPLEX
    ContinuedFraction         CONTFRAC
    DecimalExpansion         DECIMAL
    ...

categories
    AbelianGroup             ABELGRP
    AbelianMonoid            ABELMON
    AbelianMonoidRing        AMR
    AbelianSemiGroup         ABELSG
    Aggregate                AGG
    Algebra                  ALGEBRA
    AlgebraicallyClosedField ACF
    AlgebraicallyClosedFunctionSpace ACFS
    ArcHyperbolicFunctionCategory AHYP
    ...
```

For each constructor in a group, the full name and the abbreviation is given. There are other groups in **exposed.lsp** but initially only the constructors in exposure groups “basic” “categories” “naglink” and “anna” are exposed.

As an interactive user of AXIOM, you do not need to modify this file. Instead, use `)set expose` to expose, hide or query the exposure status of an individual constructor or exposure group. The reason for having exposure groups is to be able to expose or hide multiple constructors with a single command. For example, you might group together into exposure group “quantum” a number of domains and packages useful for quantum mechanical computations. These probably should not be available to every user, but you want an easy way to make the whole collection visible to AXIOM when it is looking for operations to apply.

If you wanted to hide all the basic constructors available by default, you would issue `)set expose drop group basic`. We do not recommend that you do this. If, however, you discover that you have hidden all the basic constructors, you should issue `)set expose add group basic` to restore your default environment.

It is more likely that you would want to expose or hide individual constructors. In Section 6.19 on page 224 we use several operations from `OutputForm`, a domain usually hidden. To avoid package calling every operation from `OutputForm`, we expose the domain and let AXIOM conclude that those operations should be used. Use `)set expose add constructor` and `)set expose drop constructor` to expose and hide a constructor, respectively. You should use the constructor name, not the abbreviation. The `)set expose` command guides you through these options.

If you expose a previously hidden constructor, AXIOM exhibits new behavior (that was your intention) though you might not expect the results that you get. `OutputForm` is, in fact, one of the worst offenders in this regard. This domain is meant to be used by other domains for creating a structure that AXIOM knows how to display. It has functions like “+” that form output representations rather than do mathematical calculations. Because of the order in which AXIOM looks at constructors when it is deciding what operation to apply, `OutputForm` might be used instead of what you expect.

This is a polynomial.

$$\begin{array}{r} x + x \\ 2 x \end{array} \tag{1}$$

Type: Polynomial Integer

Expose `OutputForm`.

```
)set expose add constructor OutputForm
```

```
OutputForm is now explicitly exposed in frame initial
```

This is what we get when  
OutputForm is automatically  
available.

$x + x$   
 $x + x$

(2)

Type: OutputForm

Hide OutputForm so we don't  
run into problems with any later  
examples!

)set expose drop constructor OutputForm  
OutputForm is now explicitly hidden in frame initial

Finally, exposure is done on a frame-by-frame basis. A *frame* (see Section A.11 on page 734) is one of possibly several logical AXIOM workspaces within a physical one, each having its own environment (for example, variables and function definitions). If you have several AXIOM workspace windows on your screen, they are all different frames, automatically created for you by HyperDoc. Frames can be manually created, made active and destroyed by the `)frame` system command. They do not share exposure information, so you need to use `)set expose` in each one to add or drop constructors from view.

## 2.12 Commands for Snooping

---

To conclude this chapter, we introduce you to some system commands that you can use for getting more information about domains, packages, categories, and operations. The most powerful AXIOM facility for getting information about constructors and operations is the Browse component of HyperDoc. This is discussed in Chapter 14.

Use the `)what` system command to see lists of system objects whose name contain a particular substring (uppercase or lowercase is not significant).

Issue this to see a list of all operations with “complex” in their names.

```
)what operation complex
Operations whose names satisfy the above pattern(s):
```

```
complex
complex?
complexEigenvalues
complexEigenvectors
complexElementary
complexExpand
complexForm
complexIntegrate
complexLimit
complexNormalize
complexNumeric
complexNumericIfCan
complexRoots
complexSolve
complexZeros
createLowComplexityNormalBasis
createLowComplexityTable
doubleComplex?
drawComplex
drawComplexVectorField
fortranComplex
fortranDoubleComplex
pmComplexintegrate
```

```
To get more information about an operation such as
complexExpand , issue the command )display op
complexExpand
```

If you want to see all domains with “matrix” in their names, issue this.

```
)what domain matrix
----- Domains -----

Domains with names matching patterns:
matrix

DHMATRIX DenavitHartenbergMatrix
DPMM      DirectProductMatrixModule
IMATRIX   IndexedMatrix          LSQM
LieSquareMatrix
M3D        ThreeDimensionalMatrix
MATCAT-    MatrixCategory&        MATRIX    Matrix
RMATCAT-   RectangularMatrixCategory&
RMATRIX    RectangularMatrix      SMATCAT-   SquareMatrix-
Category&
SQMATRIX   SquareMatrix
```

Similarly, if you wish to see all packages whose names contain “gauss”, enter this.

```
)what package gauss
----- Packages -----

Packages with names matching patterns:
gauss

GAUSSFAC GaussianFactorizationPackage
```

This command shows all the operations that Any provides. Wherever “\$” appears, it means “Any”.

```
)show Any
Any is a domain constructor
Abbreviation for Any is ANY
This constructor is exposed in this frame.
Issue
)edit /users/axiom/development/src/algebra/any.spad to
see algebra source code for ANY

----- Operations -----
?=? : (%,% ) -> Boolean          coerce : % ->
OutputForm
dom : % -> SExpression          domainOf : % ->
OutputForm
hash : % -> SingleInteger       latex : % -> String
obj : % -> None                 objectOf : % ->
OutputForm
?=? : (%,% ) -> Boolean
any : (SExpression,None) -> %
showTypeInOutput : Boolean -> String
```

This displays all operations with the name **complex**.

```
)display operation complex

There is one exposed function called complex :
[1] (D1,D1) -> D from D if D has COMPCAT D1 and D1
has COMRING
```

Let's analyze this output.

First we find out what some of the abbreviations mean.

```
)abbreviation query COMPCAT
COMPCAT abbreviates category ComplexCategory

)abbreviation query COMRING
COMRING abbreviates category CommutativeRing
```

So if `D1` is a commutative ring (such as the integers or floats) and `D` belongs to `ComplexCategory D1`, then there is an operation called **complex** that takes two elements of `D1` and creates an element of `D`. The primary example of a constructor implementing domains belonging to `ComplexCategory` is `Complex`. See ‘Complex’ on page 383 for more information on that and see Section 6.4 on page 183 for more information on function types.





---

## CHAPTER 3

---

# Using HyperDoc

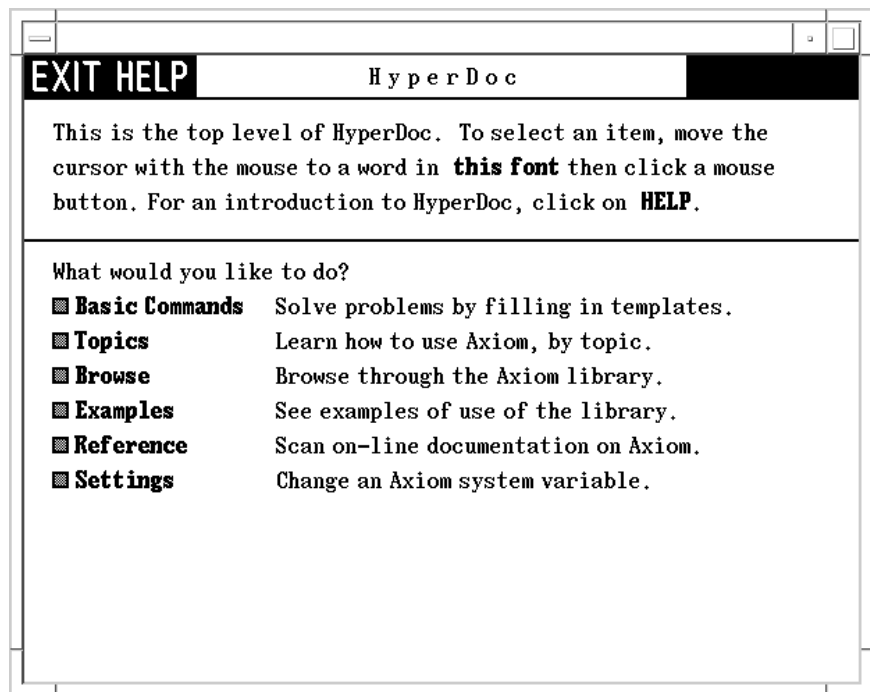


Figure 3.1: The HyperDoc root window page.

HyperDoc is the gateway to AXIOM. It's both an on-line tutorial and an on-line reference manual. It also enables you to use AXIOM simply by

using the mouse and filling in templates. HyperDoc is available to you if you are running AXIOM under the X Window System.


Pages usually have active areas, marked in **this font** (bold face). As you move the mouse pointer to an active area, the pointer changes from a filled dot to an open circle. The active areas are usually linked to other pages. When you click on an active area, you move to the linked page.

## 3.1 Headings

---

Most pages have a standard set of buttons at the top of the page. This is what they mean:

**HELP** Click on this to get help. The button only appears if there is specific help for the page you are viewing. You can get *general* help for HyperDoc by clicking the help button on the home page.

 Click here to go back one page. By clicking on this button repeatedly, you can go back several pages and then take off in a new direction.

**HOME** Go back to the home page, that is, the page on which you started. Use HyperDoc to explore, to make forays into new topics. Don't worry about how to get back. HyperDoc remembers where you came from. Just click on this button to return.

**EXIT** From the root window (the one that is displayed when you start the system) this button leaves the HyperDoc program, and it must be restarted if you want to use it again. From any other HyperDoc window, it just makes that one window go away. You *must* use this button to get rid of a window. If you use the window manager "Close" button, then all of HyperDoc goes away.

The buttons are not displayed if they are not applicable to the page you are viewing. For example, there is no **HOME** button on the top-level menu.

## 3.2 Key Definitions

---

The following keyboard definitions are in effect throughout HyperDoc. See Section 3.3 on page 133 and Section 3.4 on page 134 for some contextual key definitions.

**F1** Display the main help page.

**F3** Same as **EXIT**, makes the window go away if you are not at the top-level window or quits the HyperDoc facility if you are at the top-level.

**F5** Rereads the HyperDoc database, if necessary (for system developers).

**F9** Displays this information about key definitions.

**F12** Same as **F3**.

**Up Arrow** Scroll up one line.

**Down Arrow** Scroll down one line.

**Page Up** Scroll up one page.

**Page Down** Scroll down one page.

### 3.3 Scroll Bars

---

Whenever there is too much text to fit on a page, a *scroll bar* automatically appears along the right side.





With a scroll bar, your page becomes an aperture, that is, a window into a larger amount of text than can be displayed at one time. The scroll bar lets you move up and down in the text to see different parts. It also shows where the aperture is relative to the whole text. The aperture is indicated by a strip on the scroll bar.

Move the cursor with the mouse to the “down-arrow” at the bottom of the scroll bar and click. See that the aperture moves down one line. Do it several times. Each time you click, the aperture moves down one line. Move the mouse to the “up-arrow” at the top of the scroll bar and click. The aperture moves up one line each time you click.

Next move the mouse to any position along the middle of the scroll bar and click. HyperDoc attempts to move the top of the aperture to this point in the text.

You cannot make the aperture go off the bottom edge. When the aperture is about half the size of text, the lowest you can move the aperture is halfway down.

To move up or down one screen at a time, use the **PageUp** and **PageDown** keys on your keyboard. They move the visible part of the region up and down one page each time you press them.



If the HyperDoc page does not contain an input area (see Section 3.4 on page 134), you can also use the **Home** and  and  arrow keys to navigate. When you press the **Home** key, the screen is positioned at the very top of the page. Use the  and  arrow keys to move the screen up and down one line at a time, respectively.

## 3.4 Input Areas

---

Input areas are boxes where you can put data.

To enter characters, first move your mouse cursor to somewhere within the HyperDoc page. Characters that you type are inserted in front of the underscore. This means that when you type characters at your keyboard, they go into this first input area.

The input area grows to accommodate as many characters as you type. Use the **Backspace** key to erase characters to the left. To modify what you type, use the right-arrow  and left-arrow keys  and the keys **Insert**, **Delete**, **Home** and **End**. These keys are found immediately on the right of the standard IBM keyboard.

If you press the **Home** key, the cursor moves to the beginning of the line and if you press the **End** key, the cursor moves to the end of the line. Pressing **Ctrl-End** deletes all the text from the cursor to the end of the line.

A page may have more than one input area. Only one input area has an underscore cursor. When you first see a page, the top-most input area contains the cursor. To type information into another input area, use the **Enter** or **Tab** key to move from one input area to another. To move in the reverse order, use **Shift-Tab**.

You can also move from one input area to another using your mouse. Notice that each input area is active. Click on one of the areas. As you can see, the underscore cursor moves to that window.

## 3.5 Radio Buttons and Toggles

---

Some pages have *radio buttons* and *toggles*. Radio buttons are a group of buttons like those on car radios: you can select only one at a time. Once you have selected a button, it appears to be inverted and contains a checkmark. To change the selection, move the cursor with the mouse to a different radio button and click.

A toggle is an independent button that displays some on/off state. When “on”, the button appears to be inverted and contains a checkmark. When “off”, the button is raised. Unlike radio buttons, you can set a group of them any way you like. To change toggle the selection, move the cursor with the mouse to the button and click.

## 3.6 Search Strings

---

A *search string* is used for searching some database. To learn about search strings, we suggest that you bring up the HyperDoc glossary. To do this from the top-level page of HyperDoc:

1. Click on **Reference**, bringing up the AXIOM Reference page.
2. Click on **Glossary**, bringing up the glossary.

The glossary has an input area at its bottom. We review the various kinds of search strings you can enter to search the glossary.

The simplest search string is a word, for example, **operation**. A word only matches an entry having exactly that spelling. Enter the word **operation** into the input area above then click on **Search**. As you can see, **operation** matches only one entry, namely with **operation** itself.

Normally matching is insensitive to whether the alphabetic characters of your search string are in uppercase or lowercase. Thus **operation** and **OperAtion** both have the same effect.

You will very often want to use the wildcard “\*” in your search string so as to match multiple entries in the list. The search key “\*” matches every entry in the list. You can also use “\*” anywhere within a search string to match an arbitrary substring. Try **cat\*** for example: enter **cat\*** into the input area and click on **Search**. This matches several entries.

You use any number of wildcards in a search string as long as they are not adjacent. Try search strings such as **\*dom\***. As you see, this search string matches **domain**, **domain constructor**, **subdomain**, and so on.

### 3.6.1 Logical Searches

---

For more complicated searches, you can use “and”, “or”, and “not” with basic search strings; write logical expressions using these three operators just as in the AXIOM language. For example, **domain or package** matches the two entries **domain** and **package**. Similarly, **dom\*** and **\*con\*** matches **domain constructor** and others. Also **not \*a\*** matches every entry that does not contain the letter **a** somewhere.

Use parentheses for grouping. For example, **dom\* and (not \*con\*)** matches **domain** but not **domain constructor**.

There is no limit to how complex your logical expression can be. For example,

**a\* or b\* or c\* or d\* or e\* and (not \*a\*)**

is a valid expression.

## 3.7 Example Pages

---

Many pages have AXIOM example commands. Each command has an active “button” along the left margin. When you click on this button, the output for the command is “pasted-in.” Click again on the button and you see that the pasted-in output disappears.

Maybe you would like to run an example? To do so, just click on any part of its text! When you do, the example line is copied into a new interactive AXIOM buffer for this HyperDoc page.

Sometimes one example line cannot be run before you run an earlier one. Don’t worry—HyperDoc automatically runs all the necessary lines in the right order!

The new interactive AXIOM buffer disappears when you leave HyperDoc. If you want to get rid of it beforehand, use the **Cancel** button of the X Window manager or issue the AXIOM system command `)close`.

## 3.8 X Window Resources for HyperDoc

---

You can control the appearance of HyperDoc while running under Version 11 of the X Window System by placing the following resources in the file `.Xdefaults` in your home directory. In what follows, *font* is any valid X11 font name (for example, `Rom14`) and *color* is any valid X11 color specification (for example, `NavyBlue`). For more information about fonts and colors, refer to the X Window documentation for your system.

`Axiom.hyperdoc.RmFont: font`

This is the standard text font. The default value is `"Rom14"`.

`Axiom.hyperdoc.RmColor: color`

This is the standard text color. The default value is `"black"`.

`Axiom.hyperdoc.ActiveFont: font`

This is the font used for HyperDoc link buttons. The default value is `"Bld14"`.

`Axiom.hyperdoc.ActiveColor: color`

This is the color used for HyperDoc link buttons. The default value is `"black"`.

`Axiom.hyperdoc.AxiomFont: font`

This is the font used for active AXIOM commands.<sup>1</sup> The default value is `"Bld14"`.

`Axiom.hyperdoc.AxiomColor: color`

This is the color used for active AXIOM commands.<sup>2</sup> The default value is `"black"`.

`Axiom.hyperdoc.BoldFont: font`

---

<sup>1</sup>This was called `Axiom.hyperdoc.SpadFont` in early versions of AXIOM.

<sup>2</sup>This was called `Axiom.hyperdoc.SpadColor` in early versions of AXIOM.

This is the font used for bold face. The default value is "Bld14".

`Axiom.hyperdoc.BoldColor:` *color*  
This is the color used for bold face. The default value is "black".

`Axiom.hyperdoc.TtFont:` *font*  
This is the font used for AXIOM output in HyperDoc. This font must be fixed-width. The default value is "Rom14".

`Axiom.hyperdoc.TtColor:` *color*  
This is the color used for AXIOM output in HyperDoc. The default value is "black".

`Axiom.hyperdoc.EmphasizeFont:` *font*  
This is the font used for italics. The default value is "It14".

`Axiom.hyperdoc.EmphasizeColor:` *color*  
This is the color used for italics. The default value is "black".

`Axiom.hyperdoc.InputBackground:` *color*  
This is the color used as the background for input areas. The default value is "black".

`Axiom.hyperdoc.InputForeground:` *color*  
This is the color used as the foreground for input areas. The default value is "white".

`Axiom.hyperdoc.BorderColor:` *color*  
This is the color used for drawing border lines. The default value is "black".

`Axiom.hyperdoc.Background:` *color*  
This is the color used for the background of all windows. The default value is "white".





---

# Input Files and Output Styles

In this chapter we discuss how to collect AXIOM statements and commands into files and then read the contents into the workspace. We also show how to display the results of your computations in several different styles including T<sub>E</sub>X, FORTRAN and monospace two-dimensional format.<sup>1</sup>

The printed version of this book uses the AXIOM T<sub>E</sub>X output formatter. When we demonstrate a particular output style, we will need to turn T<sub>E</sub>X formatting off and the output style on so that the correct output is shown in the text.

## 4.1 Input Files

---

In this section we explain what an *input file* is and why you would want to know about it. We discuss where AXIOM looks for input files and how you can direct it to look elsewhere. We also show how to read the contents of an input file into the *workspace* and how to use the *history* facility to generate an input file from the statements you have entered directly into the workspace.

An *input* file contains AXIOM expressions and system commands. Anything that you can enter directly to AXIOM can be put into an input file.

---

<sup>1</sup>T<sub>E</sub>X is a trademark of the American Mathematical Society.

This is how you save input functions and expressions that you wish to read into AXIOM more than one time.

To read an input file into AXIOM, use the `)read` system command. For example, you can read a file in a particular directory by issuing

```
)read /spad/src/input/matrix.input
```

The “**.input**” is optional; this also works:

```
)read /spad/src/input/matrix
```

What happens if you just enter `)read matrix.input` or even `)read matrix`? AXIOM looks in your current working directory for input files that are not qualified by a directory name. Typically, this directory is the directory from which you invoked AXIOM. To change the current working directory, use the `)cd` system command. The command `)cd` by itself shows the current working directory. To change it to the `src/input` subdirectory for user “babar”, issue

```
)cd /u/babar/src/input
```

AXIOM looks first in this directory for an input file. If it is not found, it looks in the system’s directories, assuming you meant some input file that was provided with AXIOM.

If you have the AXIOM history facility turned on (which it is by default), you can save all the lines you have entered into the workspace by entering

```
)history )write
```

AXIOM tells you what input file to edit to see your statements. The file is in your home directory or in the directory you specified with `)cd`.

In Section 5.2 on page 153 we discuss using indentation in input files to group statements into *blocks*.

## 4.2 The **axiom.input** File

---

When AXIOM starts up, it tries to read the input file **axiom.input** from your home directory. If there is no **axiom.input** in your home directory, it reads the copy located in its own `src/input` directory. The file usually contains system commands to personalize your AXIOM environment. In the remainder of this section we mention a few things that users frequently place in their **axiom.input** files.

In order to have FORTRAN output always produced from your computations, place the system command `)set output fortran on` in **axiom.input**. If you do not want to be prompted for confirmation when you issue the

)quit system command, place `)set quit unprotected` in `axiom.input`. If you then decide that you do want to be prompted, issue `)set quit protected`. This is the default setting so that new users do not leave AXIOM inadvertently.<sup>2</sup>

To see the other system variables you can set, issue `)set` or use the HyperDoc **Settings** facility to view and change AXIOM system variables.

### 4.3 Common Features of Using Output Formats

---

In this section we discuss how to start and stop the display of the different output formats and how to send the output to the screen or to a file. To fix ideas, we use FORTRAN output format for most of the examples.

You can use the `)set output` system command to toggle or redirect the different kinds of output. The name of the kind of output follows “output” in the command. The names are

**fortran** for FORTRAN output.  
**algebra** for monospace two-dimensional mathematical output.  
**tex** for T<sub>E</sub>X output.  
**script** for IBM Script Formula Format output.

For example, issue `)set output fortran on` to turn on FORTRAN format and issue `)set output fortran off` to turn it off. By default, **algebra** is **on** and all others are **off**. When output is started, it is sent to the screen. To send the output to a file, give the file name without directory or extension. AXIOM appends a file extension depending on the kind of output being produced.

Issue this to redirect FORTRAN output to, for example, the file **linalg.sfort**.

```
)set output fortran linalg
```

FORTRAN output will be written to file `linalg.sfort` .

You must *also* turn on the creation of FORTRAN output. The above just says where it goes if it is created.

```
)set output fortran on
```

In what directory is this output placed? It goes into the directory from which you started AXIOM, or if you have used the `)cd` system command, the one that you specified with `)cd`. You should use `)cd` before you send the output to the file.

You can always direct output back to the screen by issuing this.

```
)set output fortran console
```

---

<sup>2</sup>The system command `)pquit` always prompts you for confirmation.

Let's make sure FORTRAN formatting is off so that nothing we do from now on produces FORTRAN output.

We also delete the demonstrated output file we created.

```
)set output fortran off
```

```
)system rm linalg.sfort
```

You can abbreviate the words “on,” “off” and “console” to the minimal number of characters needed to distinguish them. Because of this, you cannot send output to files called **on.sfort**, **off.sfort**, **of.sfort**, **console.sfort**, **consol.sfort** and so on.

The width of the output on the page is set by `)set output length` for all formats except FORTRAN. Use `)set fortran fortlength` to change the FORTRAN line length from its default value of 72.

## 4.4 Monospace Two-Dimensional Mathematical Format

This is the default output format for AXIOM. It is usually on when you start the system.

If it is not, issue this.

```
)set output algebra on
```

Since the printed version of this book (as opposed to the HyperDoc version) shows output produced by the T<sub>E</sub>X output formatter, let us temporarily turn off T<sub>E</sub>X output.

```
)set output tex off
```

Here is an example of what it looks like.

```
matrix [[i*x**i + j%i*y**j for i in 1..2] for j in 3..4]
```

$$(1) \begin{array}{c} \begin{array}{cc} + & 3 & 3 & 2+ \\ | & 3\%i & y & + x & 3\%i & y & + 2x & | \\ | & & 4 & & 4 & & 2 & | \\ +4\%i & y & + x & 4\%i & y & + 2x & + \end{array} \end{array} \quad (0)$$

Type: Matrix Polynomial Complex Integer

Issue this to turn off this kind of formatting.

```
)set output algebra off
```

Turn T<sub>E</sub>X output on again.

```
)set output tex on
```

The characters used for the matrix brackets above are rather ugly. You get this character set when you issue `)set output characters plain`. This character set should be used when you are running on a machine that does not support the IBM extended ASCII character set. If you are running on an IBM workstation, for example, issue `)set output`

characters default to get better looking output.

## 4.5 TeX Format

---

To turn on T<sub>E</sub>X output formatting, issue this.

```
)set output tex on
```

Here is an example of its output.

```
matrix [[i*x**i + j*%i*y**j for i in 1..2] for j in 3..4]

\[
\left[
\begin{array}{cc}
\displaystyle
\{{3 \ i \ {y \sp 3}}+ x\}&
\displaystyle
\{{3 \ i \ {y \sp 3}}+{2 \ {x \sp 2}}\} \\
\displaystyle
\{{4 \ i \ {y \sp 4}}+ x\}&
\displaystyle
\{{4 \ i \ {y \sp 4}}+{2 \ {x \sp 2}}\}
\end{array}
\right] \leqno (3)
\]
```

This formats as

$$\begin{bmatrix} 3 i y^3 + x & 3 i y^3 + 2 x^2 \\ 4 i y^4 + x & 4 i y^4 + 2 x^2 \end{bmatrix}$$

To turn T<sub>E</sub>X output formatting off, issue `)set output tex off`. The L<sup>A</sup>T<sub>E</sub>X macros in the output generated by AXIOM are all standard except for the following definitions:

```
\def\csch{\mathop{\rm csch}\nolimits}

\def\erf{\mathop{\rm erf}\nolimits}

\def\zag#1#2{
  {\hfil\left. {#1} \right|}
  \over
  {\left| {#2} \right. \hfil}
}
```

---

<sup>3</sup>See Leslie Lamport, *LaTeX: A Document Preparation System*, Reading, Massachusetts: Addison-Wesley Publishing Company, Inc., 1986.

## 4.6 IBM Script Formula Format

To turn IBM Script Formula Format on, issue this.

AXIOM can produce IBM Script Formula Format output for your expressions.

```
)set output script on
```

Here is an example of its output.

```
matrix [[i*x**i + j**i*y**j for i in 1..2] for j in 3..4]

.eq set blank @
:df.
<left 1b <<<<3 @@ %i @@ <y sup 3>>+x> here <<3 @@ %i @@
<y sup 3>>+<2 @@ <x sup 2>>>> habove <<<4 @@ %i @@
<y sup 4>>+x> here <<4 @@ %i @@ <y sup 4>>+<2 @@
<x up 2>>>>> right rb>
:edf.
```

To turn IBM Script Formula Format output formatting off, issue this.

```
)set output script off
```

## 4.7 FORTRAN Format

In addition to turning FORTRAN output on and off and stating where the output should be placed, there are many options that control the appearance of the generated code. In this section we describe some of the basic options. Issue `)set fortran` to see a full list with their current settings.

The output FORTRAN expression usually begins in column 7. If the expression needs more than one line, the ampersand character “&” is used in column 6. Since some versions of FORTRAN have restrictions on the number of lines per statement, AXIOM breaks long expressions into segments with a maximum of 1320 characters (20 lines of 66 characters) per segment. If you want to change this, say, to 660 characters, issue the system command `)set fortran explength 660`. You can turn off the line breaking by issuing `)set fortran segment off`. Various code optimization levels are available.

FORTRAN output is produced after you issue this.

```
)set output fortran on
```

For the initial examples, we set the optimization level to 0, which is the lowest level.

```
)set fortran optlevel 0
```

The output is usually in columns 7 through 72, although fewer columns are used in the following examples so that the output fits nicely on the page.

```
)set fortran fortlength 60
```

By default, the output goes to the screen and is displayed before the standard AXIOM two-dimensional output. In this example, an assignment to the variable R1 was generated because this is the result of step 1.

Here is an example that illustrates the line breaking.

$$\begin{aligned} & (x+y)^{**3} \\ & R1=y^{**3}+3*x*y*y+3*x*x*y+x^{**3} \\ & y^3 + 3 x y^2 + 3 x^2 y + x^3 \end{aligned} \tag{1}$$

Type: Polynomial Integer

$$\begin{aligned} & (x+y+z)^{**3} \\ & R2=z^{**3}+(3*y+3*x)*z*z+(3*y*y+6*x*y+3*x*x)*z+y^{**3}+3*x*y \\ & \quad \&*y+3*x*x*y+x^{**3} \\ & z^3 + (3 y + 3 x) z^2 + (3 y^2 + 6 x y + 3 x^2) z + y^3 + \\ & 3 x y^2 + 3 x^2 y + x^3 \end{aligned} \tag{2}$$

Type: Polynomial Integer

Note in the above examples that integers are generally converted to floating point numbers, except in exponents. This is the default behavior but can be turned off by issuing `)set fortran ints2floats off`. The rules governing when the conversion is done are:

1. If an integer is an exponent, convert it to a floating point number if it is greater than 32767 in absolute value, otherwise leave it as an integer.
2. Convert all other integers in an expression to floating point numbers.

These rules only govern integers in expressions. Numbers generated by AXIOM for DIMENSION statements are also integers.

To set the type of generated FORTRAN data, use one of the following:

```
)set fortran defaulttype REAL
)set fortran defaulttype INTEGER
)set fortran defaulttype COMPLEX
)set fortran defaulttype LOGICAL
)set fortran defaulttype CHARACTER
```

When temporaries are created, they are given a default type of REAL. Also, the REAL versions of functions are used by default.

$$\begin{aligned} & \sin(x) \\ & R3=DSIN(x) \\ & \sin(x) \end{aligned} \tag{3}$$

Type: Expression Integer

At optimization level 1, AXIOM removes common subexpressions.

```
)set fortran optlevel 1
```

	<pre> (x+y+z)**3 T2=y*y T3=x*x R4=z**3+(3*y+3*x)*z*z+(3*T2+6*x*y+3*T3)*z+y**3+3*x*T2+ &amp;3*T3*y+x**3 </pre> $z^3 + (3y + 3x)z^2 + (3y^2 + 6xy + 3x^2)z + y^3 + 3xy^2 + 3x^2y + x^3$ <div style="text-align: right;">(4)</div>
	Type: Polynomial Integer
This changes the precision to DOUBLE. Substitute <b>single</b> for <b>double</b> to return to single precision.	<pre> )set fortran precision double </pre>
Complex constants display the precision.	<pre> 2.3 + 5.6**%i R5=(2.3D0,5.6D0) </pre> $2.3 + 5.6i$ <div style="text-align: right;">(5)</div>
	Type: Complex Float
The function names that AXIOM generates depend on the chosen precision.	<pre> sin %e R6=DSIN(DEXP(1)) </pre> $\sin(e)$ <div style="text-align: right;">(6)</div>
	Type: Expression Integer
Reset the precision to <b>single</b> and look at these two examples again.	<pre> )set fortran precision single </pre>
	<pre> 2.3 + 5.6**%i R7=(2.3,5.6) </pre> $2.3 + 5.6i$ <div style="text-align: right;">(7)</div>
	Type: Complex Float
	<pre> sin %e R8=SIN(EXP(1)) </pre> $\sin(e)$ <div style="text-align: right;">(8)</div>
	Type: Expression Integer
Expressions that look like lists, streams, sets or matrices cause array code to be generated.	<pre> [x+1,y+1,z+1] T1(1)=x+1 T1(2)=y+1 T1(3)=z+1 R9=T1 </pre> $[x + 1, y + 1, z + 1]$ <div style="text-align: right;">(9)</div>
	Type: List Polynomial Integer



A temporary variable is generated to be the name of the array. This may have to be changed in your particular application.

```
set [2,3,4,3,5]
T1(1)=2
T1(2)=3
T1(3)=4
T1(4)=5
R10=T1
```

{2, 3, 4, 5}

(10)

Type: Set PositiveInteger

By default, the starting index for generated FORTRAN arrays is 0.

```
matrix [[2.3,9.7],[0.0,18.778]]
T1(1,1)=2.3
T1(1,2)=9.7
T1(2,1)=0.0
T1(2,2)=18.778
T1
```

$$\begin{bmatrix} 2.3 & 9.7 \\ 0.0 & 18.778 \end{bmatrix}$$

(11)

Type: Matrix Float

To change the starting index for generated FORTRAN arrays to be 1, issue this. This value can only be 0 or 1.

```
)set fortran startindex 1
```

Look at the code generated for the matrix again.

```
matrix [[2.3,9.7],[0.0,18.778]]
T1(1,1)=2.3
T1(1,2)=9.7
T1(2,1)=0.0
T1(2,2)=18.778
T1
```

$$\begin{bmatrix} 2.3 & 9.7 \\ 0.0 & 18.778 \end{bmatrix}$$

(12)

Type: Matrix Float



# Introduction to the AXIOM Interactive Language

In this chapter we look at some of the basic components of the AXIOM language that you can use interactively. We show how to create a *block* of expressions, how to form loops and list iterations, how to modify the sequential evaluation of a block and how to use `if-then-else` to evaluate parts of your program conditionally. We suggest you first read the boxed material in each section and then proceed to a more thorough reading of the chapter.

## 5.1 Immediate and Delayed Assignments

A *variable* in AXIOM refers to a value. A variable has a name beginning with an uppercase or lowercase alphabetic character, “%”, or “!”. Successive characters (if any) can be any of the above, digits, or “?”. Case is distinguished. The following are all examples of valid, distinct variable names:

a	tooBig?	a1B2c3%!?
A	%j	numberOfPoints
beta6	%J	numberofpoints

The “:=” operator is the immediate *assignment* operator. Use it to associate a value with a variable.

The syntax for immediate assignment for a single variable is

*variable := expression*

The value returned by an immediate assignment is the value of *expression*.

The right-hand side of the expression is evaluated, yielding 1. This value is then assigned to **a**.

**a := 1**  
1

(1)

Type: PositiveInteger

The right-hand side of the expression is evaluated, yielding 1. This value is then assigned to **b**. Thus **a** and **b** both have the value 1 after the sequence of assignments.

**b := a**  
1

(2)

Type: PositiveInteger

What is the value of **b** if **a** is assigned the value 2?

**a := 2**  
2

(3)

Type: PositiveInteger

As you see, the value of **b** is left unchanged.

**b**  
1

(4)

Type: PositiveInteger

This is what we mean when we say this kind of assignment is *immediate*; **b** has no dependency on **a** after the initial assignment. This is the usual notion of assignment found in programming languages such as C, PASCAL and FORTRAN.

AXIOM provides delayed assignment with “==”. This implements a delayed evaluation of the right-hand side and dependency checking.

The syntax for delayed assignment is

*variable == expression*

The value returned by a delayed assignment is the unique value of Void.

Using **a** and **b** as above, these are the corresponding delayed assignments.

**a == 1**

Type: Void

**b == a**

Type: Void

The right-hand side of each delayed assignment is left unevaluated until the variables on the left-hand sides are evaluated. Therefore this evaluation and ...

**a**

Compiling body of rule a to compute value of type  
PositiveInteger

1

(7)

Type: PositiveInteger

this evaluation seem the same as before.

**b**

Compiling body of rule b to compute value of type  
PositiveInteger

1

(8)

Type: PositiveInteger

If we change **a** to 2

**a == 2**

Compiled code for a has been cleared.  
Compiled code for b has been cleared.  
1 old definition(s) deleted for function or rule a

Type: Void

then **a** evaluates to 2, as expected, but

**a**

Compiling body of rule a to compute value of type  
PositiveInteger

+++ |\*0;a;1;initial| redefined

2

(10)

Type: PositiveInteger

the value of **b** reflects the change to **a**.

**b**  
Compiling body of rule **b** to compute value of type  
PositiveInteger

+++ |\*0;b;1;initial| redefined

2

(11)

Type: PositiveInteger

It is possible to set several variables at the same time by using a *tuple* of variables and a tuple of expressions.<sup>1</sup>

The syntax for multiple immediate assignments is

$$(var_1, var_2, \dots, var_N) := (expr_1, expr_2, \dots, expr_N)$$

The value returned by an immediate assignment is the value of  $expr_N$ .

This sets **x** to 1 and **y** to 2.

(**x**,**y**) := (1,2)

2

(12)

Type: PositiveInteger

Multiple immediate assignments are parallel in the sense that the expressions on the right are all evaluated before any assignments on the left are made. However, the order of evaluation of these expressions is undefined.

You can use multiple immediate assignment to swap the values held by variables.

(**x**,**y**) := (**y**,**x**)

1

(13)

Type: PositiveInteger

**x** has the previous value of **y**.

**x**

2

(14)

Type: PositiveInteger

**y** has the previous value of **x**.

**y**

1

(15)

Type: PositiveInteger

There is no syntactic form for multiple delayed assignments. See the discussion in Section 6.8 on page 190 about how AXIOM differentiates between delayed assignments and user functions of no arguments.

---

<sup>1</sup>A *tuple* is a collection of things separated by commas, often surrounded by parentheses.

## 5.2 Blocks

A *block* is a sequence of expressions evaluated in the order that they appear, except as modified by control expressions such as **break**, **return**, **iterate** and **if-then-else** constructions. The value of a block is the value of the expression last evaluated in the block.

To leave a block early, use “=>”. For example, `i < 0 => x`. The expression before the “=>” must evaluate to **true** or **false**. The expression following the “=>” is the return value for the block.

A block can be constructed in two ways:

1. the expressions can be separated by semicolons and the resulting expression surrounded by parentheses, and
2. the expressions can be written on succeeding lines with each line indented the same number of spaces (which must be greater than zero). A block entered in this form is called a *pile*.

Only the first form is available if you are entering expressions directly to AXIOM. Both forms are available in **.input** files.

The syntax for a simple block of expressions entered interactively is

$$( \textit{expression}_1; \textit{expression}_2; \dots; \textit{expression}_N )$$

The value returned by a block is the value of an “=>” expression, or  $\textit{expression}_N$  if no “=>” is encountered.

In **.input** files, blocks can also be written using *piles*. The examples throughout this book are assumed to come from **.input** files.

In this example, we assign a rational number to **a** using a block consisting of three expressions. This block is written as a pile. Each expression in the pile has the same indentation, in this case two spaces to the right of the first line.

```
a :=
  i := gcd(234,672)
  i := 3*i**5 - i + 1
  1 / i
23323
```

(1)

Type: Fraction Integer

Here is the same block written on one line. This is how you are required to enter it at the input prompt.

```
a := (i := gcd(234,672); i := 3*i**5 - i + 1; 1 / i)
23323
```

(2)

Type: Fraction Integer

Blocks can be used to put several expressions on one line. The value returned is that of the last expression.

```
(a := 1; b := 2; c := 3; [a,b,c])
```

[1, 2, 3] (3)

Type: List PositiveInteger

AXIOM gives you two ways of writing a block and the preferred way in an **.input** file is to use a pile. Roughly speaking, a pile is a block whose constituent expressions are indented the same amount. You begin a pile by starting a new line for the first expression, indenting it to the right of the previous line. You then enter the second expression on a new line, vertically aligning it with the first line. And so on. If you need to enter an inner pile, further indent its lines to the right of the outer pile. AXIOM knows where a pile ends. It ends when a subsequent line is indented to the left of the pile or the end of the file.

Blocks can be used to perform several steps before an assignment (immediate or delayed) is made.

```
d :=
  c := a**2 + b**2
  sqrt(c * 1.3)
```

2.549509756796392415 (4)

Type: Float

Blocks can be used in the arguments to functions. (Here **h** is assigned **2.1 + 3.5**.)

```
h := 2.1 +
  1.0
  3.5
```

5.6 (5)

Type: Float

Here the second argument to **eval** is **x = z**, where the value of **z** is computed in the first line of the block starting on the second line.

```
eval(x**2 - x*y**2,
  z := %pi/2.0 - exp(4.1)
  x = z
)
```

58.769491270567072878  $y^2 + 3453.853104201259382$  (6)

Type: Polynomial Float

Blocks can be used in the clauses of **if-then-else** expressions (see Section 5.3 on page 156).

```
if h > 3.1 then 1.0 else (z := cos(h); max(z,0.5))
```

1.0 (7)

Type: Float

This is the pile version of the last block.

```
if h > 3.1 then
  1.0
else
  z := cos(h)
  max(z,0.5)
```

1.0 (8)

Type: Float



Blocks can be nested.

```
a := (b := factorial(12); c := (d := eulerPhi(22);  
    factorial(d));b+c)
```

482630400

(9)

Type: PositiveInteger

This is the pile version of the last block.

```
a :=  
  b := factorial(12)  
  c :=  
    d := eulerPhi(22)  
    factorial(d)  
  b+c
```

482630400

(10)

Type: PositiveInteger

Since `c + d` does equal 3628855, `a` has the value of `c` and the last line is never evaluated.

```
a :=  
  c := factorial 10  
  d := fibonacci 10  
  c + d = 3628855 => c  
  d
```

3628800

(11)

Type: PositiveInteger

## 5.3 if-then-else

Like many other programming languages, AXIOM uses the three keywords **if**, **then** and **else** to form conditional expressions. The **else** part of the conditional is optional. The expression between the **if** and **then** keywords is a *predicate*: an expression that evaluates to or is convertible to either **true** or **false**, that is, a Boolean.

The syntax for conditional expressions is

**if** *predicate* **then** *expression*<sub>1</sub> **else** *expression*<sub>2</sub>

where the **else** *expression*<sub>2</sub> part is optional. The value returned from a conditional expression is *expression*<sub>1</sub> if the predicate evaluates to **true** and *expression*<sub>2</sub> otherwise. If no **else** clause is given, the value is always the unique value of Void.

An **if-then-else** expression always returns a value. If the **else** clause is missing then the entire expression returns the unique value of Void. If both clauses are present, the type of the value returned by **if** is obtained by resolving the types of the values of the two clauses. See Section 2.10 on page 122 for more information.

The predicate must evaluate to, or be convertible to, an object of type Boolean: **true** or **false**. By default, the equal sign “=” creates an equation.

This is an equation. In particular, it is an object of type Equation Polynomial Integer.

$x + 1 = y$

$x + 1 = y$

(1)

Type: Equation Polynomial Integer

However, for predicates in **if** expressions, AXIOM places a default target type of Boolean on the predicate and equality testing is performed. Thus you need not qualify the “=” in any way. In other contexts you may need to tell AXIOM that you want to test for equality rather than create an equation. In those cases, use “@” and a target type of Boolean. See Section 2.9 on page 119 for more information.

The compound symbol meaning “not equal” in AXIOM is “~=". This can be used directly without a package call or a target specification. The expression **a ~ = b** is directly translated into **not (a = b)**.

Many other functions have return values of type Boolean. These include **<**, **<=**, **>**, **>=**, **~ =** and **member?**. By convention, operations with names ending in “?” return Boolean values.

The usual rules for piles are suspended for conditional expressions. In

**.input** files, the **then** and **else** keywords can begin in the same column as the corresponding **if** but may also appear to the right. Each of the following styles of writing **if-then-else** expressions is acceptable:

```
if i>0 then output("positive") else output("nonpositive")

if i > 0 then output("positive")
  else output("nonpositive")

if i > 0 then output("positive")
else output("nonpositive")

if i > 0
then output("positive")
else output("nonpositive")

if i > 0
  then output("positive")
  else output("nonpositive")
```

A block can follow the **then** or **else** keywords. In the following two assignments to **a**, the **then** and **else** clauses each are followed by two-line piles. The value returned in each is the value of the second line.

```
a :=
  if i > 0 then
    j := sin(i * pi())
    exp(j + 1/j)
  else
    j := cos(i * 0.5 * pi())
    log(abs(j)**5 + 1)

a :=
  if i > 0
  then
    j := sin(i * pi())
    exp(j + 1/j)
  else
    j := cos(i * 0.5 * pi())
    log(abs(j)**5 + 1)
```

These are both equivalent to the following:

```
a :=
  if i > 0 then (j := sin(i * pi()); exp(j + 1/j))
  else (j := cos(i * 0.5 * pi()); log(abs(j)**5 + 1))
```

## 5.4 Loops

A *loop* is an expression that contains another expression, called the *loop body*, which is to be evaluated zero or more times. All loops contain the **repeat** keyword and return the unique value of `Void`. Loops can contain inner loops to any depth.

The most basic loop is of the form

**repeat** *loopBody*

Unless *loopBody* contains a **break** or **return** expression, the loop repeats forever. The value returned by the loop is the unique value of `Void`.

### 5.4.1 Compiling vs. Interpreting Loops

AXIOM tries to determine completely the type of every object in a loop and then to translate the loop body to LISP or even to machine code. This translation is called *compilation*.

If AXIOM decides that it cannot compile the loop, it issues a message stating the problem and then the following message:

**We will attempt to step through and interpret the code.**

It is still possible that AXIOM can evaluate the loop but in *interpret-code mode*. See Section 6.10 on page 193 where this is discussed in terms of compiling versus interpreting functions.

### 5.4.2 return in Loops

A **return** expression is used to exit a function with a particular value. In particular, if a **return** is in a loop within the function, the loop is terminated whenever the **return** is evaluated.

Suppose we start with this.

```
f() ==  
  i := 1  
  repeat  
    if factorial(i) > 1000 then return i  
    i := i + 1
```

Type: Void

When `factorial(i)` is big enough, control passes from inside the loop all the way outside the function, returning the value of `i` (or so we think).

`f()`

Compiling function `f` with type `() -> Void`

Type: Void

What went wrong? Isn't it obvious that this function should return an integer? Well, AXIOM makes no attempt to analyze the structure of a

loop to determine if it always returns a value because, in general, this is impossible. So AXIOM has this simple rule: the type of the function is determined by the type of its body, in this case a block. The normal value of a block is the value of its last expression, in this case, a loop. And the value of every loop is the unique value of Void! So the return type of **f** is Void.

There are two ways to fix this. The best way is for you to tell AXIOM what the return type of **f** is. You do this by giving **f** a declaration **f: () -> Integer** prior to calling for its value. This tells AXIOM: “trust me—an integer is returned.” We’ll explain more about this in the next chapter. Another clumsy way is to add a dummy expression as follows.

Since we want an integer, let’s stick in a dummy final expression that is an integer and will never be evaluated.

```
f() ==
  i := 1
  repeat
    if factorial(i) > 1000 then return i
    i := i + 1
  0
```

```
Compiled code for f has been cleared.
1 old definition(s) deleted for function or rule f
```

Type: Void

When we try **f** again we get what we wanted. See Section 6.15 on page 210 for more information.

```
f()
Compiling function f with type () ->
  NonNegativeInteger
```

```
+++ |*0;f;1;initial| redefined
```

```
7
```

(4)

Type: PositiveInteger

### 5.4.3 break in Loops

The **break** keyword is often more useful in terminating a loop. A **break** causes control to transfer to the expression immediately following the loop. As loops always return the unique value of Void, you cannot return a value with **break**. That is, **break** takes no argument.

This example is a modification of the last example in the previous section. Instead of using **return**, we’ll use **break**.

```
f() ==
  i := 1
  repeat
    if factorial(i) > 1000 then break
    i := i + 1
  i
```

Type: Void

The loop terminates when `factorial(i)` gets big enough, the last line of the function evaluates to the corresponding “good” value of `i`, and the function terminates, returning that value.

```
f()
Compiling function f with type () -> PositiveInteger
+++ |*0;f;1;initial| redefined
7
(2)
Type: PositiveInteger
```

You can only use `break` to terminate the evaluation of one loop. Let’s consider a loop within a loop, that is, a loop with a nested loop. First, we initialize two counter variables.

```
(i,j) := (1, 1)
1
(3)
Type: PositiveInteger
```

Nested loops must have multiple `break` expressions at the appropriate nesting level. How would you rewrite this so `(i + j) > 10` is only evaluated once?

```
repeat
  repeat
    if (i + j) > 10 then break
    j := j + 1
  if (i + j) > 10 then break
  i := i + 1
Type: Void
```

#### 5.4.4 break vs. => in Loop Bodies

Compare the following two loops:

<pre>i := 1 repeat   i := i + 1   i &gt; 3 =&gt; i   output(i)</pre>	<pre>i := 1 repeat   i := i + 1   if i &gt; 3 then break   output(i)</pre>
--	--

In the example on the left, the values 2 and 3 for `i` are displayed but then the “=>” does not allow control to reach the call to **output** again. The loop will not terminate until you run out of space or interrupt the execution. The variable `i` will continue to be incremented because the “=>” only means to leave the *block*, not the loop.

In the example on the right, upon reaching 4, the **break** will be executed, and both the block and the loop will terminate. This is one of the reasons why both “=>” and **break** are provided. Using a **while** clause (see below) with the “=>” lets you simulate the action of **break**.

#### 5.4.5 More Examples of break

First, initialize `i` as the loop counter.

```
Here we give four examples of repeat loops that terminate when a value exceeds a given bound.
i := 0
0
(1)
Type: NonNegativeInteger
```

```
repeat
  i := i + 1
  if i**2 > 100 then break
```

(3)

i  
11

Type: NonNegativeInteger

$$\begin{array}{l} i := 0 \\ 0 \end{array}$$

(4)

Type: NonNegativeInteger

```
repeat
  i := i + 1
  i**2 > 100 => break
```

Type: Void

(6)

Type: NonNegativeInteger

```
(n, i, f) := (100, 1, 1)
```

(7)

Type: `PositiveInteger`

```
repeat
  if i > n then break
  f := f * i
  i := i + 1
```

Type: Void

f

(9)

Type: `PositiveInteger`

```
m := matrix [[21,37,53,14], [8,-24,22,-16], [2,10,15,14],
[26,33,55,-13]]
```

(10)

Type: Matrix Integer

Next, set row counter `r` and column counter `c` to 1. Note: if we were writing a function, these would all be local variables rather than global workspace variables.

```
(r, c) := (1, 1)
```

(11)  
Type: PositiveInteger

Also, let `lastrow` and `lastcol` be the final row and column index.

```
(lastrow, lastcol) := (nrows(m), ncols(m))
```

(12)  
Type: PositiveInteger

Scan the rows looking for the first negative element. We remark that you can reformulate this example in a better, more concise form by using a `for` clause with `repeat`. See Section 5.4.8 on page 164 for more information.

```
repeat
  if r > lastrow then break
  c := 1
  repeat
    if c > lastcol then break
    if elt(m,r,c) < 0 then
      output [r, c, elt(m,r,c)]
      r := lastrow
      break -- don't look any further
    c := c + 1
  r := r + 1
[2,2,- 24]
```

Type: Void

### 5.4.6 iterate in Loops

AXIOM provides an `iterate` expression that skips over the remainder of a loop body and starts the next loop iteration.

We first initialize a counter.

```
i := 0
```

(1)  
Type: NonNegativeInteger

Display the even integers from 2 to 5.

```
repeat
  i := i + 1
  if i > 5 then break
  if odd?(i) then iterate
  output(i)
```

Type: Void

### 5.4.7 while Loops

The `repeat` in a loop can be modified by adding one or more `while` clauses. Each clause contains a *predicate* immediately following the `while` keyword. The predicate is tested *before* the evaluation of the body of the loop. The loop body is evaluated whenever the predicates in a `while` clause are all `true`.



The syntax for a simple loop using **while** is

**while** *predicate* **repeat** *loopBody*

The *predicate* is evaluated before *loopBody* is evaluated. A **while** loop terminates immediately when *predicate* evaluates to **false** or when a **break** or **return** expression is evaluated in *loopBody*. The value returned by the loop is the unique value of Void.

Here is a simple example of using **while** in a loop. We first initialize the counter.

```
i := 1
1
```

(1)

Type: PositiveInteger

The steps involved in computing this example are (1) set *i* to 1, (2) test the condition *i* < 1 and determine that it is not true, and (3) do not evaluate the loop body and therefore do not display "hello".

```
while i < 1 repeat
  output "hello"
  i := i + 1
```

Type: Void

If you have multiple predicates to be tested use the logical **and** operation to separate them. AXIOM evaluates these predicates from left to right.

```
(x, y) := (1, 1)
1
```

(3)

Type: PositiveInteger

```
while x < 4 and y < 10 repeat
  output [x,y]
  x := x + 1
  y := y + 2
```

```
[1,1]
[2,3]
[3,5]
```

Type: Void

A **break** expression can be included in a loop body to terminate a loop even if the predicate in any **while** clauses are not **false**.

```
(x, y) := (1, 1)
1
```

(5)

Type: PositiveInteger

This loop has multiple **while** clauses and the loop terminates before any one of their conditions evaluates to **false**.

```
while x < 4 while y < 10 repeat
  if x + y > 7 then break
  output [x,y]
  x := x + 1
  y := y + 2
```

```
[1,1]
[2,3]
```

Type: Void

Here's a different version of the nested loops that looked for the first negative element in a matrix.

```
m := matrix [[21,37,53,14], [8,-24,22,-16], [2,10,15,14],
             [26,33,55,-13]]
```

$$\begin{bmatrix} 21 & 37 & 53 & 14 \\ 8 & -24 & 22 & -16 \\ 2 & 10 & 15 & 14 \\ 26 & 33 & 55 & -13 \end{bmatrix}$$
(7)

Type: Matrix Integer

Initialized the row index to 1 and get the number of rows and columns. If we were writing a function, these would all be local variables.

```
r := 1
```

```
1
```

(8)

Type: PositiveInteger

```
(lastrow, lastcol) := (nrows(m), ncols(m))
```

```
4
```

(9)

Type: PositiveInteger

Scan the rows looking for the first negative element.

```
while r <= lastrow repeat
  c := 1 -- index of first column
  while c <= lastcol repeat
    if elt(m,r,c) < 0 then
      output [r, c, elt(m,r,c)]
      r := lastrow
      break -- don't look any further
    c := c + 1
  r := r + 1
```

```
[2,2,- 24]
```

Type: Void

### 5.4.8 for Loops

---

AXIOM provides the **for** and **in** keywords in **repeat** loops, allowing you to iterate across all elements of a list, or to have a variable take on integral values from a lower bound to an upper bound. We shall refer to these modifying clauses of **repeat** loops as **for** clauses. These clauses can be present in addition to **while** clauses. As with all other types of **repeat** loops, **break** can be used to prematurely terminate the evaluation of the loop.

The syntax for a simple loop using **for** is

**for** *iterator* **repeat** *loopBody*

The *iterator* has several forms. Each form has an end test which is evaluated before *loopBody* is evaluated. A **for** loop terminates immediately when the end test succeeds (evaluates to **true**) or when a **break** or **return** expression is evaluated in *loopBody*. The value returned by the loop is the unique value of Void.

### 5.4.9 for i in n..m repeat

If `for` is followed by a variable name, the `in` keyword and then an integer segment of the form `n..m`, the end test for this loop is the predicate `i > m`. The body of the loop is evaluated `m-n+1` times if this number is greater than 0. If this number is less than or equal to 0, the loop body is not evaluated at all.

The variable `i` has the value `n`, `n+1`, ..., `m` for successive iterations of the loop body. The loop variable is a *local variable* within the loop body: its value is not available outside the loop body and its value and type within the loop body completely mask any outer definition of a variable with the same name.

This loop prints the values of  $10^3$ ,  $11^3$ , and  $12^3$ :

```
for i in 10..12 repeat output(i**3)
1000
1331
1728
```

Type: Void

Here is a sample list.

```
a := [1,2,3]
[1, 2, 3]
```

(2)

Type: List PositiveInteger

Iterate across this list, using “.” to access the elements of a list and the `#` operation to count its elements.

```
for i in 1..#a repeat output(a.i)
1
2
3
```

Type: Void

This type of iteration is applicable to anything that uses “.”. You can also use it with functions that use indices to extract elements.

Define `m` to be a matrix.

```
m := matrix [[1,2],[4,3],[9,0]]
[ 1  2 ]
[ 4  3 ]
[ 9  0 ]
```

(4)

Type: Matrix Integer

Display the rows of `m`.

```
for i in 1..nrows(m) repeat output row(m,i)
[1,2]
[4,3]
[9,0]
```

Type: Void

You can use `iterate` with `for`-loops.

Display the even integers in a segment.

```
for i in 1..5 repeat
  if odd?(i) then iterate
  output(i)

2
4
```

Type: Void

See ‘Segment’ on page 559 for more information about segments.

#### 5.4.10 for i in n..m by s repeat

---

By default, the difference between values taken on by a variable in loops such as `for i in n..m repeat ...` is 1. It is possible to supply another, possibly negative, step value by using the `by` keyword along with `for` and `in`. Like the upper and lower bounds, the step value following the `by` keyword must be an integer. Note that the loop `for i in 1..2 by 0 repeat output(i)` will not terminate by itself, as the step value does not change the index from its initial value of 1.

This expression displays the odd integers between two bounds.

```
for i in 1..5 by 2 repeat output(i)

1
3
5
```

Type: Void

Use this to display the numbers in reverse order.

```
for i in 5..1 by -2 repeat output(i)

5
3
1
```

Type: Void

#### 5.4.11 for i in n.. repeat

---

If the value after the “.” is omitted, the loop has no end test. A potentially infinite loop is thus created. The variable is given the successive values `n`, `n+1`, `n+2`, ... and the loop is terminated only if a `break` or `return` expression is evaluated in the loop body. However you may also add some other modifying clause on the `repeat` (for example, a `while` clause) to stop the loop.

This loop displays the integers greater than or equal to 15 and less than the first prime greater than 15.

```
for i in 15.. while not prime?(i) repeat output(i)

15
16
```

Type: Void

### 5.4.12 for x in l repeat

---

Another variant of the `for` loop has the form:

`for x in list repeat loopBody`

This form is used when you want to iterate directly over the elements of a list. In this form of the `for` loop, the variable `x` takes on the value of each successive element in `l`. The end test is most simply stated in English: “are there no more `x` in `l`?”

If `l` is this list,

```
l := [0, -5, 3]
```

```
[0, -5, 3] (1)
```

Type: List Integer

display all elements of `l`, one per line.

```
for x in l repeat output(x)
```

```
0  
- 5  
3
```

Type: Void

Since the list constructing expression `expand [n..m]` creates the list `[n, n+1, ..., m]`<sup>2</sup>, you might be tempted to think that the loops

```
for i in n..m repeat output(i)
```

and

```
for x in expand [n..m] repeat output(x)
```

are equivalent. The second form first creates the list `expand [n..m]` (no matter how large it might be) and then does the iteration. The first form potentially runs in much less space, as the index variable `i` is simply incremented once per loop and the list is not actually created. Using the first form is much more efficient.

Of course, sometimes you really want to iterate across a specific list. This displays each of the factors of 2400000.

```
for f in factors(factor(2400000)) repeat output(f)
```

```
[factor= 2,exponent= 8]  
[factor= 3,exponent= 1]  
[factor= 5,exponent= 5]
```

Type: Void

### 5.4.13 “Such that” Predicates

---

This loop expression prints out the integers `n` in the given segment such that `n` is odd.

```
for n in 0..4 | odd? n repeat output n
```

```
1  
3
```

Type: Void

---

<sup>2</sup>This list is empty if `n > m`.

A **for** loop can also be written

*for* *iterator* | *predicate* **repeat** *loopBody*

which is equivalent to:

*for* *iterator* **repeat** *if* *predicate* **then** *loopBody* **else** *iterate*

The predicate need not refer only to the variable in the **for** clause: any variable in an outer scope can be part of the predicate.

In this example, the predicate on the inner **for** loop uses *i* from the outer loop and the *j* from the **for** clause that it directly modifies.

```
for i in 1..50 repeat
  for j in 1..50 | factorial(i+j) < 25 repeat
    output [i,j]
[1,1]
[1,2]
[1,3]
[2,1]
[2,2]
[3,1]
```

Type: Void

#### 5.4.14 Parallel Iteration

The last example of the previous section gives an example of *nested iteration*: a loop is contained in another loop. Sometimes you want to iterate across two lists in parallel, or perhaps you want to traverse a list while incrementing a variable.

The general syntax of a repeat loop is

*iterator*<sub>1</sub> *iterator*<sub>2</sub> ... *iterator*<sub>N</sub> **repeat** *loopBody*

where each *iterator* is either a **for** or a **while** clause. The loop terminates immediately when the end test of any *iterator* succeeds or when a **break** or **return** expression is evaluated in *loopBody*. The value returned by the loop is the unique value of Void.

Here we write a loop to iterate across two lists, computing the sum of the pairwise product of elements. Here is the first list.

```
l := [1,3,5,7]
[1, 3, 5, 7] (1)
```

Type: List PositiveInteger

And the second.

```
m := [100,200]
[100, 200] (2)
```

Type: List PositiveInteger

The initial value of the sum counter.	<pre>sum := 0</pre> <pre>0</pre>	(3)	Type: NonNegativeInteger
The last two elements of <code>l</code> are not used in the calculation because <code>m</code> has two fewer elements than <code>l</code> .	<pre>for x in l for y in m repeat</pre> <pre>    sum := sum + x*y</pre>		Type: Void
Display the “dot product.”	<pre>sum</pre> <pre>700</pre>	(5)	Type: NonNegativeInteger
Next, we write a loop to compute the sum of the products of the loop elements with their positions in the loop.	<pre>l := [2,3,5,7,11,13,17,19,23,29,31,37]</pre> <pre>[2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37]</pre>	(6)	Type: List PositiveInteger
The initial sum.	<pre>sum := 0</pre> <pre>0</pre>	(7)	Type: NonNegativeInteger
Here looping stops when the list <code>l</code> is exhausted, even though the <code>for i in 0..</code> specifies no terminating condition.	<pre>for i in 0.. for x in l repeat sum := i * x</pre>		Type: Void
Display this weighted sum.	<pre>sum</pre> <pre>407</pre>	(9)	Type: NonNegativeInteger
<p>When “ ” is used to qualify any of the <code>for</code> clauses in a parallel iteration, the variables in the predicates can be from an outer scope or from a <code>for</code> clause in or to the left of a modified clause.</p> <p>This is correct:</p> <pre>for i in 1..10 repeat</pre> <pre>    for j in 200..300   odd? (i+j) repeat</pre> <pre>        output [i,j]</pre> <p>This is not correct since the variable <code>j</code> has not been defined outside the inner loop.</p> <pre>for i in 1..10   odd? (i+j) repeat -- wrong, j not defined</pre> <pre>    for j in 200..300 repeat</pre> <pre>        output [i,j]</pre>			

This example shows that it is possible to mix several of the forms of **repeat** modifying clauses on a loop.

```
for i in 1..10
  for j in 151..160 | odd? j
    while i + j < 160 repeat
      output [i,j]

[1,151]
[3,153]
```

Type: Void

Here are useful rules for composing loop expressions:

1. **while** predicates can only refer to variables that are global (or in an outer scope) or that are defined in **for** clauses to the left of the predicate.
2. A “such that” predicate (something following “|”) must directly follow a **for** clause and can only refer to variables that are global (or in an outer scope) or defined in the modified **for** clause or any **for** clause to the left.



## 5.5 Creating Lists and Streams with Iterators

This creates a simple list of the integers from 1 to 10.

```
list := [i for i in 1..10]
```

```
[1, 2, 3, 4, 5, 6, 7, 8, 9, 10]
```

(1)

Type: List PositiveInteger

Create a stream of the integers greater than or equal to 1.

```
stream := [i for i in 1..]
```

```
[1, 2, 3, 4, 5, 6, 7, ...]
```

(2)

Type: Stream PositiveInteger

This is a list of the prime integers between 1 and 10, inclusive.

```
[i for i in 1..10 | prime? i]
```

```
[2, 3, 5, 7]
```

(3)

Type: List PositiveInteger

This is a stream of the prime integers greater than or equal to 1.

```
[i for i in 1.. | prime? i]
```

```
[2, 3, 5, 7, 11, 13, 17, ...]
```

(4)

Type: Stream PositiveInteger

This is a list of the integers between 1 and 10, inclusive, whose squares are less than 700.

```
[i for i in 1..10 while i*i < 700]
```

```
[1, 2, 3, 4, 5, 6, 7, 8, 9, 10]
```

(5)

Type: List PositiveInteger

This is a stream of the integers greater than or equal to 1 whose squares are less than 700.

```
[i for i in 1.. while i*i < 700]
```

```
[1, 2, 3, 4, 5, 6, 7, ...]
```

(6)

Type: Stream PositiveInteger

Got the idea? Here is the general rule.

The general syntax of a collection is

$$[ \textit{collectExpression} \textit{iterator}_1 \textit{iterator}_2 \dots \textit{iterator}_N ]$$

where each  $\textit{iterator}_i$  is either a **for** or a **while** clause. The loop terminates immediately when the end test of any  $\textit{iterator}_i$  succeeds or when a **return** expression is evaluated in  $\textit{collectExpression}$ . The value returned by the collection is either a list or a stream of elements, one for each iteration of the  $\textit{collectExpression}$ .

Be careful when you use **while** to create a stream. By default, AXIOM

tries to compute and display the first ten elements of a stream. If the **while** condition is not satisfied quickly, AXIOM can spend a long (possibly infinite) time trying to compute the elements. Use **)set streams calculate** to change the default to something else. This also affects the number of terms computed and displayed for power series. For the purposes of this book, we have used this system command to display fewer than ten terms.

Use nested iterators to create lists of lists which can then be given as an argument to **matrix**.

```
matrix [[x**i+j for i in 1..3] for j in 10..12]
```

$$\begin{bmatrix} x+10 & x^2+10 & x^3+10 \\ x+11 & x^2+11 & x^3+11 \\ x+12 & x^2+12 & x^3+12 \end{bmatrix} \quad (7)$$

Type: Matrix Polynomial Integer

You can also create lists of streams, streams of lists and streams of streams. Here is a stream of streams.

```
[[i/j for i in j+1..] for j in 1..]
```

$$\begin{aligned} & \left[ \left[ 2, 3, 4, 5, 6, 7, 8, \dots \right], \left[ \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}, 4, \frac{9}{2}, \dots \right], \right. \\ & \left[ \frac{4}{3}, \frac{5}{3}, 2, \frac{7}{3}, \frac{8}{3}, 3, \frac{10}{3}, \dots \right], \left[ \frac{5}{4}, \frac{3}{2}, \frac{7}{4}, 2, \frac{9}{4}, \frac{5}{2}, \frac{11}{4}, \dots \right], \\ & \left[ \frac{6}{5}, \frac{7}{5}, \frac{8}{5}, \frac{9}{5}, 2, \frac{11}{5}, \frac{12}{5}, \dots \right], \left[ \frac{7}{6}, \frac{4}{3}, \frac{3}{2}, \frac{5}{3}, \frac{11}{6}, 2, \frac{13}{6}, \dots \right], \\ & \left. \left[ \frac{8}{7}, \frac{9}{7}, \frac{10}{7}, \frac{11}{7}, \frac{12}{7}, \frac{13}{7}, 2, \dots \right], \dots \right] \end{aligned} \quad (8)$$

Type: Stream Stream Fraction Integer

You can use parallel iteration across lists and streams to create new lists.

```
[i/j for i in 3.. by 10 for j in 2..]
```

$$\left[ \frac{3}{2}, \frac{13}{3}, \frac{23}{4}, \frac{33}{5}, \frac{43}{6}, \frac{53}{7}, \frac{63}{8}, \dots \right] \quad (9)$$

Type: Stream Fraction Integer

Iteration stops if the end of a list or stream is reached.

```
[i**j for i in 1..7 for j in 2.. ]
```

$$[1, 8, 81, 1024, 15625, 279936, 5764801] \quad (10)$$

Type: Stream Integer

As with loops, you can combine these modifiers to make very complicated conditions.

```
[[[i,j] for i in 10..15 | prime? i] for j in 17..22 | j = squareFreePart j]
```

$$\begin{aligned} & [[11, 17], [13, 17]], [[11, 19], [13, 19]], [[11, 21], [13, 21]], \\ & [[11, 22], [13, 22]] \end{aligned} \quad (11)$$

Type: List List List PositiveInteger

See ‘List’ on page 489 and ‘Stream’ on page 575 for more information on creating and manipulating lists and streams, respectively.

## 5.6 An Example: Streams of Primes

We conclude this chapter with an example of the creation and manipulation of infinite streams of prime integers. This might be useful for experiments with numbers or other applications where you are using sequences of primes over and over again. As for all streams, the stream of primes is only computed as far out as you need. Once computed, however, all the primes up to that point are saved for future reference.

Two useful operations provided by the AXIOM library are **prime?** and **nextPrime**. A straight-forward way to create a stream of prime numbers is to start with the stream of positive integers `[2, ...]` and filter out those that are prime.

Create a stream of primes.

```
primes : Stream Integer := [i for i in 2.. | prime? i]
[2, 3, 5, 7, 11, 13, 17, ...] (1)
```

Type: Stream Integer

A more elegant way, however, is to use the **generate** operation from Stream. Given an initial value **a** and a function **f**, **generate** constructs the stream `[a, f(a), f(f(a)), ...]`. This function gives you the quickest method of getting the stream of primes.

This is how you use **generate** to generate an infinite stream of primes.

```
primes := generate(nextPrime, 2)
[2, 3, 5, 7, 11, 13, 17, ...] (2)
```

Type: Stream Integer

Once the stream is generated, you might only be interested in primes starting at a particular value.

```
smallPrimes := [p for p in primes | p > 1000]
[1009, 1013, 1019, 1021, 1031, 1033, 1039, ...] (3)
```

Type: Stream Integer

Here are the first 11 primes greater than 1000.

```
[p for p in smallPrimes for i in 1..11]
[1009, 1013, 1019, 1021, 1031, 1033, 1039, ...] (4)
```

Type: Stream Integer

Here is a stream of primes between 1000 and 1200.

```
[p for p in smallPrimes while p < 1200]
[1009, 1013, 1019, 1021, 1031, 1033, 1039, ...] (5)
```

Type: Stream Integer

To get these expanded into a finite stream, you call **complete** on the stream.

```
complete %
[1009, 1013, 1019, 1021, 1031, 1033, 1039, ...] (6)
```

Type: Stream Integer

Twin primes are consecutive odd number pairs which are prime. Here is the stream of twin primes.

```
twinPrimes := [[p,p+2] for p in primes | prime?(p + 2)]
[[3, 5], [5, 7], [11, 13], [17, 19], [29, 31], [41, 43], [59, 61], ...] (7)
```

Type: Stream List Integer

Since we already have the primes computed we can avoid the call to **prime?** by using a double iteration. This time we'll just generate a stream of the first of the twin primes.

```
firstOfTwins:= [p for p in primes for q in rest primes |
               q=p+2]
[3, 5, 11, 17, 29, 41, 59, ...]
```

(8)

Type: Stream Integer

Let's try to compute the infinite stream of triplet primes, the set of primes  $p$  such that  $[p, p+2, p+4]$  are primes. For example,  $[3, 5, 7]$  is a triplet prime. We could do this by a triple **for** iteration. A more economical way is to use **firstOfTwins**. This time however, put a semicolon at the end of the line.

Put a semicolon at the end so that no elements are computed.

```
firstTriplets := [p for p in firstOfTwins for q in rest
                 firstOfTwins | q = p+2];
```

(9)

Type: Stream Integer

What happened? As you know, by default AXIOM displays the first ten elements of a stream when you first display it. And, therefore, it needs to compute them! If you want *no* elements computed, just terminate the expression by a semicolon (“;”).<sup>3</sup>

Compute the first triplet prime.

```
firstTriplets.1
```

3

(10)

Type: PositiveInteger

If you want to compute another, just ask for it. But wait a second! Given three consecutive odd integers, one of them must be divisible by 3. Thus there is only one triplet prime. But suppose that you did not know this and wanted to know what was the tenth triplet prime.

```
firstTriples.10
```

To compute the tenth triplet prime, AXIOM first must compute the second, the third, and so on. But since there isn't even a second triplet prime, AXIOM will compute forever. Nonetheless, this effort can produce a useful result. After waiting a bit, hit **Ctrl**-**c**. The system responds as follows.

```
>> System error:
Console interrupt.
You are being returned to the top level of
the interpreter.
```

Let's say that you want to know how many primes have been computed. Issue

---

<sup>3</sup>Why does this happen? The semi-colon prevents the display of the result of evaluating the expression. Since no stream elements are needed for display (or anything else, so far), none are computed.

```
numberOfComputedEntries primes
```

and, for this discussion, let's say that the result is 2045.

How big is the 2045<sup>th</sup> prime?

```
primes.2045
```

```
17837
```

(11)

Type: PositiveInteger

What you have learned is that there are no triplet primes between 5 and 17837. Although this result is well known (some might even say trivial), there are many experiments you could make where the result is not known. What you see here is a paradigm for testing of hypotheses. Here our hypothesis could have been: “there is more than one triplet prime.” We have tested this hypothesis for 17837 cases. With streams, you can let your machine run, interrupt it to see how far it has progressed, then start it up and let it continue from where it left off.



---

# User-Defined Functions, Macros and Rules

In this chapter we show you how to write functions and macros, and we explain how AXIOM looks for and applies them. We show some simple one-line examples of functions, together with larger ones that are defined piece-by-piece or through the use of piles.

## 6.1 Functions vs. Macros

---

This is a use of the “absolute value” library function for integers.

```
abs(-8)
```

```
8
```

(1)

Type: PositiveInteger

This is an unnamed function that does the same thing, using the “maps-to” syntax “ $\mapsto$ ” that we discuss in Section 6.17 on page 218.

```
(x  $\mapsto$  if x < 0 then -x else x)(-8)
```

```
8
```

(2)

Type: PositiveInteger

Functions can be used alone or serve as the building blocks for larger programs. Usually they return a value that you might want to use in the next stage of a computation, but not always (for example, see ‘Exit’ on page 409 and ‘Void’ on page 603). They may also read data from your keyboard, move information from one place to another, or format and display results on your screen.

In AXIOM, as in mathematics, functions are usually *parameterized*. Each time you *call* (some people say *apply* or *invoke*) a function, you give values to the parameters (variables). Such a value is called an *argument* of the function. AXIOM uses the arguments for the computation. In this way you get different results depending on what you “feed” the function.

Functions can have local variables or refer to global variables in the workspace. AXIOM can often *compile* functions so that they execute very efficiently. Functions can be passed as arguments to other functions.

Macros are textual substitutions. They are used to clarify the meaning of constants or expressions and to be templates for frequently used expressions. Macros can be parameterized but they are not objects that can be passed as arguments to functions. In effect, macros are extensions to the AXIOM expression parser.



## 6.2 Macros

A *macro* provides general textual substitution of an AXIOM expression for a name. You can think of a macro as being a generalized abbreviation. You can only have one macro in your workspace with a given name, no matter how many arguments it has.

The two general forms for macros are

$$\begin{aligned} \text{macro name} &== \text{body} \\ \text{macro name}(\text{arg1}, \dots) &== \text{body} \end{aligned}$$

where the body of the macro can be any AXIOM expression.

For example, suppose you decided that you like to use **df** for **D**. You define the macro **df** like this.

```
macro df == D
```

Type: Void

Whenever you type **df**, the system expands it to **D**.

```
df(x**2 + x + 1, x)
```

$$2x + 1$$

(2)

Type: Polynomial Integer

Macros can be parameterized and so can be used for many different kinds of objects.

```
macro ff(x) == x**2 + 1
```

Type: Void

Apply it to a number, a symbol, or an expression.

```
ff z
```

$$z^2 + 1$$

(4)

Type: Polynomial Integer

Macros can also be nested, but you get an error message if you run out of space because of an infinite nesting loop.

```
macro gg(x) == ff(2*x - 2/3)
```

Type: Void

This new macro is fine as it does not produce a loop.

```
gg(1/w)
```

$$\frac{13w^2 - 24w + 36}{9w^2}$$

(6)

Type: Fraction Polynomial Integer

This, however, loops since **gg** is defined in terms of **ff**.

```
macro ff(x) == gg(-x)
```

Type: Void

The body of a macro can be a block.	<pre>macro next == (past := present; present := future; future                := past + present)</pre>	Type: Void
Before entering <b>next</b> , we need values for <b>present</b> and <b>future</b> .	<pre>present : Integer := 0 0</pre>	(9) Type: Integer
	<pre>future : Integer := 1 1</pre>	(10) Type: Integer
Repeatedly evaluating <b>next</b> produces the next Fibonacci number.	<pre>next 1</pre>	(11) Type: Integer
And the next one.	<pre>next 2</pre>	(12) Type: Integer
Here is the infinite stream of the rest of the Fibonacci numbers.	<pre>[next for i in 1..] [3, 5, 8, 13, 21, 34, 55, ...]</pre>	(13) Type: Stream Integer
Bundle all the above lines into a single macro.	<pre>macro fibStream ==   present : Integer := 1   future : Integer := 1   [next for i in 1..] where     macro next ==       past := present       present := future       future := past + present</pre>	Type: Void
Use <b>concat</b> to start with the first two Fibonacci numbers.	<pre>concat([0,1],fibStream) [0, 1, 2, 3, 5, 8, 13, ...]</pre>	(15) Type: Stream Integer
An easier way to compute these numbers is to use the library operation <b>fibonacci</b> .	<pre>[fibonacci i for i in 1..] [1, 1, 2, 3, 5, 8, 13, ...]</pre>	(16) Type: Stream Integer

## 6.3 Introduction to Functions

---

Each name in your workspace can refer to a single object. This may be any kind of object including a function. You can use interactively any function from the library or any that you define in the workspace. In the library the same name can have very many functions, but you can have only one function with a given name, although it can have any number of arguments that you choose.

If you define a function in the workspace that has the same name and number of arguments as one in the library, then your definition takes precedence. In fact, to get the library function you must *package-call* it (see Section 2.9 on page 119).

To use a function in AXIOM, you apply it to its arguments. Most functions are applied by entering the name of the function followed by its argument or arguments.

```
factor(12)
```

$2^2 \cdot 3$  (1)

Type: Factored Integer

Some functions like “+” have *infix operators* as names.

```
3 + 4
```

7 (2)

Type: PositiveInteger

The function “+” has two arguments. When you give it more than two arguments, AXIOM groups the arguments to the left. This expression is equivalent to  $(1 + 2) + 7$ .

```
1 + 2 + 7
```

10 (3)

Type: PositiveInteger

All operations, including infix operators, can be written in prefix form, that is, with the operation name followed by the arguments in parentheses. For example,  $2 + 3$  can alternatively be written as  $+(2, 3)$ . But  $+(2, 3, 4)$  is an error since “+” takes only two arguments.

Prefix operations are generally applied before the infix operation. Thus `factorial 3 + 1` means `factorial(3) + 1` producing 7, and `- 2 + 5` means  $(-2) + 5$  producing 3. An example of a prefix operator is prefix “-”. For example, `- 2 + 5` converts to  $(- 2) + 5$  producing the value 3. Any prefix function taking two arguments can be written in an infix manner by putting an ampersand (“&”) before the name. Thus `D(2*x, x)` can be written as `2*x &D x` returning 2.

Every function in AXIOM is identified by a *name* and *type*.<sup>1</sup> The type of a function is always a mapping of the form **Source**  $\rightarrow$  **Target** where

---

<sup>1</sup>An exception is an “anonymous function” discussed in Section 6.17 on page 218.

`Source` and `Target` are types. To enter a type from the keyboard, enter the arrow by using a hyphen “-” followed by a greater-than sign “>”, e.g. `Integer -> Integer`.

Let’s go back to “+”. There are many “+” functions in the AXIOM library: one for integers, one for floats, another for rational numbers, and so on. These “+” functions have different types and thus are different functions. You’ve seen examples of this *overloading* before—using the same name for different functions. Overloading is the rule rather than the exception. You can add two integers, two polynomials, two matrices or two power series. These are all done with the same function name but with different functions.

## 6.4 Declaring the Type of Functions

In Section 2.3 on page 103 we discussed how to declare a variable to restrict the kind of values that can be assigned to it. In this section we show how to declare a variable that refers to function objects.

A function is an object of type

$$\text{Source} \rightarrow \text{Type}$$

where **Source** and **Target** can be any type. A common type for **Source** is  $\text{Tuple}(T_1, \dots, T_n)$ , usually written  $(T_1, \dots, T_n)$ , to indicate a function of  $n$  arguments.

If **g** takes an Integer, a Float and another Integer, and returns a String, the declaration is written this way.

```
g: (Integer,Float,Integer) -> String
```

Type: Void

The types need not be written fully; using abbreviations, the above declaration is:

```
g: (INT,FLOAT,INT) -> STRING
```

Type: Void

It is possible for a function to take no arguments. If **h** takes no arguments but returns a Polynomial Integer, any of the following declarations is acceptable.

```
h: () -> POLY INT
```

Type: Void

```
h: () -> Polynomial INT
```

Type: Void

```
h: () -> POLY Integer
```

Type: Void

Functions can also be declared when they are being defined. The syntax for combined declaration/definition is:

$$\text{functionName}(\text{parm}_1: \text{parmType}_1, \dots, \text{parm}_N: \text{parmType}_N): \\ \text{functionReturnType}$$

The following definition fragments show how this can be done for the

functions `g` and `h` above.

```
g(arg1: INT, arg2: FLOAT, arg3: INT): STRING == ...
```

```
h(): POLY INT == ...
```

A current restriction on function declarations is that they must involve fully specified types (that is, cannot include modes involving explicit or implicit “?”). For more information on declaring things in general, see Section 2.3 on page 103.

## 6.5 One-Line Functions

This is a simple recursive factorial function for positive integers.

As you use AXIOM, you will find that you will write many short functions to codify sequences of operations that you often perform. In this section we write some simple one-line functions.

```
fac n == if n < 3 then n else n * fac(n-1)
```

Type: Void

```
fac 10
```

```
Compiling function fac with type Integer -> Integer
```

```
3628800
```

(2)

Type: PositiveInteger

This function computes  $1 + 1/2 + 1/3 + \dots + 1/n$ .

```
s n == reduce(+,[1/i for i in 1..n])
```

Type: Void

```
s 50
```

```
Compiling function s with type PositiveInteger ->
Fraction Integer
```

```
13943237577224054960759
3099044504245996706400
```

(4)

Type: Fraction Integer

This function computes a Mersenne number, several of which are prime.

```
mersenne i == 2**i - 1
```

Type: Void

If you type `mersenne`, AXIOM shows you the function definition.

```
mersenne
```

```
mersenne i == 2i - 1
```

(6)

Type: FunctionCalled mersenne

Generate a stream of Mersenne numbers.

```
[mersenne i for i in 1..]
```

```
Compiling function mersenne with type PositiveInteger
-> Integer
```

```
[1, 3, 7, 15, 31, 63, 127, ...]
```

(7)

Type: Stream Integer

Create a stream of those values of  $i$  such that `mersenne(i)` is prime.

```
mersenneIndex := [n for n in 1.. | prime?(mersenne(n))]
```

```
[2, 3, 5, 7, 13, 17, 19, ...]
```

(8)

Type: Stream PositiveInteger

Finally, write a function that returns the  $n^{\text{th}}$  Mersenne prime.

```
mersennePrime n == mersenne mersenneIndex(n)
```

Type: Void

```
mersennePrime 5
```

```
Compiling function mersennePrime with type
  PositiveInteger -> Integer
```

```
8191
```

(10)

Type: PositiveInteger



## 6.6 Declared vs. Undeclared Functions

Define `f` with type `Integer → Integer`.

If you declare the type of a function, you can apply it to any data that can be converted to the source type of the function.

```
f(x: Integer): Integer == x + 1
```

```
Function declaration f : Integer -> Integer has been
added to workspace.
```

Type: Void

The function `f` can be applied to integers, ...

```
f 9
```

```
Compiling function f with type Integer -> Integer
```

```
10
```

(2)

Type: PositiveInteger

and to values that convert to integers, ...

```
f(-2.0)
```

```
-1
```

(3)

Type: Integer

but not to values that cannot be converted to integers.

```
f(2/3)
```

```
Conversion failed in the compiled user function f .
```

```
Cannot convert from type Fraction Integer to Integer
for value
```

```
2
-
3
```

To make the function over a wide range of types, do not declare its type.

Give the same definition with no declaration.

```
g x == x + 1
```

Type: Void

If `x + 1` makes sense, you can apply `g` to `x`.

```
g 9
```

```
Compiling function g with type PositiveInteger ->
PositiveInteger
```

```
10
```

(5)

Type: PositiveInteger

A version of `g` with different argument types get compiled for each new kind of argument used.

`g(2/3)`

Compiling function `g` with type Fraction Integer ->  
Fraction Integer

$\frac{5}{3}$

(6)

Type: Fraction Integer

Here `x+1` for `x = "axiom"` makes no sense.

`g("axiom")`

There are 11 exposed and 5 unexposed library operations named `+` having 2 argument(s) but none was determined to be applicable. Use HyperDoc Browse, or issue

`)display op +`

to learn more about the available operations.

Perhaps package-calling the operation or using coercions on the arguments will allow you to apply the operation.

Cannot find a definition or applicable library operation named `+` with argument type(s)

String

PositiveInteger

Perhaps you should use `"@"` to indicate the required return type, or `"$"` to specify which version of the function you need.

AXIOM will attempt to step through and interpret the code.

There are 11 exposed and 5 unexposed library operations named `+` having 2 argument(s) but none was determined to be applicable. Use HyperDoc Browse, or issue

`)display op +`

to learn more about the available operations.

Perhaps package-calling the operation or using coercions on the arguments will allow you to apply the operation.

Cannot find a definition or applicable library operation named `+` with argument type(s)

String

PositiveInteger

Perhaps you should use `"@"` to indicate the required return type, or `"$"` to specify which version of the function you need.

As you will see in Chapter 12, AXIOM has a formal idea of categories for what “makes sense.”

## 6.7 Functions vs. Operations

---

A function is an object that you can create, manipulate, pass to, and return from functions (for some interesting examples of library functions that manipulate functions, see ‘MappingPackage1’ on page 496). Yet, we often seem to use the term *operation* and function interchangeably in AXIOM. What is the distinction?

First consider values and types associated with some variable  $n$  in your workspace. You can make the declaration  $n : \text{Integer}$ , then assign  $n$  an integer value. You then speak of the integer  $n$ . However, note that the integer is not the name  $n$  itself, but the value that you assign to  $n$ .

Similarly, you can declare a variable  $f$  in your workspace to have type  $\text{Integer} \rightarrow \text{Integer}$ , then assign  $f$ , through a definition or an assignment of an anonymous function. You then speak of the function  $f$ . However, the function is not  $f$ , but the value that you assign to  $f$ .

A function is a value, in fact, some machine code for doing something. Doing what? Well, performing some *operation*. Formally, an operation consists of the constituent parts of  $f$  in your workspace, excluding the value; thus an operation has a name and a type. An operation is what domains and packages export. Thus Ring exports one operation “+”. Every ring also exports this operation. Also, the author of every ring in the system is obliged under contract (see Section 11.3 on page 651) to provide an implementation for this operation.

This chapter is all about functions—how you create them interactively and how you apply them to meet your needs. In Chapter 11 you will learn how to create them for the AXIOM library. Then in Chapter 12, you will learn about categories and exported operations.

## 6.8 Delayed Assignments vs. Functions with No Arguments

A function of no arguments is sometimes called a *nullary function*.

You must use the parentheses (“()”) to evaluate it. Like a delayed assignment, the right-hand-side of a function evaluation is not evaluated until the left-hand-side is used.

If you omit the parentheses, you just get the function definition.

You do not use the parentheses “()” in a delayed assignment...

nor in the evaluation.

In Section 5.1 on page 150 we discussed the difference between immediate and delayed assignments. In this section we show the difference between delayed assignments and functions of no arguments.

```
sin24() == sin(24.0)
```

Type: Void

```
sin24()
```

```
Compiling function sin24 with type () -> Float
```

```
-0.90557836200662384514
```

(2)

Type: Float

```
sin24
```

```
sin24 () == sin(24.0)
```

(3)

Type: FunctionCalled sin24

```
cos24 == cos(24.0)
```

Type: Void

```
cos24
```

```
Compiling body of rule cos24 to compute value of type
Float
```

```
0.42417900733699697594
```

(5)

Type: Float

The only syntactic difference between delayed assignments and nullary functions is that you use “()” in the latter case.

## 6.9 How AXIOM Determines What Function to Use

What happens if you define a function that has the same name as a library function? Well, if your function has the same name and number of arguments (we sometimes say *arity*) as another function in the library, then your function covers up the library function. If you want then to call the library function, you will have to package-call it. AXIOM can use both the functions you write and those that come from the library. Let's do a simple example to illustrate this.

Suppose you (wrongly!) define `sin` in this way.

```
sin x == 1.0
```

Type: Void

The value 1.0 is returned for any argument.

```
sin 4.3
Compiling function sin with type Float -> Float
1.0
```

(2)

Type: Float

If you want the library operation, we have to package-call it (see Section 2.9 on page 119 for more information).

```
sin(4.3)$Float
-0.91616593674945498404
```

(3)

Type: Float

```
sin(34.6)$Float
-0.042468034716950101543
```

(4)

Type: Float

Even worse, say we accidentally used the same name as a library function in the function.

```
sin x == sin x
Compiled code for sin has been cleared.
1 old definition(s) deleted for function or rule sin
```

Type: Void

Then AXIOM definitely does not understand us.

```
sin 4.3
AXIOM cannot determine the type of sin because it
cannot analyze the non-recursive part, if that
exists. This may be remedied by declaring the
function.
```

Again, we could package-call the inside function.

```
sin x == sin(x)$Float
1 old definition(s) deleted for function or rule sin
```

Type: Void

```

sin 4.3
Compiling function sin with type Float -> Float
+++ |*1;sin;1;initial| redefined
-0.91616593674945498404

```

(7)

Type: Float

Of course, you are unlikely to make such obvious errors. It is more probable that you would write a function and in the body use a function that you think is a library function. If you had also written a function by that same name, the library function would be invisible.

How does AXIOM determine what library function to call? It very much depends on the particular example, but the simple case of creating the polynomial  $x + 2/3$  will give you an idea.

1. The  $x$  is analyzed and its default type is `Variable(x)`.
2. The 2 is analyzed and its default type is `PositiveInteger`.
3. The 3 is analyzed and its default type is `PositiveInteger`.
4. Because the arguments to “/” are integers, AXIOM gives the expression  $2/3$  a default target type of `Fraction(Integer)`.
5. AXIOM looks in `PositiveInteger` for “/”. It is not found.
6. AXIOM looks in `Fraction(Integer)` for “/”. It is found for arguments of type `Integer`.
7. The 2 and 3 are converted to objects of type `Integer` (this is trivial) and “/” is applied, creating an object of type `Fraction(Integer)`.
8. No “+” for arguments of types `Variable(x)` and `Fraction(Integer)` are found in either domain.
9. AXIOM resolves (see Section 2.10 on page 122) the types and gets `Polynomial (Fraction (Integer))`.
10. The  $x$  and the  $2/3$  are converted to objects of this type and “+” is applied, yielding the answer, an object of type `Polynomial (Fraction (Integer))`.

## 6.10 Compiling vs. Interpreting

---

When possible, AXIOM completely determines the type of every object in a function, then translates the function definition to Common LISP or to machine code (see next section). This translation, called *compilation*, happens the first time you call the function and results in a computational delay. Subsequent function calls with the same argument types use the compiled version of the code without delay.

If AXIOM cannot determine the type of everything, the function may still be executed but in *interpret-code mode*: each statement in the function is analyzed and executed as the control flow indicates. This process is slower than executing a compiled function, but it allows the execution of code that may involve objects whose types change.

If AXIOM decides that it cannot compile the code, it issues a message stating the problem and then the following message:

**We will attempt to step through and interpret the code.**

This is not a time to panic. Rather, it just means that what you gave to AXIOM is somehow ambiguous: either it is not specific enough to be analyzed completely, or it is beyond AXIOM's present interactive compilation abilities.

This function runs in interpret-code mode, but it does not compile.

```
varPolys(vars) ==  
  for var in vars repeat  
    output(1 :: UnivariatePolynomial(var,Integer))
```

Type: Void

For vars equal to ['x, 'y, 'z], this function displays 1 three times.

```
varPolys ['x,'y,'z]  
Cannot compile conversion for types involving local  
  variables. In particular, could not compile the  
  expression involving :: UnivariatePolynomial(var,  
  Integer)  
AXIOM will attempt to step through and interpret the  
  code.  
1  
1  
1
```

Type: Void

The type of the argument to **output** changes in each iteration, so AXIOM cannot compile the function. In this case, even the inner loop by itself would have a problem:

```
for var in ['x','y','z'] repeat
  output(1 :: UnivariatePolynomial(var,Integer))
Cannot compile conversion for types involving local
variables. In particular, could not compile the
expression involving :: UnivariatePolynomial(var,
Integer)
AXIOM will attempt to step through and interpret the
code.
1
1
1
```

Type: Void

Sometimes you can help a function to compile by using an extra conversion or by using **pretend**. See Section 2.8 on page 116 for details.

When a function is compilable, you have the choice of whether it is compiled to Common LISP and then interpreted by the Common LISP interpreter or then further compiled from Common LISP to machine code. The option is controlled via **)set functions compile**. Issue **)set functions compile on** to compile all the way to machine code. With the default setting **)set functions compile off**, AXIOM has its Common LISP code interpreted because the overhead of further compilation is larger than the run-time of most of the functions our users have defined. You may find that selectively turning this option on and off will give you the best performance in your particular application. For example, if you are writing functions for graphics applications where hundreds of points are being computed, it is almost certainly true that you will get the best performance by issuing **)set functions compile on**.



## 6.11 Piece-Wise Function Definitions

---

To move beyond functions defined in one line, we introduce in this section functions that are defined piece-by-piece. That is, we say “use this definition when the argument is such-and-such and use this other definition when the argument is that-and-that.”

### 6.11.1 A Basic Example

---

There are many other ways to define a factorial function for nonnegative integers. You might say factorial of 0 is 1, otherwise factorial of  $n$  is  $n$  times factorial of  $n-1$ . Here is one way to do this in AXIOM.

Here is the value for  $n = 0$ .

```
fact(0) == 1
```

Type: Void

Here is the value for  $n > 0$ . The vertical bar “|” means “such that”.

```
fact(n | n > 0) == n * fact(n - 1)
```

Type: Void

What is the value for  $n = 3$ ?

```
fact(3)
```

```
Compiling function fact with type Integer -> Integer
Compiling function fact as a recurrence relation.
```

```
6
```

(3)

Type: PositiveInteger

What is the value for  $n = -3$ ?

```
fact(-3)
```

```
You did not define fact for argument -3 .
```

Now for a second definition.  
Here is the value for  $n = 0$ .

```
facto(0) == 1
```

Type: Void

Give an error message if  $n < 0$ .

```
facto(n | n < 0) == error "arguments to facto must be non-
negative"
```

Type: Void

Here is the value otherwise.

```
facto(n) == n * facto(n - 1)
```

Type: Void

What is the value for $n = 7$ ?	<pre>facto(3)</pre> <p>Compiling function facto with type Integer -&gt; Integer</p> <p>6</p> <p style="text-align: right;">(7)</p> <p style="text-align: right;">Type: PositiveInteger</p>
What is the value for $n = -7$ ?	<pre>facto(-7)</pre> <p>Error signalled from user code in function facto: arguments to facto must be non-negative</p>
To see the current piece-wise definition of a function, use )display value.	<pre>)display value facto</pre> <p>Definition:</p> <pre>facto 0 == 1 facto (n   n &lt; 0) ==   error(arguments to facto must be non-negative) facto n == n facto(n - 1)</pre> <p>In general a <i>piece-wise definition</i> of a function consists of two or more parts. Each part gives a “piece” of the entire definition. AXIOM collects the pieces of a function as you enter them. When you ask for a value of the function, it then “glues” the pieces together to form a function.</p> <p>The two piece-wise definitions for the factorial function are examples of recursive functions, that is, functions that are defined in terms of themselves. Here is an interesting doubly-recursive function. This function returns the value 11 for all positive integer arguments.</p>
Here is the first of two pieces.	<pre>eleven(n   n &lt; 1) == n + 11</pre> <p style="text-align: right;">Type: Void</p>
And the general case.	<pre>eleven(m) == eleven(eleven(m - 12))</pre> <p style="text-align: right;">Type: Void</p>
Compute <b>elevens</b> , the infinite stream of values of <b>eleven</b> .	<pre>elevens := [eleven(i) for i in 0..]</pre> <p>Compiling function eleven with type Integer -&gt; Integer</p> <p>[11, 11, 11, 11, 11, 11, ...]</p> <p style="text-align: right;">(10)</p> <p style="text-align: right;">Type: Stream Integer</p>

What is the value at <code>n = 200</code> ?	<pre> eleven 200  11 </pre>	(11) Type: PositivelInteger
What is the AXIOM's definition of <code>eleven</code> ?	<pre> )display value eleven </pre>	

Definition:

```

eleven (m | m < 1) == m + 11
eleven m == eleven(eleven(m - 12))

```

### 6.11.2 Picking Up the Pieces

Here are the details about how AXIOM creates a function from its pieces. AXIOM converts the  $i^{\text{th}}$  piece of a function definition into a conditional expression of the form: `if  $pred_i$  then  $expression_i$` . If any new piece has a  $pred_i$  that is identical<sup>2</sup> to an earlier  $pred_j$ , the earlier piece is removed. Otherwise, the new piece is always added at the end.

If there are `n` pieces to a function definition for `f`, the function defined `f` is:

```

if  $pred_1$  then  $expression_1$  else
    . . .
if  $pred_n$  then  $expression_n$  else
error "You did not define f for argument <arg>."

```

You can give definitions of any number of mutually recursive function definitions, piece-wise or otherwise. No computation is done until you ask for a value. When you do ask for a value, all the relevant definitions are gathered, analyzed, and translated into separate functions and compiled.

Let's recall the definition of `eleven` from the previous section.

```

eleven(n | n < 1) == n + 11

eleven(m) == eleven(eleven(m - 12))

```

Type: Void

Type: Void

A similar doubly-recursive function below produces `-11` for all negative positive integers. If you haven't worked out why or how `eleven` works, the structure of this definition gives a clue.

---

<sup>2</sup>after all variables are uniformly named

This definition we write as a block.

```
minusEleven(n) ==
  n >= 0 => n - 11
  minusEleven (5 + minusEleven(n + 7))
```

Type: Void

Define  $s(n)$  to be the sum of plus and minus “eleven” functions divided by  $n$ . Since  $11 - 11 = 0$ , we define  $s(0)$  to be 1.

```
s(0) == 1
```

Type: Void

And the general term.

```
s(n) == (eleven(n) + minusEleven(n))/n
```

Type: Void

What are the first ten values of  $s$ ?

```
[s(n) for n in 0..]
Compiling function eleven with type Integer ->
  Integer
+++ |*1;eleven;1;initial| redefined
Compiling function minusEleven with type Integer ->
  Integer
Compiling function s with type NonNegativeInteger ->
  Fraction Integer
+++ |*1;s;1;initial| redefined
[1, 1, 1, 1, 1, 1, 1, ...]
```

(6)

Type: Stream Fraction Integer

AXIOM can create infinite streams in the positive direction (for example, for index values 0,1, ...) or negative direction (for example, for index values 0,-1,-2, ...). Here we would like a stream of values of  $s(n)$  that is infinite in both directions. The function  $t(n)$  below returns the  $n^{\text{th}}$  term of the infinite stream  $[s(0), s(1), s(-1), s(2), s(-2), \dots]$ . Its definition has three pieces.

Define the initial term.

```
t(1) == s(0)
```

Type: Void

The even numbered terms are the  $s(i)$  for positive  $i$ . We use “quo” rather than “/” since we want the result to be an integer.

```
t(n | even?(n)) == s(n quo 2)
```

Type: Void

Finally, the odd numbered terms are the  $s(i)$  for negative  $i$ . In piece-wise definitions, you can use different variables to define different pieces. AXIOM will not get confused.

Look at the definition of  $t$ . In the first piece, the variable  $n$  was used; in the second piece,  $p$ . AXIOM always uses your last variable to display your definitions back to you.

Create a series of values of  $s$  applied to alternating positive and negative arguments.

Evidently  $t(n) = 1$  for all  $i$ . Check it at  $n = 100$ .

```
t(p) == s(- p quo 2)
```

Type: Void

```
)display value t
Definition:
  t 1 == s(0)
  t (p | even?(p)) == s(p quo 2)
  t p == s(- p quo 2)
```

```
[t(i) for i in 1..]
Compiling function s with type Integer -> Fraction
Integer
Compiling function t with type PositiveInteger ->
Fraction Integer
```

```
[1, 1, 1, 1, 1, 1, 1, ...] (10)
```

Type: Stream Fraction Integer

```
t(100)
1 (11)
```

Type: Fraction Integer

### 6.11.3 Predicates

We have already seen some examples of predicates (Section 6.11.1 on page 195). Predicates are Boolean-valued expressions and AXIOM uses them for filtering collections (see Section 5.5 on page 171) and for placing constraints on function arguments. In this section we discuss their latter usage.

The simplest use of a predicate is one you don't see at all.

```
opposite 'right == 'left
```

Type: Void

Here is a longer way to give the "opposite definition."

```
opposite (x | x = 'left) == 'right
```

Type: Void

Try it out.

```
for x in ['right','left','inbetween'] repeat output opposite
x
Compiling function opposite with type
  OrderedVariableList [right,left,inbetween] ->
  Symbol
left
right

The function opposite is not defined for the given
argument(s).
```

Explicit predicates tell AXIOM that the given function definition piece is to be applied if the predicate evaluates to **true** for the arguments to the function. You can use such “constant” arguments for integers, strings, and quoted symbols. The Boolean values **true** and **false** can also be used if qualified with “@” or “\$” and Boolean. The following are all valid function definition fragments using constant arguments.

```
a(1) == ...
b("unramified") == ...
c('untested') == ...
d(true@Boolean) == ...
```

If a function has more than one argument, each argument can have its own predicate. However, if a predicate involves two or more arguments, it must be given *after* all the arguments mentioned in the predicate have been given. You are always safe to give a single predicate at the end of the argument list.

A function involving predicates on two arguments.

```
inFirstHalfQuadrant(x | x > 0,y | y < x) == true
```

Type: Void

This is incorrect as it gives a predicate on y before the argument y is given.

```
inFirstHalfQuadrant(x | x > 0 and y < x,y) == true
1 old definition(s) deleted for function or rule
inFirstHalfQuadrant
```

Type: Void

It is always correct to write the predicate at the end.

```
inFirstHalfQuadrant(x,y | x > 0 and y < x) == true
1 old definition(s) deleted for function or rule
inFirstHalfQuadrant
```

Type: Void

Here is the rest of the definition.

```
inFirstHalfQuadrant(x,y) == false
```

Type: Void

Try it out.

```
[inFirstHalfQuadrant(i,3) for i in 1..5]
```

```
Compiling function inFirstHalfQuadrant with type (  
  PositiveInteger,PositiveInteger) -> Boolean
```

```
[false, false, false, true, true]
```

(7)

Type: List Boolean

**Remark:** Very old versions of AXIOM allowed predicates to be given after a **when** keyword as in `inFirstHalfQuadrant(x ,y) == true when x >0 and y < x`. This is no longer supported, is **WRONG**, and will cause a syntax error or strange behavior.

## 6.12 Caching Previously Computed Results

By default, AXIOM does not save the values of any function. You can cause it to save values and not to recompute unnecessarily by using `)set functions cache`. This should be used before the functions are defined or, at least, before they are executed. The word following “cache” should be 0 to turn off caching, a positive integer `n` to save the last `n` computed values or “all” to save all computed values. If you then give a list of names of functions, the caching only affects those functions. Use no list of names or “all” when you want to define the default behavior for functions not specifically mentioned in other `)set functions cache` statements. If you give no list of names, all functions will have the caching behavior. If you explicitly turn on caching for one or more names, you must explicitly turn off caching for those names when you want to stop saving their values.

This causes the functions `f` and `g` to have the last three computed values saved.

```
)set functions cache 3 f g
function f will cache the last 3 values.
function g will cache the last 3 values.
```

This is a sample definition for `f`.

```
f x == factorial(2**x)
```

Type: Void

A message is displayed stating what `f` will cache.

```
f(4)
Compiling function f with type PositiveInteger ->
Integer
f will cache 3 most recently computed value(s).
+++ |*1;f;1;initial| redefined
20922789888000
```

(2)

Type: PositiveInteger

This causes all other functions to have all computed values saved by default.

```
)set functions cache all
In general, interpreter functions will cache all
values.
```

This causes all functions that have not been specifically cached in some way to have no computed values saved.

```
)set functions cache 0
In general, functions will cache no returned values.
```

We also make `f` and `g` uncached.

```
)set functions cache 0 f g
Caching for function f is turned off
Caching for function g is turned off
```



Be careful about caching functions that have *side effects*. Such a function might destructively modify the elements of an array or issue a **draw** command, for example. A function that you expect to execute every time it is called should not be cached. Also, it is highly unlikely that a function with no arguments should be cached.

You should also be careful about caching functions that depend on free variables. See Section 6.16 on page 213 for an example.

## 6.13 Recurrence Relations

---

One of the most useful classes of function are those defined via a “recurrence relation.” A *recurrence relation* makes each successive value depend on some or all of the previous values. A simple example is the ordinary “factorial” function:

```
fact(0) == 1
fact(n | n > 0) == n * fact(n-1)
```

The value of `fact(10)` depends on the value of `fact(9)`, `fact(9)` on `fact(8)`, and so on. Because it depends on only one previous value, it is usually called a *first order recurrence relation*. You can easily imagine a function based on two, three or more previous values. The Fibonacci numbers are probably the most famous function defined by a second order recurrence relation.

The library function **fibonacci** computes Fibonacci numbers. It is obviously optimized for speed.

```
[fibonacci(i) for i in 0..]
```

```
[0, 1, 1, 2, 3, 5, 8, ...] (1)
```

Type: Stream Integer

Define the Fibonacci numbers ourselves using a piece-wise definition.

```
fib(1) == 1
```

Type: Void

```
fib(2) == 1
```

Type: Void

```
fib(n) == fib(n-1) + fib(n-2)
```

Type: Void

As defined, this recurrence relation is obviously doubly-recursive. To compute `fib(10)`, we need to compute `fib(9)` and `fib(8)`. And to `fib(9)`, we need to compute `fib(8)` and `fib(7)`. And so on. It seems that to compute `fib(10)` we need to compute `fib(9)` once, `fib(8)` twice, `fib(7)` three times. Look familiar? The number of function calls needed to compute *any* second order recurrence relation in the obvious way is exactly `fib(n)`. These numbers grow! For example, if AXIOM actually did this, then `fib(500)` requires more than  $10^{104}$  function calls. And, given all this, our definition of **fib** obviously could not be used to calculate the five-hundredth Fibonacci number.

Let's try it anyway.

```
fib(500)
Compiling function fib with type Integer ->
  PositiveInteger
Compiling function fib as a recurrence relation.
1394232245616978801397243828704072839500702565876973072
64108962948325571622863290691557658876222521294125
Type: PositiveInteger
```

Since this takes a short time to compute, it obviously didn't do as many as  $10^{104}$  operations! By default, AXIOM transforms any recurrence relation it recognizes into an iteration. Iterations are efficient. To compute the value of the  $n^{\text{th}}$  term of a recurrence relation using an iteration requires only  $n$  function calls.<sup>3</sup>

To turn off this special recurrence relation compilation, issue

```
)set functions recurrence off
```

To turn it back on, substitute "on" for "off".

The transformations that AXIOM uses for **fib** caches the last two values.<sup>4</sup> If, after computing a value for **fib**, you ask for some larger value, AXIOM picks up the cached values and continues computing from there. See Section 6.16 on page 213 for an example of a function definition that has this same behavior. Also see Section 6.12 on page 202 for a more general discussion of how you can cache function values.

Recurrence relations can be used for defining recurrence relations involving polynomials, rational functions, or anything you like. Here we compute the infinite stream of Legendre polynomials.

The Legendre polynomial of degree 0.

```
p(0) == 1
```

Type: Void

The Legendre polynomial of degree 1.

```
p(1) == x
```

Type: Void

The Legendre polynomial of degree  $n$ .

```
p(n) == ((2*n-1)*x*p(n-1) - (n-1)*p(n-2))/n
```

Type: Void

---

<sup>3</sup>If you compare the speed of our **fib** function to the library function, our version is still slower. This is because the library **fibonacci** uses a "powering algorithm" with a computing time proportional to  $\log^3(n)$  to compute **fibonacci**( $n$ ).

<sup>4</sup>For a more general  $k^{\text{th}}$  order recurrence relation, AXIOM caches the last  $k$  values.

Compute the Legendre  
polynomial of degree 6.

`p(6)`

Compiling function p with type Integer -> Polynomial  
Fraction Integer

Compiling function p as a recurrence relation.

$$\frac{231}{16} x^6 - \frac{315}{16} x^4 + \frac{105}{16} x^2 - \frac{5}{16} \quad (9)$$

Type: Polynomial Fraction Integer

## 6.14 Making Functions from Objects

There are many times when you compute a complicated expression and then wish to use that expression as the body of a function. AXIOM provides an operation called **function** to do this. It creates a function object and places it into the workspace. There are several versions, depending on how many arguments the function has. The first argument to **function** is always the expression to be converted into the function body, and the second is always the name to be used for the function. For more information, see ‘MakeFunction’ on page 494.

Start with a simple example of a polynomial in three variables.

```
p := -x + y**2 - z**3
```

$$-z^3 + y^2 - x$$
(1)

Type: Polynomial Integer

To make this into a function of no arguments that simply returns the polynomial, use the two argument form of **function**.

```
function(p, 'f0)
```

$$f0$$
(2)

Type: Symbol

To avoid possible conflicts (see below), it is a good idea to quote always this second argument.

```
f0
```

$$f0 () == -z^3 + y^2 - x$$
(3)

Type: FunctionCalled f0

This is what you get when you evaluate the function.

```
f0()
```

Compiling function f0 with type () -> Polynomial Integer

$$-z^3 + y^2 - x$$
(4)

Type: Polynomial Integer

To make a function in **x**, use a version of **function** that takes three arguments. The last argument is the name of the variable to use as the parameter. Typically, this variable occurs in the expression and, like the function name, you should quote it to avoid possible confusion.

```
function(p, 'f1, 'x)
```

$$f1$$
(5)

Type: Symbol

This is what the new function looks like.

```
f1
```

$$f1 x == -z^3 + y^2 - x$$
(6)

Type: FunctionCalled f1

This is the value of **f1** at **x = 3**. Notice that the return type of the function is Polynomial (Integer), the same as **p**.

```
f1(3)
```

Compiling function f1 with type PositiveInteger -> Polynomial Integer

$$-z^3 + y^2 - 3$$
(7)

Type: Polynomial Integer

To use **x** and **y** as parameters, use the four argument form of **function**.

```
function(p,'f2','x','y')
f2
Type: Symbol
```

(8)

```
f2
f2 (x, y) == -z3 + y2 - x
Type: FunctionCalled f2
```

(9)

Evaluate **f2** at **x** = 3 and **y** = 0. The return type of **f2** is still Polynomial(Integer) because the variable **z** is still present and not one of the parameters.

```
f2(3,0)
Compiling function f2 with type (PositiveInteger,
NonNegativeInteger) -> Polynomial Integer
-z3 - 3
Type: Polynomial Integer
```

(10)

Finally, use all three variables as parameters. There is no five argument form of **function**, so use the one with three arguments, the third argument being a list of the parameters.

```
function(p,'f3,['x','y','z'])
f3
Type: Symbol
```

(11)

Evaluate this using the same values for **x** and **y** as above, but let **z** be -6. The result type of **f3** is Integer.

```
f3
f3 (x, y, z) == -z3 + y2 - x
Type: FunctionCalled f3
```

(12)

```
f3(3,0,-6)
Compiling function f3 with type (PositiveInteger,
NonNegativeInteger,Integer) -> Integer
213
Type: PositiveInteger
```

(13)

The four functions we have defined via **p** have been undeclared. To declare a function whose body is to be generated by **function**, issue the declaration *before* the function is created.

```
g: (Integer, Integer) -> Float
Type: Void
```

```
D(sin(x-y)/cos(x+y),x)
-sin(y-x) sin(y+x) + cos(y-x) cos(y+x)
cos(y+x)2
Type: Expression Integer
```

(15)

```
function(%, 'g, 'x, 'y)
```

$g$  (16)

Type: Symbol

```
g
```

$$g(x, y) == \frac{-\sin(y-x) \sin(y+x) + \cos(y-x) \cos(y+x)}{\cos(y+x)^2} \quad (17)$$

Type: FunctionCalled g

It is an error to use `g` without the quote in the penultimate expression since `g` had been declared but did not have a value. Similarly, since it is common to overuse variable names like `x`, `y`, and so on, you avoid problems if you always quote the variable names for **function**. In general, if `x` has a value and you use `x` without a quote in a call to **function**, then AXIOM does not know what you are trying to do.

What kind of object is allowable as the first argument to **function**? Let's use the Browse facility of HyperDoc to find out. At the main Browse menu, enter the string **function** and then click on **Operations**. The exposed operations called **function** all take an object whose type belongs to category `ConvertibleTo InputForm`. What domains are those? Go back to the main Browse menu, erase **function**, enter `ConvertibleTo` in the input area, and click on **categories** on the **Constructors** line. At the bottom of the page, enter `InputForm` in the input area following **S =**. Click on **Cross Reference** and then on **Domains**. The list you see contains over forty domains that belong to the category `ConvertibleTo InputForm`. Thus you can use **function** for Integer, Float, String, Complex, Expression, and so on.

## 6.15 Functions Defined with Blocks

Here is a short function that swaps two elements of a list, array or vector.

```
swap(m,i,j) ==
  temp := m.i
  m.i := m.j
  m.j := temp
```

Type: Void

The significance of **swap** is that it has a destructive effect on its first argument.

```
k := [1,2,3,4,5]
[1, 2, 3, 4, 5]
```

(2)

Type: List PositiveInteger

```
swap(k,2,4)
```

```
Compiling function swap with type (List
  PositiveInteger,PositiveInteger,PositiveInteger)
  -> PositiveInteger
```

```
2
```

(3)

Type: PositiveInteger

You see that the second and fourth elements are interchanged.

```
k
[1, 4, 3, 2, 5]
```

(4)

Type: List PositiveInteger

Using this, we write a couple of different sort functions. First, a simple bubble sort. The operation “#” returns the number of elements in an aggregate.

```
bubbleSort(m) ==
  n := #m
  for i in 1..(n-1) repeat
    for j in n..(i+1) by -1 repeat
      if m.j < m.(j-1) then swap(m,j,j-1)
  m
```

Type: Void

Let this be the list we want to sort.

```
m := [8,4,-3,9]
[8, 4, -3, 9]
```

(6)

Type: List Integer

This is the result of sorting.

```
bubbleSort(m)
Compiling function swap with type (List Integer,
  Integer,Integer) -> Integer
Compiling function bubbleSort with type List Integer
  -> List Integer
[-3, 4, 8, 9]
```

(7)

Type: List Integer



Moreover, `m` is destructively changed to be the sorted version.

```
m
[-3, 4, 8, 9]
(8)
Type: List Integer
```

This function implements an insertion sort. The basic idea is to traverse the list and insert the  $i^{\text{th}}$  element in its correct position among the  $i-1$  previous elements. Since we start at the beginning of the list, the list elements before the  $i^{\text{th}}$  element have already been placed in ascending order.

```
insertionSort(m) ==
  for i in 2..#m repeat
    j := i
    while j > 1 and m.j < m.(j-1) repeat
      swap(m,j,j-1)
      j := j - 1
  m
Type: Void
```

As with our bubble sort, this is a destructive function.

```
m := [8,4,-3,9]
[8, 4, -3, 9]
(10)
Type: List Integer
```

```
insertionSort(m)
Compiling function swap with type (List Integer,
  NonNegativeInteger,Integer) -> Integer
Compiling function insertionSort with type List
  Integer -> List Integer
[-3, 4, 8, 9]
(11)
Type: List Integer
```

```
m
[-3, 4, 8, 9]
(12)
Type: List Integer
```

Neither of the above functions is efficient for sorting large lists since they reference elements by asking for the  $j^{\text{th}}$  element of the structure `m`.

Here is a more efficient bubble sort for lists.

```
bubbleSort2(m: List Integer): List Integer ==
  null m => m
  l := m
  while not null (r := l.rest) repeat
    r := bubbleSort2 r
    x := l.first
    if x < r.first then
      l.first := r.first
      r.first := x
    l.rest := r
    l := l.rest
  m
Function declaration bubbleSort2 : List Integer ->
  List Integer has been added to workspace.
Type: Void
```

Try it out.

```
bubbleSort2 [3,7,2]
```

```
Compiling function bubbleSort2 with type List Integer  
-> List Integer
```

```
[7, 3, 2]
```

(14)

Type: List Integer

This definition is both recursive and iterative, and is tricky! Unless you are *really* curious about this definition, we suggest you skip immediately to the next section.

Here are the key points in the definition. First notice that if you are sorting a list with less than two elements, there is nothing to do: just return the list. This definition returns immediately if there are zero elements, and skips the entire **while** loop if there is just one element.

The second point to realize is that on each outer iteration, the bubble sort ensures that the minimum element is propagated leftmost. Each iteration of the **while** loop calls **bubbleSort2** recursively to sort all but the first element. When finished, the minimum element is either in the first or second position. The conditional expression ensures that it comes first. If it is in the second, then a swap occurs. In any case, the **rest** of the original list must be updated to hold the result of the recursive call.

## 6.16 Free and Local Variables

---

This is a global workspace variable.

```
counter := 0
0
```

(1)  
Type: NonNegativeInteger

This function refers to the global `counter`.

```
f() ==
  free counter
  counter := counter + 1
```

Type: Void

The global `counter` is incremented by 1.

```
f()
Compiling function f with type () ->
  NonNegativeInteger
1
```

(3)  
Type: PositiveInteger

```
counter
1
```

(4)  
Type: NonNegativeInteger

Usually AXIOM can tell that you mean to refer to a global variable and so `free` isn't always necessary. However, for clarity and the sake of self-documentation, we encourage you to use it.

Declare a variable to be “local” when you do not want to refer to a global variable by the same name.

This function uses `counter` as a local variable.

```
g() ==
  local counter
  counter := 7
```

Type: Void

Apply the function.

```
g()
Compiling function g with type () -> PositiveInteger
7
```

(6)  
Type: PositiveInteger

Check that the global value of `counter` is unchanged.

```
counter
1
```

(7)  
Type: NonNegativeInteger

Parameters to a function are local variables in the function. Even if you issue a **free** declaration for a parameter, it is still local.

What happens if you do not declare that a variable **x** in the body of your function is **local** or **free**? Well, AXIOM decides on this basis:

1. AXIOM scans your function line-by-line, from top-to-bottom. The right-hand side of an assignment is looked at before the left-hand side.
2. If **x** is referenced before it is assigned a value, it is a **free** (global) variable.
3. If **x** is assigned a value before it is referenced, it is a **local** variable.

Set two global variables to 1.

```
a := b := 1
```

```
1
```

(8)

Type: PositiveInteger

Refer to **a** before it is assigned a value, but assign a value to **b** before it is referenced.

```
h() ==
  b := a + 1
  a := b + a
```

Type: Void

Can you predict this result?

```
h()
```

```
Compiling function h with type () -> PositiveInteger
```

```
3
```

(10)

Type: PositiveInteger

How about this one?

```
[a, b]
```

```
[3, 1]
```

(11)

Type: List PositiveInteger

What happened? In the first line of the function body for **h**, **a** is referenced on the right-hand side of the assignment. Thus **a** is a free variable. The variable **b** is not referenced in that line, but it is assigned a value. Thus **b** is a local variable and is given the value  $a + 1 = 2$ . In the second line, the free variable **a** is assigned the value  $b + a$  which equals  $2 + 1 = 3$ . This is the value returned by the function. Since **a** was free in **h**, the global variable **a** has value 3. Since **b** was local in **h**, the global variable **b** is unchanged—it still has the value 1.

It is good programming practice always to declare global variables. However, by far the most common situation is to have local variables in your functions. No declaration is needed for this situation, but be sure to initialize their values.

Be careful if you use free variables and you cache the value of your function (see Section 6.12 on page 202). Caching *only* checks if the values of the function arguments are the same as in a function call previously seen. It does not check if any of the free variables on which the function depends have changed between function calls.

Turn on caching for **p**.

```
)set fun cache all p
function p will cache all values.
```

Define **p** to depend on the free variable **N**.

```
p(i,x) == ( free N; reduce( + , [ (x-i)**n for n in 1..N ]
) )
```

Type: Void

Set the value of **N**.

```
N := 1
1
```

(13)

Type: PositiveInteger

Evaluate **p** the first time.

```
p(0, x)
Compiling function p with type (NonNegativeInteger,
Variable x) -> Polynomial Integer
p will cache all previously computed values.
```

$x$  (14)

Type: Polynomial Integer

Change the value of **N**.

```
N := 2
2
```

(15)

Type: PositiveInteger

Evaluate **p** the second time.

```
p(0, x)
 $x$ 
```

(16)

Type: Polynomial Integer

If caching had been turned off, the second evaluation would have reflected the changed value of **N**.

Turn off caching for **p**.

```
)set fun cache 0 p
Caching for function p is turned off
```

AXIOM does not allow *fluid variables*, that is, variables *bound* by a function **f** that can be referenced by functions called by **f**.

Values are passed to functions by *reference*: a pointer to the value is passed rather than a copy of the value or a pointer to a copy.

This is a global variable that is bound to a record object.

```
r : Record(i : Integer) := [1]
[i = 1]
(17)
```

Type: Record(i: Integer)

This function first modifies the one component of its record argument and then rebinds the parameter to another record.

```
resetRecord rr ==
  rr.i := 2
  rr := [10]
```

Type: Void

Pass **r** as an argument to **resetRecord**.

```
resetRecord r
Compiling function resetRecord with type Record(i:
  Integer) -> Record(i: Integer)
[i = 10]
(19)
```

Type: Record(i: Integer)

The value of **r** was changed by the expression **rr.i := 2** but not by **rr := [10]**.

```
r
[i = 2]
(20)
```

Type: Record(i: Integer)

To conclude this section, we give an iterative definition of a function that computes Fibonacci numbers. This definition approximates the definition into which AXIOM transforms the recurrence relation definition of **fib** in Section 6.13 on page 204.

Global variables **past** and **present** are used to hold the last computed Fibonacci numbers.

```
past := present := 1
1
(21)
```

Type: PositiveInteger

Global variable **index** gives the current index of **present**.

```
index := 2
2
(22)
```

Type: PositiveInteger

Here is a recurrence relation defined in terms of these three global variables.

```
fib(n) ==
  free past, present, index
  n < 3 => 1
  n = index - 1 => past
  if n < index-1 then
    (past,present) := (1,1)
    index := 2
  while (index < n) repeat
    (past,present) := (present, past+present)
    index := index + 1
  present
```

Type: Void

Compute the infinite stream of Fibonacci numbers.

```

fibs := [fib(n) for n in 1..]
Compiling function fib with type PositiveInteger ->
PositiveInteger

+++ |*1;fib;1;initial| redefined

[1, 1, 2, 3, 5, 8, 13, ...]
Type: Stream PositiveInteger

```

(24)

What is the 1000th Fibonacci number?

```

fibs 1000
4346655768693745643568852767504062580256466051737178040
2481729089536555417949051890403879840079255169295922593
0803226347752096896232398733224711616429964409065331879
38298969649928516003704476137795166849228875
Type: PositiveInteger

```

(25)

As an exercise, we suggest you write a function in an iterative style that computes the value of the recurrence relation  $p(n) = p(n-1) - 2p(n-2) + 4p(n-3)$  having the initial values  $p(1) = 1$ ,  $p(2) = 3$  and  $p(3) = 9$ . How would you write the function using an element `OneDimensionalArray` or `Vector` to hold the previously computed values?

## 6.17 Anonymous Functions

An *anonymous function* is a function that is defined by giving a list of parameters, the “maps-to” compound symbol “+>” (from the mathematical symbol  $\mapsto$ ), and by an expression involving the parameters, the evaluation of which determines the return value of the function.

$$( \textit{parm}_1, \textit{parm}_2, \dots, \textit{parm}_N ) +> \textit{expression}$$

You can apply an anonymous function in several ways.

1. Place the anonymous function definition in parentheses directly followed by a list of arguments.
2. Assign the anonymous function to a variable and then use the variable name when you would normally use a function name.
3. Use “==” to use the anonymous function definition as the arguments and body of a regular function definition.
4. Have a named function contain a declared anonymous function and use the result returned by the named function.

### 6.17.1 Some Examples

Anonymous functions are particularly useful for defining functions “on the fly.” That is, they are handy for simple functions that are used only in one place. In the following examples, we show how to write some simple anonymous functions.

This is a simple absolute value function.

```
x +> if x < 0 then -x else x
x ↦ if x < 0 then -x
      else x
```

Type: AnonymousFunction

```
abs1 := %
x ↦ if x < 0 then -x
      else x
```

Type: AnonymousFunction

This function returns **true** if the absolute value of the first argument is greater than the absolute value of the second, **false** otherwise.

```
(x,y) +> abs1(x) > abs1(y)
(x, y) ↦ abs1(y) < abs1(x)
```

Type: AnonymousFunction

We use the above function to “sort” a list of integers.

```
sort(%, [3,9,-4,10,-3,-1,-9,5])
[10, -9, 9, 5, -4, -3, 3, -1]
```

Type: List Integer



This function returns 1 if  $i + j$  is even, -1 otherwise.

```
ev := ( (i,j) +-> if even?(i+j) then 1 else -1)
```

$$(i, j) \mapsto \begin{cases} 1 & \text{if } \text{even?}(i+j) \\ -1 & \text{otherwise} \end{cases} \quad (5)$$

Type: AnonymousFunction

We create a four-by-four matrix containing 1 or -1 depending on whether the row plus the column index is even or not.

```
matrix([[ev(row,col) for row in 1..4] for col in 1..4])
```

$$\begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix} \quad (6)$$

Type: Matrix Integer

This function returns **true** if a polynomial in  $x$  has multiple roots, **false** otherwise. It is defined and applied in the same expression.

```
( p +-> not one?(gcd(p,D(p,x))) ) (x**2+4*x+4)
```

$$\text{true} \quad (7)$$

Type: Boolean

This and the next expression are equivalent.

```
g(x,y,z) == cos(x + sin(y + tan(z)))
```

Type: Void

The one you use is a matter of taste.

```
g == (x,y,z) +-> cos(x + sin(y + tan(z)))
```

```
1 old definition(s) deleted for function or rule g
```

Type: Void

## 6.17.2 Declaring Anonymous Functions

This is an example of a fully declared anonymous function. The output shown just indicates that the object you created is a particular kind of map, that is, function.

AXIOM allows you to declare the arguments and not declare the return type.

If you declare any of the arguments you must declare all of them. Thus,

```
(x: INT,y): FRAC INT +-> (x + 2*y)/(y - 1)
```

is not legal.

```
(x: INT,y: INT): FRAC INT +-> (x + 2*y)/(y - 1)
```

```
theMap (...) \quad (1)
```

Type: ((Integer, Integer) → Fraction Integer)

```
(x: INT,y: INT) +-> (x + 2*y)/(y - 1)
```

```
theMap (...) \quad (2)
```

Type: ((Integer, Integer) → Fraction Integer)

The return type is computed from the types of the arguments and the body of the function. You cannot declare the return type if you do not declare the arguments. Therefore,

```
(x,y): FRAC INT +-> (x + 2*y)/(y - 1)
```

is not legal.

This and the next expression are equivalent.

```
h(x: INT,y: INT): FRAC INT == (x + 2*y)/(y - 1)
Function declaration h : (Integer,Integer) ->
  Fraction Integer has been added to workspace.
```

Type: Void

The one you use is a matter of taste.

```
h == (x: INT,y: INT): FRAC INT +-> (x + 2*y)/(y - 1)
Function declaration h : (Integer,Integer) ->
  Fraction Integer has been added to workspace.
1 old definition(s) deleted for function or rule h
```

Type: Void

When should you declare an anonymous function?

1. If you use an anonymous function and AXIOM can't figure out what you are trying to do, declare the function.
2. If the function has nontrivial argument types or a nontrivial return type that AXIOM may be able to determine eventually, but you are not willing to wait that long, declare the function.
3. If the function will only be used for arguments of specific types and it is not too much trouble to declare the function, do so.
4. If you are using the anonymous function as an argument to another function (such as **map** or **sort**), consider declaring the function.
5. If you define an anonymous function inside a named function, you *must* declare the anonymous function.

This is an example of a named function for integers that returns a function.

```
addx x == ((y: Integer): Integer +-> x + y)
```

Type: Void

We define **g** to be a function that adds 10 to its argument.

```
g := addx 10
Compiling function addx with type PositiveInteger ->
  (Integer -> Integer)
```

```
theMap (...) (6)
```

Type: (Integer → Integer)

Try it out.

```
g 3
```

```
13 (7)
```

Type: PositiveInteger

```
g (-4)
```

```
6 (8)
```

Type: PositiveInteger

An anonymous function cannot be recursive: since it does not have a name, you cannot even call it within itself! If you place an anonymous function inside a named function, the anonymous function must be declared.

## 6.18

### Example: A Database

---

The database is entered as “assertions” that are really pieces of a function definition.

This example shows how you can use AXIOM to organize a database of lineage data and then query the database for relationships.

```
children("albert") == ["albertJr","richard","diane"]
```

Type: Void

Each piece `children(x) == y` means “the children of `x` are `y`”.

```
children("richard") == ["douglas","daniel","susan"]
```

Type: Void

This family tree thus spans four generations.

```
children("douglas") == ["dougie","valerie"]
```

Type: Void

Say “no one else has children.”

```
children(x) == []
```

Type: Void

We need some functions for computing lineage. Start with `childOf`.

```
childOf(x,y) == member?(x,children(y))
```

Type: Void

To find the `parentOf` someone, you have to scan the database of people applying `children`.

```
parentOf(x) ==  
  for y in people repeat  
    (if childOf(x,y) then return y)  
  "unknown"
```

Type: Void

And a grandparent of `x` is just a parent of a parent of `x`.

```
grandParentOf(x) == parentOf parentOf x
```

Type: Void

The grandchildren of `x` are the people `y` such that `x` is a grandparent of `y`.

```
grandchildren(x) == [y for y in people | grandParentOf(y)  
  = x]
```

Type: Void

Suppose you want to make a list of all great-grandparents. Well, a great-grandparent is a grandparent of a person who has children.

```
greatGrandParents == [x for x in people |  
  reduce(_or,[not empty? children(y) for y in  
  grandchildren(x)],false)]
```

Type: Void

Define `descendants` to include the parent as well.

```
descendants(x) ==
  kids := children(x)
  null kids => [x]
  concat(x, reduce(concat, [descendants(y)
    for y in kids], []))
```

Type: Void

Finally, we need a list of people. Since all people are descendants of "albert", let's say so.

```
people == descendants "albert"
```

Type: Void

We have used "==" to define the database and some functions to query the database. But no computation is done until we ask for some information. Then, once and for all, the functions are analyzed and compiled to machine code for run-time efficiency. Notice that no types are given anywhere in this example. They are not needed.

Who are the grandchildren of "richard"?

```
grandchildren "richard"
Compiling function children with type String -> List
String
Compiling function descendants with type String ->
List String
Compiling body of rule people to compute value of
type List String
Compiling function childOf with type (String,String)
-> Boolean
Compiling function parentOf with type String ->
String
Compiling function grandParentOf with type String ->
String
Compiling function grandchildren with type String ->
List String
```

```
["dougie", "valerie"] (12)
```

Type: List String

Who are the great-grandparents?

```
greatGrandParents
Compiling body of rule greatGrandParents to compute
value of type List String
```

```
["albert"] (13)
```

Type: List String

## 6.19

### Example: A Famous Triangle

---

To make these output operations available, we have to *expose* the domain `OutputForm`. See Section 2.11 on page 124 for more information about exposing domains and packages.

Define the values along the first row and any column `i`.

Define the values for when the row and column index `i` are equal. Repeating the argument name indicates that the two index values are equal.

First, define a function that gives the  $n^{\text{th}}$  row.

Next, we write the function **displayRow** to display the row, separating entries by blanks and centering.

In this example we write some functions that display Pascal's triangle. It demonstrates the use of piece-wise definitions and some output operations you probably haven't seen before.

```
)set expose add constructor OutputForm
```

`OutputForm` is now explicitly exposed in frame `initial`

```
pascal(1,i) == 1
```

Type: Void

```
pascal(n,n) == 1
```

Type: Void

```
pascal(i,j | 1 < i and i < j) ==  
  pascal(i-1,j-1)+pascal(i,j-1)
```

Type: Void

Now that we have defined the coefficients in Pascal's triangle, let's write a couple of one-liners to display it.

```
pascalRow(n) == [pascal(i,n) for i in 1..n]
```

Type: Void

```
displayRow(n) == output center blankSeparate pascalRow(n)
```

Type: Void

Here we have used three output operations. Operation **output** displays the printable form of objects on the screen, **center** centers a printable form in the width of the screen, and **blankSeparate** takes a list of printable forms and inserts a blank between successive elements.

Look at the result.

```
for i in 1..7 repeat displayRow i
Compiling function pascal with type (Integer,Integer)
-> PositiveInteger
Compiling function pascalRow with type
PositiveInteger -> List PositiveInteger
Compiling function displayRow with type
PositiveInteger -> Void
```

```
      1
     1 1
    1 2 1
   1 3 3 1
  1 4 6 4 1
 1 5 10 10 5 1
1 6 15 20 15 6 1
```

Type: Void

Being purists, we find this less than satisfactory. Traditionally, elements of Pascal's triangle are centered between the left and right elements on the line above.

To fix this misalignment, we go back and redefine **pascalRow** to right adjust the entries within the triangle within a width of four characters.

```
pascalRow(n) == [right(pascal(i,n),4) for i in 1..n]
Compiled code for pascalRow has been cleared.
Compiled code for displayRow has been cleared.
1 old definition(s) deleted for function or rule
pascalRow
```

Type: Void

Finally let's look at our purely reformatted triangle.

```
for i in 1..7 repeat displayRow i
Compiling function pascalRow with type
PositiveInteger -> List OutputForm

+++ |*1;pascalRow;1;initial| redefined
Compiling function displayRow with type
PositiveInteger -> Void
```

```
+++ |*1;displayRow;1;initial| redefined
      1
     1 1
    1 2 1
   1 3 3 1
  1 4 6 4 1
 1 5 10 10 5 1
1 6 15 20 15 6 1
```

Type: Void

Unexpose OutputForm so we don't get unexpected results later.

```
)set expose drop constructor OutputForm
OutputForm is now explicitly hidden in frame initial
```

## 6.20

### Example: Testing for Palindromes

Here is the definition for **pal?**. It is simply a call to an auxiliary function called **palAux?**. We are following the convention of ending a function's name with “?” if the function returns a Boolean value.

Here is **palAux?**. It works by comparing elements that are equidistant from the start and end of the object.

Try **pal?** on some examples. First, a string.

A list of polynomials.

A list of integers from the example in the last section.

To use **pal?** on an integer, first convert it to a string.

In this section we define a function **pal?** that tests whether its argument is a *palindrome*, that is, something that reads the same backwards and forwards. For example, the string “Madam I’m Adam” is a palindrome (excluding blanks and punctuation) and so is the number 123454321. The definition works for any datatype that has **n** components that are accessed by the indices **1...n**.

```
pal? s == palAux?(s,1,#s)
```

Type: Void

```
palAux?(s,i,j) ==
  j > i =>
    (s.i = s.j) and palAux?(s,i+1,i-1)
  true
```

Type: Void

```
pal? "Oxford"
```

```
Compiling function palAux? with type (String,Integer,
Integer) -> Boolean
```

```
Compiling function pal? with type String -> Boolean
```

```
false
```

(3)

Type: Boolean

```
pal? [4,a,x-1,0,x-1,a,4]
```

```
Compiling function palAux? with type (List Polynomial
Integer,Integer,Integer) -> Boolean
```

```
Compiling function pal? with type List Polynomial
Integer -> Boolean
```

```
true
```

(4)

Type: Boolean

```
pal? [1,6,15,20,15,6,1]
```

```
Compiling function palAux? with type (List
PositiveInteger,Integer,Integer) -> Boolean
```

```
Compiling function pal? with type List
PositiveInteger -> Boolean
```

```
true
```

(5)

Type: Boolean

```
pal?(1441::String)
```

```
true
```

(6)

Type: Boolean



Compute an infinite stream of decimal numbers, each of which is an obvious palindrome.

```
ones := [reduce(+,[10**j for j in 0..i]) for i in 1..]
```

[11, 111, 1111, 11111, 111111, 1111111, 11111111, ...] (7)

Type: Stream PositiveInteger

How about their squares?

```
squares := [x**2 for x in ones]
```

[121, 12321, 1234321, 123454321, 12345654321, 1234567654321,  
123456787654321, 12345678987654321, 1234567900987654321, ...] (8)

Type: Stream PositiveInteger

Well, let's test them all!

```
[pal?(x::String) for x in squares]
```

[true, true, true, true, true, true, true, true, true, ...] (9)

Type: Stream Boolean

## 6.21 Rules and Pattern Matching

---

A common mathematical formula is

$$\log(x) + \log(y) = \log(xy) \quad \forall x \text{ and } y.$$

The presence of “ $\forall$ ” indicates that  $x$  and  $y$  can stand for arbitrary mathematical expressions in the above formula. You can use such mathematical formulas in AXIOM to specify “rewrite rules”. Rewrite rules are objects in AXIOM that can be assigned to variables for later use, often for the purpose of simplification. Rewrite rules look like ordinary function definitions except that they are preceded by the reserved word **rule**. For example, a rewrite rule for the above formula is:

```
rule log(x) + log(y) == log(x * y)
```

Like function definitions, no action is taken when a rewrite rule is issued. Think of rewrite rules as functions that take one argument. When a rewrite rule  $A = B$  is applied to an argument  $f$ , its meaning is: “rewrite every subexpression of  $f$  that *matches*  $A$  by  $B$ .” The left-hand side of a rewrite rule is called a *pattern*; its right-side side is called its *substitution*.

```
logrule := rule log(x) + log(y) == log(x * y)
```

```
log(y) + log(x) + %B == log(x y) + %B
```

Type: RewriteRule(Integer, Integer, Expression Integer)

Create a rewrite rule named **logrule**. The generated symbol beginning with a “%” is a place-holder for any other terms that might occur in the sum.

Create an expression with logarithms.

```
f := log sin x + log x
```

```
log(sin(x)) + log(x)
```

Type: Expression Integer

Apply **logrule** to  $f$ .

```
logrule f
```

```
log(x sin(x))
```

Type: Expression Integer

The meaning of our example rewrite rule is: “for all expressions  $x$  and  $y$ , rewrite  $\log(x) + \log(y)$  by  $\log(x * y)$ .” Patterns generally have both operation names (here, **log** and “+”) and variables (here,  $x$  and  $y$ ). By default, every operation name stands for itself. Thus **log** matches only “log” and not any other operation such as **sin**. On the other hand, variables do not stand for themselves. Rather, a variable denotes a *pattern variable* that is free to match any expression whatsoever.

When a rewrite rule is applied, a process called *pattern matching* goes to work by systematically scanning the subexpressions of the argument. When a subexpression is found that “matches” the pattern, the subexpression is replaced by the right-hand side of the rule. The details of what happens will be covered later.

The customary AXIOM notation for patterns is actually a shorthand for a

longer, more general notation. Pattern variables can be made explicit by using a percent (“%”) as the first character of the variable name. To say that a name stands for itself, you can prefix that name with a quote operator (“’”). Although the current AXIOM parser does not let you quote an operation name, this more general notation gives you an alternate way of giving the same rewrite rule:

```
rule log(%x) + log(%y) == log(x * y)
```

This longer notation gives you patterns that the standard notation won’t handle. For example, the rule

```
rule %f(c * ’x) == c*f(x)
```

means “for all *f* and *c*, replace *f*(*y*) by *c* \* *f*(*x*) when *y* is the product of *c* and the explicit variable *x*.”

Thus the pattern can have several adornments on the names that appear there. Normally, all these adornments are dropped in the substitution on the right-hand side.

To summarize:

To enter a single rule in AXIOM, use the following syntax:

```
rule leftHandSide == rightHandSide
```

The *leftHandSide* is a pattern to be matched and the *rightHandSide* is its substitution. The rule is an object of type RewriteRule that can be assigned to a variable and applied to expressions to transform them.

Rewrite rules can be collected into rulesets so that a set of rules can be applied at once. Here is another simplification rule for logarithms.

$$y \log(x) = \log(x^y) \quad \forall x \text{ and } y.$$

If instead of giving a single rule following the reserved word **rule** you give a “pile” of rules, you create what is called a *ruleset*. Like rules, rulesets are objects in AXIOM and can be assigned to variables. You will find it useful to group commonly used rules into input files, and read them in as needed.

Create a ruleset named  
**logrules**.

```
logrules := rule
  log(x) + log(y) == log(x * y)
  y * log x      == log(x ** y)
{log(y) + log(x) + %C == log(x y) + %C, y log(x) == log(x^y)}
Type: Ruleset(Integer, Integer, Expression Integer)
```

(4)

Again, create an expression **f** containing logarithms.

$$f := a * \log(\sin x) - 2 * \log x$$

$$a \log(\sin(x)) - 2 \log(x) \quad (5)$$

Type: Expression Integer

Apply the ruleset **logrules** to **f**.

$$\log\left(\frac{\sin(x)^a}{x^2}\right) \quad (6)$$

Type: Expression Integer

We have allowed pattern variables to match arbitrary expressions in the above examples. Often you want a variable only to match expressions satisfying some predicate. For example, we may want to apply the transformation

$$y \log(x) = \log(x^y)$$

only when **y** is an integer. The way to restrict a pattern variable **y** by a predicate **f(y)** is by using a vertical bar “|”, which means “such that,” in much the same way it is used in function definitions. You do this only once, but at the earliest (meaning deepest and leftmost) part of the pattern.

This restricts the logarithmic rule to create integer exponents only.

$$\begin{aligned} \text{logrules2} &:= \text{rule} \\ &\quad \log(x) + \log(y) == \log(x * y) \\ &\quad (y \mid \text{integer? } y) * \log x == \log(x ** y) \\ &\quad \{\log(y) + \log(x) + \%E == \log(x y) + \%E, y \log(x) == \log(x^y)\} \end{aligned} \quad (7)$$

Type: Ruleset(Integer, Integer, Expression Integer)

Compare this with the result of applying the previous set of rules.

$$f$$

$$a \log(\sin(x)) - 2 \log(x) \quad (8)$$

Type: Expression Integer

$$\text{logrules2 } f$$

$$a \log(\sin(x)) + \log\left(\frac{1}{x^2}\right) \quad (9)$$

Type: Expression Integer

You should be aware that you might need to apply a function like **integer** within your predicate expression to actually apply the test function.

Here we use **integer** because **n** has type Expression Integer but **even?** is an operation defined on integers.

$$\begin{aligned} \text{evenRule} &:= \text{rule } \cos(x)**(n \mid \text{integer? } n \text{ and even? integer} \\ &\quad n) == (1 - \sin(x)**2)**(n/2) \\ \cos(x)^n &= \left(-\sin(x)^2 + 1\right)^{\frac{n}{2}} \end{aligned} \quad (10)$$

Type: RewriteRule(Integer, Integer, Expression Integer)

Here is the application of the rule.

$$\text{evenRule}(\cos(x)^2) = -\sin(x)^2 + 1 \quad (11)$$

Type: Expression Integer

This is an example of some of the usual identities involving products of sines and cosines.

```
sinCosProducts == rule
  sin(x) * sin(y) == (cos(x-y) - cos(x + y))/2
  cos(x) * cos(y) == (cos(x-y) + cos(x+y))/2
  sin(x) * cos(y) == (sin(x-y) + sin(x + y))/2
```

Type: Void

$$g := \sin(a)\sin(b) + \cos(b)\cos(a) + \sin(2a)\cos(2a)$$

$$\sin(a)\sin(b) + \cos(2a)\sin(2a) + \cos(a)\cos(b) \quad (13)$$

Type: Expression Integer

```
sinCosProducts g
```

```
Compiling body of rule sinCosProducts to compute
  value of type Ruleset(Integer,Integer,Expression
  Integer)
```

$$\frac{\sin(4a) + 2\cos(b-a)}{2} \quad (14)$$

Type: Expression Integer

Another qualification you will often want to use is to allow a pattern to match an identity element. Using the pattern  $x + y$ , for example, neither  $x$  nor  $y$  matches the expression 0. Similarly, if a pattern contains a product  $x*y$  or an exponentiation  $x**y$ , then neither  $x$  or  $y$  matches 1.

If identical elements were matched, pattern matching would generally loop. Here is an expansion rule for exponentials.

```
exprule := rule exp(a + b) == exp(a) * exp(b)
```

$$e^{(b+a)} = e^a e^b \quad (15)$$

Type: RewriteRule(Integer, Integer, Expression Integer)

This rule would cause infinite rewriting on this if either  $a$  or  $b$  were allowed to match 0.

```
exprule exp x
```

$$e^x \quad (16)$$

Type: Expression Integer

There are occasions when you do want a pattern variable in a sum or product to match 0 or 1. If so, prefix its name with a “?” whenever it appears in a left-hand side of a rule. For example, consider the following rule for the exponential integral:

$$\int \left( \frac{y + e^x}{x} \right) dx = \int \frac{y}{x} dx + \text{Ei}(x) \quad \forall x \text{ and } y.$$

This rule is valid for  $y = 0$ . One solution is to create a Ruleset with two rules, one with and one without  $y$ . A better solution is to use an “optional” pattern variable.

Define rule `eirule` with a pattern variable `?y` to indicate that an expression may or may not occur.

```
eirule := rule integral((?y + exp x)/x,x) ==
integral(y/x,x) + Ei x
```

$$\int^x \frac{e^{\%N} + y}{\%N} d\%N = 'integral\left(\frac{y}{x}, x\right) + 'Ei(x) \quad (17)$$

Type: RewriteRule(Integer, Integer, Expression Integer)

Apply rule `eirule` to an integral without this term.

```
eirule integral(exp u/u, u)
```

$$Ei(u) \quad (18)$$

Type: Expression Integer

Apply rule `eirule` to an integral with this term.

```
eirule integral(sin u + exp u/u, u)
```

$$\int^u \sin(\%N) d\%N + Ei(u) \quad (19)$$

Type: Expression Integer

Here is one final adornment you will find useful. When matching a pattern of the form `x + y` to an expression containing a long sum of the form `a + ... + b`, there is no way to predict in advance which subset of the sum matches `x` and which matches `y`. Aside from efficiency, this is generally unimportant since the rule holds for any possible combination of matches for `x` and `y`. In some situations, however, you may want to say which pattern variable is a sum (or product) of several terms, and which should match only a single term. To do this, put a prefix colon “:” before the pattern variable that you want to match multiple terms.

The remaining rules involve operators `u` and `v`.

```
u := operator 'u
```

$$u \quad (20)$$

Type: BasicOperator

These definitions tell AXIOM that `u` and `v` are formal operators to be used in expressions.

```
v := operator 'v
```

$$v \quad (21)$$

Type: BasicOperator

First define `myRule` with no restrictions on the pattern variables `x` and `y`.

```
myRule := rule u(x + y) == u x + v y
```

$$u(y + x) = 'v(y) + 'u(x) \quad (22)$$

Type: RewriteRule(Integer, Integer, Expression Integer)

Apply `myRule` to an expression.

```
myRule u(a + b + c + d)
```

$$v(d + c + b) + u(a) \quad (23)$$

Type: Expression Integer

Define `myOtherRule` to match several terms so that the rule gets applied recursively.

```
myOtherRule := rule u(:x + y) == u x + v y
```

$$u(y + x) = 'v(y) + 'u(x) \quad (24)$$

Type: RewriteRule(Integer, Integer, Expression Integer)

Apply `myOtherRule` to the same expression.

$$\begin{aligned} &\text{myOtherRule } u(a + b + c + d) \\ &v(c) + v(b) + v(a) + u(d) \end{aligned} \tag{25}$$

Type: Expression Integer

Summary of pattern variable adornments:

<code>(x   predicate?(x))</code>	means that the substitution <code>s</code> for <code>x</code> must satisfy <code>predicate?(s) = true</code> .
<code>?x</code>	means that <code>x</code> can match an identity element (0 or 1).
<code>:x</code>	means that <code>x</code> can match several terms in a sum.

Here are some final remarks on pattern matching. Pattern matching provides a very useful paradigm for solving certain classes of problems, namely, those that involve transformations of one form to another and back. However, it is important to recognize its limitations.

First, pattern matching slows down as the number of rules you have to apply increases. Thus it is good practice to organize the sets of rules you use optimally so that irrelevant rules are never included.

Second, careless use of pattern matching can lead to wrong answers. You should avoid using pattern matching to handle hidden algebraic relationships that can go undetected by other programs. As a simple example, a symbol such as “J” can easily be used to represent the square root of -1 or some other important algebraic quantity. Many algorithms branch on whether an expression is zero or not, then divide by that expression if it is not. If you fail to simplify an expression involving powers of J to -1, algorithms may incorrectly assume an expression is non-zero, take a wrong branch, and produce a meaningless result.

Pattern matching should also not be used as a substitute for a domain. In AXIOM, objects of one domain are transformed to objects of other domains using well-defined **coerce** operations. Pattern matching should be used on objects that are all the same type. Thus if your application can be handled by type Expression in AXIOM and you think you need pattern matching, consider this choice carefully. You may well be better served by extending an existing domain or by building a new domain of objects for your application.





---

# Graphics

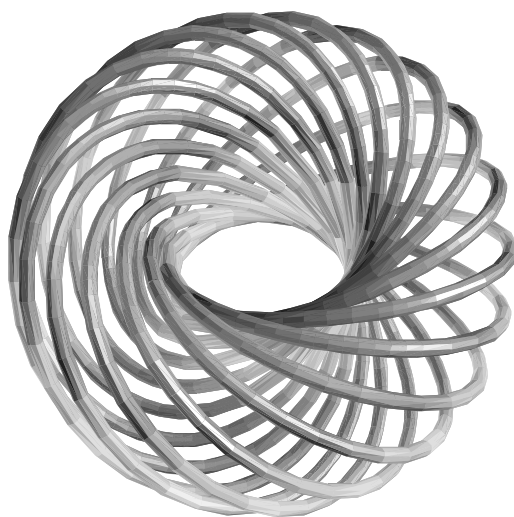


Figure 7.1: Torus knot of type (15,17).

This chapter shows how to use the AXIOM graphics facilities under the X Window System. AXIOM has two-dimensional and three-dimensional drawing and rendering packages that allow the drawing, coloring, transforming, mapping, clipping, and combining of graphic output from AXIOM computations. This facility is particularly useful for investigating problems in areas such as topology. The graphics package is capable of plotting functions of one or more variables or plotting parametric surfaces and curves. Various coordinate systems are also available, such as polar and spherical.

A graph is displayed in a viewport window and it has a control-panel that uses interactive mouse commands. PostScript and other output forms are available so that AXIOM images can be printed or used by other

programs.<sup>1</sup>

## 7.1 Two-Dimensional Graphics

---

The AXIOM two-dimensional graphics package provides the ability to display

- curves defined by functions of a single real variable
- curves defined by parametric equations
- implicit non-singular curves defined by polynomial equations
- planar graphs generated from lists of point components.

These graphs can be modified by specifying various options, such as calculating points in the polar coordinate system or changing the size of the graph viewport window.

### 7.1.1 Plotting Two-Dimensional Functions of One Variable

---

The first kind of two-dimensional graph is that of a curve defined by a function  $y = f(x)$  over a finite interval of the  $x$  axis.

The general format for drawing a function defined by a formula  $f(x)$  is:

`draw(f(x), x = a..b, options)`

where `a..b` defines the range of  $x$ , and where *options* prescribes zero or more options as described in Section 7.1.4 on page 243. An example of an option is `curveColor == bright red()`. An alternative format involving functions  $f$  and  $g$  is also available.

A simple way to plot a function is to use a formula. The first argument is the formula. For the second argument, write the name of the independent variable (here,  $x$ ), followed by an “=”, and the range of values.

---

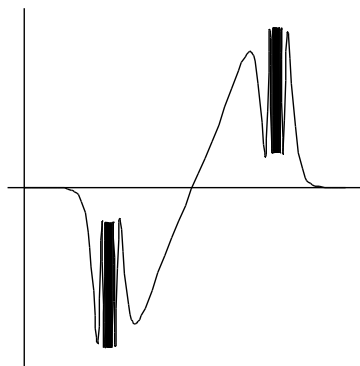
<sup>1</sup>PostScript is a trademark of Adobe Systems Incorporated, registered in the United States.

Display this formula over the range  $0 \leq x \leq 6$ . AXIOM converts your formula to a compiled function so that the results can be computed quickly and efficiently.

```
draw(sin(tan(x)) - tan(sin(x)),x = 0..6)
Compiling function %B with type DoubleFloat ->
DoubleFloat
Graph data being transmitted to the viewport
manager...
AXIOM2D data being transmitted to the viewport
manager...
```

TwoDimensionalViewport: "(-DTAN(DSIN(x)))+DSIN(DTAN(x))" (1)

Type: TwoDimensionalViewport



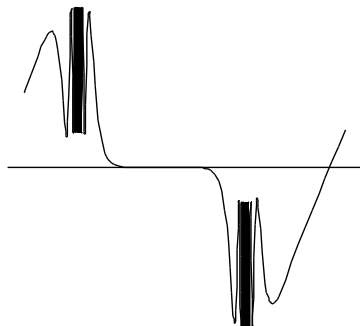
Notice that AXIOM compiled the function before the graph was put on the screen.

Here is the same graph on a different interval. This time we give the graph a title.

```
draw(sin(tan(x)) - tan(sin(x)),x = 10..16)
Compiling function %D with type DoubleFloat ->
DoubleFloat
Graph data being transmitted to the viewport
manager...
AXIOM2D data being transmitted to the viewport
manager...
```

TwoDimensionalViewport: "(-DTAN(DSIN(x)))+DSIN(DTAN(x))" (2)

Type: TwoDimensionalViewport



Once again the formula is converted to a compiled function before any points were computed. If you want to graph the same function on several intervals, it is a good idea to define the function first so that the function has to be compiled only once.

This time we first define the function.

```
f(x) == (x-1)*(x-2)*(x-3)
```

Type: Void

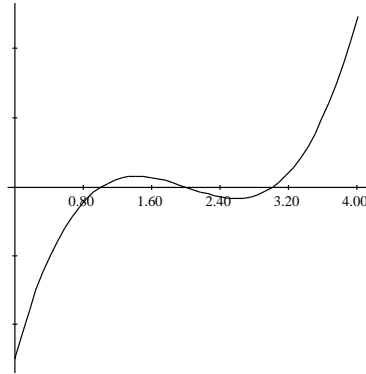
To draw the function, the first argument is its name and the second is just the range with no independent variable.

```
draw(f, 0..4)
Compiling function f with type DoubleFloat ->
  DoubleFloat
Graph data being transmitted to the viewport
manager...
AXIOM2D data being transmitted to the viewport
manager...
```

TwoDimensionalViewport: "AXIOM2D"

(4)

Type: TwoDimensionalViewport



### 7.1.2 Plotting Two-Dimensional Parametric Plane Curves

---

The second kind of two-dimensional graph is that of curves produced by parametric equations. Let  $x = f(t)$  and  $y = g(t)$  be formulas or two functions  $f$  and  $g$  as the parameter  $t$  ranges over an interval  $[a,b]$ . The function `curve` takes the two functions  $f$  and  $g$  as its parameters.

The general format for drawing a two-dimensional plane curve defined by parametric formulas  $x = f(t)$  and  $y = g(t)$  is:

```
draw(curve(f(t), g(t)), t = a..b, options)
```

where `a..b` defines the range of the independent variable  $t$ , and where *options* prescribes zero or more options as described in Section ?? on page ???. An example of an option is `curveColor == bright red()`.

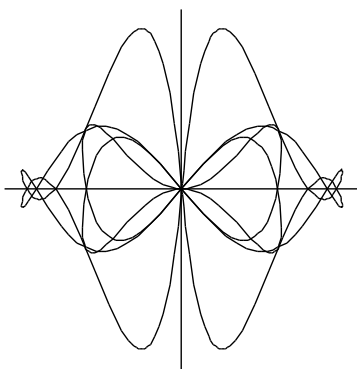
Here's an example:

Define a parametric curve using a range involving `%pi`, AXIOM's way of saying  $\pi$ . For parametric curves, AXIOM compiles two functions, one for each of the functions  $f$  and  $g$ .

```
draw(curve(sin(t)*sin(2*t)*sin(3*t),
sin(4*t)*sin(5*t)*sin(6*t)), t = 0..2*pi)
Compiling function %F with type DoubleFloat ->
DoubleFloat
Compiling function %H with type DoubleFloat ->
DoubleFloat
Graph data being transmitted to the viewport
manager...
AXIOM2D data being transmitted to the viewport
manager...
```

TwoDimensionalViewport: "DSIN(t)\*DSIN(2\*t)\*DSIN(3\*t)" (1)

Type: TwoDimensionalViewport



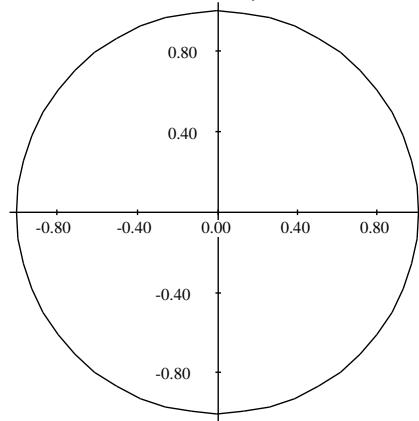
The title may be an arbitrary string and is an optional argument to the **draw** command.

```
draw(curve(cos(t), sin(t)), t = 0..2*pi)
Compiling function %J with type DoubleFloat ->
  DoubleFloat
Compiling function %L with type DoubleFloat ->
  DoubleFloat
Graph data being transmitted to the viewport
manager...
AXIOM2D data being transmitted to the viewport
manager...
```

TwoDimensionalViewport: "cos t"

(2)

Type: TwoDimensionalViewport



If you plan on plotting  $x = f(t)$ ,  $y = g(t)$  as  $t$  ranges over several intervals, you may want to define functions  $f$  and  $g$  first, so that they need not be recompiled every time you create a new graph. Here's an example:

As before, you can first define the functions you wish to draw.

```
f(t:DFLOAT):DFLOAT == sin(3*t/4)
Function declaration f : DoubleFloat -> DoubleFloat
  has been added to workspace.
```

Type: Void

AXIOM compiles them to map DoubleFloat values to DoubleFloat values.

```
g(t:DFLOAT):DFLOAT == sin(t)
Function declaration g : DoubleFloat -> DoubleFloat
  has been added to workspace.
```

Type: Void

Give to **curve** the names of the functions, then write the range without the name of the independent variable.

```
draw(curve(f,g),0..%pi)
```

```
Compiling function f with type DoubleFloat ->
DoubleFloat
```

```
+++ |*1;f;1;initial| redefined
```

```
Compiling function g with type DoubleFloat ->
DoubleFloat
```

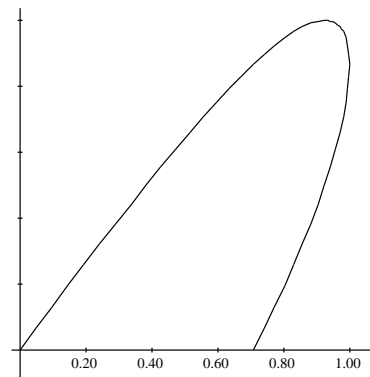
```
Graph data being transmitted to the viewport
manager...
```

```
AXIOM2D data being transmitted to the viewport
manager...
```

```
TwoDimensionalViewport: "AXIOM2D"
```

(5)

Type: TwoDimensionalViewport



Here is another look at the same curve but over a different range. Notice that **f** and **g** are not recompiled. Also note that AXIOM provides a default title based on the first function specified in **curve**.

```
draw(curve(f,g),-4*%pi..4*%pi)
```

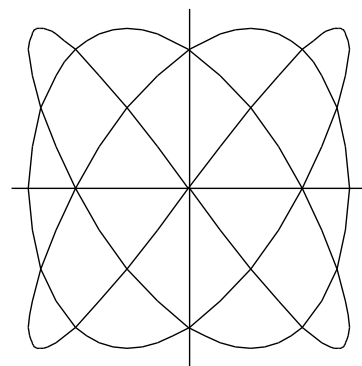
```
Graph data being transmitted to the viewport
manager...
```

```
AXIOM2D data being transmitted to the viewport
manager...
```

```
TwoDimensionalViewport: "AXIOM2D"
```

(6)

Type: TwoDimensionalViewport



### 7.1.3 Plotting Plane Algebraic Curves

A third kind of two-dimensional graph is a non-singular “solution curve” in a rectangular region of the plane. A solution curve is a curve defined by a polynomial equation  $p(x,y) = 0$ . Non-singular means that the curve is “smooth” in that it does not cross itself or come to a point (cusp). Algebraically, this means that for any point  $(x,y)$  on the curve, that is, a point such that  $p(x,y) = 0$ , the partial derivatives  $\frac{\partial p}{\partial x}(x,y)$  and  $\frac{\partial p}{\partial y}(x,y)$  are not both zero.

The general format for drawing a non-singular solution curve given by a polynomial of the form  $p(x,y) = 0$  is:

```
draw(p(x,y) = 0, x, y, range == [a..b, c..d], options)
```

where the second and third arguments name the first and second independent variables of  $p$ . A **range** option is always given to designate a bounding rectangular region of the plane  $a \leq x \leq b, c \leq y \leq d$ . Zero or more additional options as described in Section 7.1.4 on page 243 may be given.

We require that the polynomial has rational or integral coefficients. Here is an algebraic curve example (“Cartesian ovals”):

The first argument is always expressed as an equation of the form  $p = 0$  where  $p$  is a polynomial.

$$p := ((x^{**2} + y^{**2} + 1) - 8*x)^{**2} - (8*(x^{**2} + y^{**2} + 1) - 4*x - 1) \\ y^4 + (2x^2 - 16x - 6)y^2 + x^4 - 16x^3 + 58x^2 - 12x - 6 \quad (1)$$

Type: Polynomial Integer

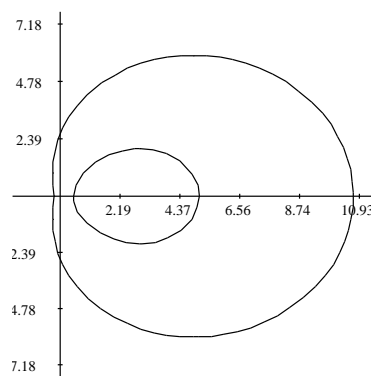
```
draw(p = 0, x, y, range == [-1..11, -7..7])
```

```
Graph data being transmitted to the viewport
manager...
AXIOM2D data being transmitted to the viewport
manager...
```

TwoDimensionalViewport: "AXIOM2D"

(2)

Type: TwoDimensionalViewport





### 7.1.4 Two-Dimensional Options

---

The **adaptive** option is normally on. Here we turn it off.

The **draw** commands take an optional list of options, such as **title** shown above. Each option is given by the syntax: *name == value*. Here is a list of the available options in the order that they are described below.

adaptive	clip	unit
clip	curveColor	range
toScale	pointColor	coordinates

The **adaptive** option turns adaptive plotting on or off. Adaptive plotting uses an algorithm that traverses a graph and computes more points for those parts of the graph with high curvature. The higher the curvature of a region is, the more points the algorithm computes.

```
draw(sin(1/x),x=-2*pi..2*pi, adaptive == false)
```

```
Compiling function %N with type DoubleFloat ->
```

```
DoubleFloat
```

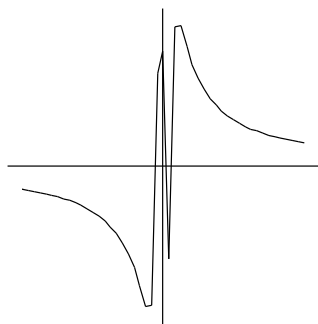
```
Graph data being transmitted to the viewport  
manager...
```

```
AXIOM2D data being transmitted to the viewport  
manager...
```

```
TwoDimensionalViewport: "sin 1/x"
```

(1)

Type: TwoDimensionalViewport



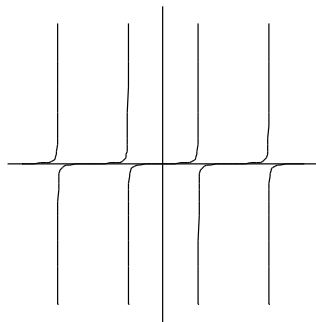
The `clip` option turns clipping on or off. If on, large values are cut off according to `clipPointsDefault`.

```
draw(tan(x),x=-2*pi..2*pi, clip == true)
Compiling function %P with type DoubleFloat ->
DoubleFloat
Graph data being transmitted to the viewport
manager...
AXIOM2D data being transmitted to the viewport
manager...
```

TwoDimensionalViewport: "tan x"

(2)

Type: TwoDimensionalViewport



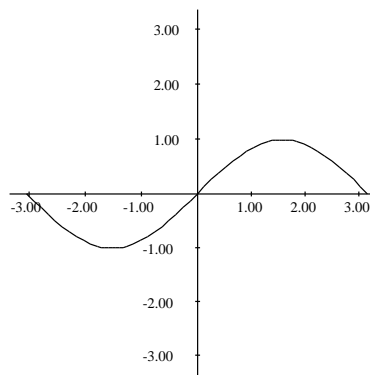
Option `toScale` does plotting to scale if `true` or uses the entire viewport if `false`. The default can be determined using `drawToScale`.

```
draw(sin(x),x=-%pi..%pi, toScale == true, unit ==
[1.0,1.0])
Compiling function %R with type DoubleFloat ->
DoubleFloat
Graph data being transmitted to the viewport
manager...
AXIOM2D data being transmitted to the viewport
manager...
```

TwoDimensionalViewport: "sin x"

(3)

Type: TwoDimensionalViewport



Option `clip` with a range sets point clipping of a graph within the ranges specified in the list `[x range,y range]`. If only one range is specified, clipping applies to the y-axis.

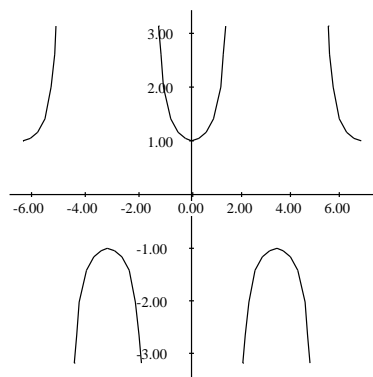
```
draw(sec(x),x=-2*%pi..2*%pi, clip == [-2*%pi..2*%pi,-
%pi..%pi], unit == [1.0,1.0])
```

```
Compiling function %S with type DoubleFloat ->
DoubleFloat
Graph data being transmitted to the viewport
manager...
AXIOM2D data being transmitted to the viewport
manager...
```

TwoDimensionalViewport: "sec x"

(4)

Type: TwoDimensionalViewport



Option `curveColor` sets the color of the graph curves or lines to be the indicated palette color (see Section 7.1.5 on page 248 and Section 7.1.6 on page 250).

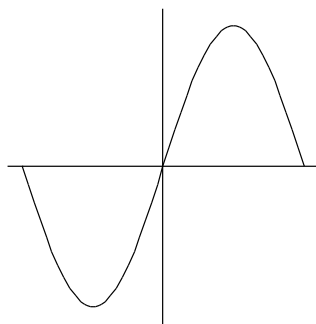
```
draw(sin(x),x=-%pi..%pi, curveColor == bright red())
```

```
Compiling function with type DoubleFloat ->
DoubleFloat
Graph data being transmitted to the viewport
manager...
AXIOM2D data being transmitted to the viewport
manager...
```

TwoDimensionalViewport: "sin x"

(5)

Type: TwoDimensionalViewport



Option `pointColor` sets the color of the graph points to the indicated palette color (see Section 7.1.5 on page 248 and Section 7.1.6 on page 250).

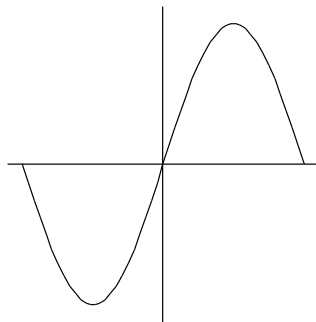
```
draw(sin(x),x=-%pi..%pi, pointColor == pastel yellow())
```

```
Compiling function %W with type DoubleFloat ->
DoubleFloat
Graph data being transmitted to the viewport
manager...
AXIOM2D data being transmitted to the viewport
manager...
```

TwoDimensionalViewport: "sin x"

(6)

Type: TwoDimensionalViewport



Option `unit` sets the intervals at which the axis units are plotted according to the indicated steps [x interval, y interval].

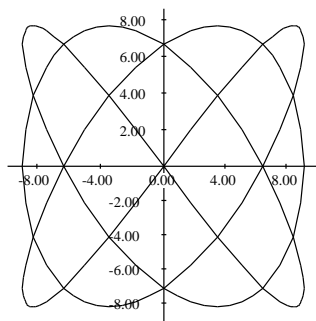
```
draw(curve(9*sin(3*t/4),8*sin(t)), t = -4*%pi..4*%pi, unit
== [2.0,1.0])
```

```
Compiling function %Y with type DoubleFloat ->
DoubleFloat
Compiling function %BA with type DoubleFloat ->
DoubleFloat
Graph data being transmitted to the viewport
manager...
AXIOM2D data being transmitted to the viewport
manager...
```

TwoDimensionalViewport: "9\*DSIN((3\*t)/4)"

(7)

Type: TwoDimensionalViewport



Option **range** sets the range of variables in a graph to be within the ranges for solving plane algebraic curve plots.

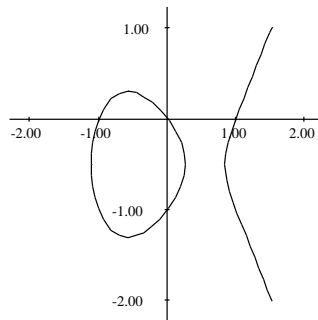
```
draw(y**2 + y - (x**3 - x) = 0, x, y, range == [-2..2, -2..1], unit==[1.0,1.0])
```

Graph data being transmitted to the viewport manager...  
AXIOM2D data being transmitted to the viewport manager...

TwoDimensionalViewport: "AXIOM2D"

(8)

Type: TwoDimensionalViewport



A second example of a solution plot.

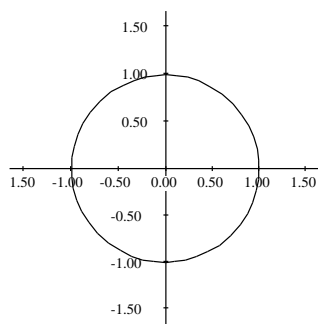
```
draw(x**2 + y**2 = 1, x, y, range == [-3/2..3/2, -3/2..3/2], unit==[0.5,0.5])
```

Graph data being transmitted to the viewport manager...  
AXIOM2D data being transmitted to the viewport manager...

TwoDimensionalViewport: "AXIOM2D"

(9)

Type: TwoDimensionalViewport

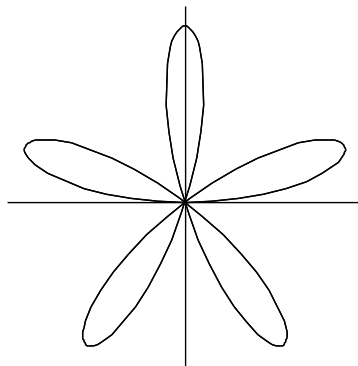


Option `coordinates` indicates the coordinate system in which the graph is plotted. The default is to use the Cartesian coordinate system. For more details, see Section ?? on page ??? .

```
draw(curve(sin(5*t),t),t=0..2*pi, coordinates == polar)
Compiling function %BC with type DoubleFloat ->
  DoubleFloat
Compiling function %BE with type DoubleFloat ->
  DoubleFloat
Graph data being transmitted to the viewport
manager...
AXIOM2D data being transmitted to the viewport
manager...
```

TwoDimensionalViewport: "sin 5\*t" (10)

Type: TwoDimensionalViewport



### 7.1.5 Color

The domain `Color` provides operations for manipulating colors in two-dimensional graphs. Colors are objects of `Color`. Each color has a *hue* and a *weight*. Hues are represented by integers that range from 1 to the `numberOfHues()`, normally 27. Weights are floats and have the value 1.0 by default.

**color** (*integer*)

creates a color of hue *integer* and weight 1.0.

**hue** (*color*)

returns the hue of *color* as an integer.

**red** ()

, **blue**(), **green**(), and **yellow**() create colors of that hue with weight 1.0.

*color*<sub>1</sub> + *color*<sub>2</sub> returns the color that results from additively combining the indicated *color*<sub>1</sub> and *color*<sub>2</sub>. Color addition is not commutative: changing the order of the arguments produces different results.

*integer* \* *color* changes the weight of *color* by *integer* without affecting its hue. For example, **red**() + 3\***yellow**() produces a color closer to yellow than to red. Color multiplication is not associative: changing

the order of grouping produces different results.

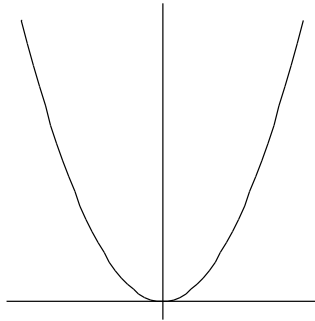
These functions can be used to change the point and curve colors for two- and three-dimensional graphs. Use the `pointColor` option for points.

```
draw(x**2,x=-1..1,pointColor == green())
Compiling function %BG with type DoubleFloat ->
DoubleFloat
Graph data being transmitted to the viewport
manager...
AXIOM2D data being transmitted to the viewport
manager...
```

TwoDimensionalViewport: "x\*x"

(1)

Type: TwoDimensionalViewport



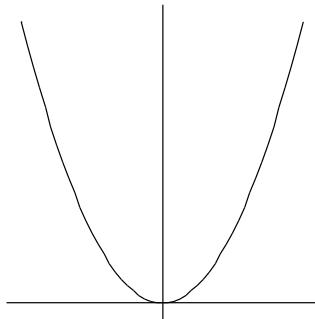
Use the `curveColor` option for curves.

```
draw(x**2,x=-1..1,curveColor == color(13) + 2*blue())
Compiling function %BI with type DoubleFloat ->
DoubleFloat
Graph data being transmitted to the viewport
manager...
AXIOM2D data being transmitted to the viewport
manager...
```

TwoDimensionalViewport: "x\*x"

(2)

Type: TwoDimensionalViewport



### 7.1.6 Palette

Colors are normally “bright.”

Domain Palette is the domain of shades of colors: **dark**, **dim**, **bright**, **pastel**, and **light**, designated by the integers 1 through 5, respectively.

```
shade red()
```

```
3
```

(1)

Type: PositiveInteger

To change the shade of a color, apply the name of a shade to it.

```
myFavoriteColor := dark blue()
```

```
[Hue: 22 Weight: 1.0] from the Dark palette
```

(2)

Type: Palette

The expression `shade(color)` returns the value of a shade of `color`.

```
shade myFavoriteColor
```

```
1
```

(3)

Type: PositiveInteger

The expression `hue(color)` returns its hue.

```
hue myFavoriteColor
```

```
Hue: 22 Weight: 1.0
```

(4)

Type: Color

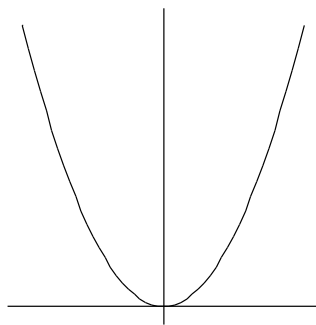
Palettes can be used in specifying colors in two-dimensional graphs.

```
draw(x**2,x=-1..1,curveColor == dark blue())
Compiling function %BK with type DoubleFloat ->
DoubleFloat
Graph data being transmitted to the viewport
manager...
AXIOM2D data being transmitted to the viewport
manager...
```

```
TwoDimensionalViewport: "x*x"
```

(5)

Type: TwoDimensionalViewport





### 7.1.7 Two-Dimensional Control-Panel

Once you have created a viewport, move your mouse to the viewport and click with your left mouse button to display a control-panel. The panel is displayed on the side of the viewport closest to where you clicked. Each of the buttons which toggle on and off show the current state of the graph.

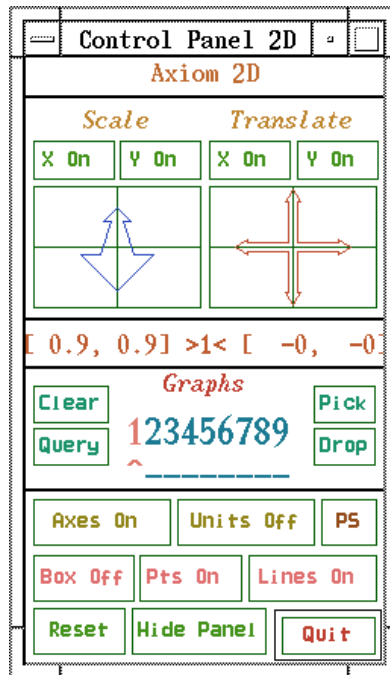


Figure 7.2: Two-dimensional control-panel.

#### Transformations

Object transformations are executed from the control-panel by mouse-activated potentiometer windows.

**Scale:** To scale a graph, click on a mouse button within the **Scale** window in the upper left corner of the control-panel. The axes along which the scaling is to occur are indicated by setting the toggles above the arrow. With **X On** and **Y On** appearing, both axes are selected and scaling is uniform. If either is not selected, for example, if **X Off** appears, scaling is non-uniform.

**Translate:** To translate a graph, click the mouse in the **Translate** window in the direction you wish the graph to move. This window is located in the upper right corner of the control-panel. Along the top of the **Translate** window are two buttons for selecting the direction of translation. Translation along both coordinate axes results when **X On** and **Y On** appear or along one axis when one is on, for example, **X On** and **Y Off** appear.

## Messages

The window directly below the transformation potentiometer windows is used to display system messages relating to the viewport and the control-panel. The following format is displayed:

[scaleX, scaleY] >graph< [translateX, translateY]

The two values to the left show the scale factor along the X and Y coordinate axes. The two values to the right show the distance of translation from the center in the X and Y directions. The number in the center shows which graph in the viewport this data pertains to. When multiple graphs exist in the same viewport, the graph must be selected (see “Multiple Graphs,” below) in order for its transformation data to be shown, otherwise the number is 1.

## Multiple Graphs

The **Graphs** window contains buttons that allow the placement of two-dimensional graphs into one of nine available slots in any other two-dimensional viewport. In the center of the window are numeral buttons from one to nine that show whether a graph is displayed in the viewport. Below each number button is a button showing whether a graph that is present is selected for application of some transformation. When the caret symbol is displayed, then the graph in that slot will be manipulated. Initially, the graph for which the viewport is created occupies the first slot, is displayed, and is selected.

**Clear:** The **Clear** button deselects every viewport graph slot. A graph slot is reselected by selecting the button below its number.

**Query:** The **Query** button is used to display the scale and translate data for the indicated graph. When this button is selected the message “Click on the graph to query” appears. Select a slot number button from the **Graphs** window. The scaling factor and translation offset of the graph are then displayed in the message window.

**Pick:** The **Pick** button is used to select a graph to be placed or dropped into the indicated viewport. When this button is selected, the message “Click on the graph to pick” appears. Click on the slot with the graph number of the desired graph. The graph information is held waiting for you to execute a **Drop** in some other graph.

**Drop:** Once a graph has been picked up using the **Pick** button, the **Drop** button places it into a new viewport slot. The message “Click on the graph to drop” appears in the message window when the **Drop** button is selected. By selecting one of the slot number buttons in the **Graphs** window, the graph currently being held is dropped into this slot and displayed.

## Buttons

**Axes** turns the coordinate axes on or off.

**Units** turns the units along the *x* and *y* axis on or off.

**Box** encloses the area of the viewport graph in a bounding box, or removes the box if already enclosed.

**Pts** turns on or off the display of points.

**Lines** turns on or off the display of lines connecting points.

**PS** writes the current viewport contents to a file **axiom2D.ps** or to a name specified in the user's **.Xdefaults** file. The file is placed in the directory from which AXIOM or the **viewAlone** program was invoked.

**Reset** resets the object transformation characteristics and attributes back to their initial states.

**Hide** makes the control-panel disappear.

**Quit** queries whether the current viewport session should be terminated.

### 7.1.8 Operations for Two-Dimensional Graphics

---

Here is a summary of useful AXIOM operations for two-dimensional graphics. Each operation name is followed by a list of arguments. Each argument is written as a variable informally named according to the type of the argument (for example, *integer*). If appropriate, a default value for an argument is given in parentheses immediately following the name.

**adaptive** (*[[boolean(true)]]*)

sets or indicates whether graphs are plotted according to the adaptive refinement algorithm.

**axesColorDefault** (*[[color(dark blue())]]*)

sets or indicates the default color of the axes in a two-dimensional graph viewport.

**clipPointsDefault** (*[[boolean(false)]]*)

sets or indicates whether point clipping is to be applied as the default for graph plots.

**drawToScale** (*[[boolean(false)]]*)

sets or indicates whether the plot of a graph is “to scale” or uses the entire viewport space as the default.

**lineColorDefault** (*[[color(pastel yellow())]]*)

sets or indicates the default color of the lines or curves in a two-dimensional graph viewport.

**maxPoints** (*[[integer(500)]]*)

sets or indicates the default maximum number of possible points to be used when constructing a two-dimensional graph.

**minPoints** (*[integer(21)]*)  
 sets or indicates the default minimum number of possible points to be used when constructing a two-dimensional graph.

**pointColorDefault** (*[color(bright red())]*)  
 sets or indicates the default color of the points in a two-dimensional graph viewport.

**pointSizeDefault** (*[integer(5)]*)  
 sets or indicates the default size of the dot used to plot points in a two-dimensional graph.

**screenResolution** (*[integer(600)]*)  
 sets or indicates the default screen resolution constant used in setting the computation limit of adaptively generated curve plots.

**unitsColorDefault** (*[color(dim green())]*)  
 sets or indicates the default color of the unit labels in a two-dimensional graph viewport.

**viewDefaults** ()  
 resets the default settings for the following attributes: point color, line color, axes color, units color, point size, viewport upper left-hand corner position, and the viewport size.

**viewPosDefault** (*[list([100,100])]*)  
 sets or indicates the default position of the upper left-hand corner of a two-dimensional viewport, relative to the display root window. The upper left-hand corner of the display is considered to be at the (0, 0) position.

**viewSizeDefault** (*[list([200,200])]*)  
 sets or indicates the default size in which two dimensional viewport windows are shown. It is defined by a width and then a height.

**viewWriteAvailable** (*[list(["pixmap", "bitmap", "postscript", "image"])]*)  
 indicates the possible file types that can be created with the **write** function.

**viewWriteDefault** (*[list([])]*)  
 sets or indicates the default types of files, in addition to the **data** file, that are created when a **write** function is executed on a viewport.

**units** (*viewport, integer(1), string("off")*)  
 turns the units on or off for the graph with index *integer*.

**axes** (*viewport, integer(1), string("on")*)  
 turns the axes on or off for the graph with index *integer*.

**close** (*viewport*)  
 closes *viewport*.

**connect** (*viewport, integer(1), string("on")*)

declares whether lines connecting the points are displayed or not.

**controlPanel** (*viewport*, *string*("off"))  
declares whether the two-dimensional control-panel is automatically displayed or not.

**graphs** (*viewport*)  
returns a list describing the state of each graph. If the graph state is not being used this is shown by "undefined", otherwise a description of the graph's contents is shown.

**graphStates** (*viewport*)  
displays a list of all the graph states available for *viewport*, giving the values for every property.

**key** (*viewport*)  
returns the process ID number for *viewport*.

**move** (*viewport*, *integer<sub>x</sub>*(*viewPosDefault*), *integer<sub>y</sub>*(*viewPosDefault*))  
moves *viewport* on the screen so that the upper left-hand corner of *viewport* is at the position (*x*,*y*).

**options** (*viewport*)  
returns a list of all the DrawOptions used by *viewport*.

**points** (*viewport*, *integer*(1), *string*("on"))  
specifies whether the graph points for graph *integer* are to be displayed or not.

**region** (*viewport*, *integer*(1), *string*("off"))  
declares whether graph *integer* is or is not to be displayed with a bounding rectangle.

**reset** (*viewport*)  
resets all the properties of *viewport*.

**resize** (*viewport*, *integer<sub>width</sub>*, *integer<sub>height</sub>*)  
resizes *viewport* with a new *width* and *height*.

**scale** (*viewport*, *integer<sub>n</sub>*(1), *integer<sub>x</sub>*(0.9), *integer<sub>y</sub>*(0.9))  
scales values for the *x* and *y* coordinates of graph *n*.

**show** (*viewport*, *integer<sub>n</sub>*(1), *string*("on"))  
indicates if graph *n* is shown or not.

**title** (*viewport*, *string*("Axiom 2D"))  
designates the title for *viewport*.

**translate** (*viewport*, *integer<sub>n</sub>*(1), *float<sub>x</sub>*(0.0), *float<sub>y</sub>*(0.0))  
causes graph *n* to be moved *x* and *y* units in the respective directions.

**write** (*viewport*, *string<sub>directory</sub>*, [*strings*])  
if no third argument is given, writes the **data** file onto the directory with extension **data**. The third argument can be a single string or a list of strings with some or all the entries "pixmap", "bitmap", "postscript", and "image".

### 7.1.9 Addendum: Building Two-Dimensional Graphs

#### Creating a Two-Dimensional Viewport from a List of Points

The following expressions create a list of lists of points which will be read by AXIOM and made into a two-dimensional viewport.

In this section we demonstrate how to create two-dimensional graphs from lists of points and give an example showing how to read the lists of points from a file.

AXIOM creates lists of points in a two-dimensional viewport by utilizing the `GraphImage` and `TwoDimensionalViewport` domains. In this example, the **`makeGraphImage`** function takes a list of lists of points parameter, a list of colors for each point in the graph, a list of colors for each line in the graph, and a list of sizes for each point in the graph.

```
p1 := point [1,1]$(Point DFLOAT)
[1.0, 1.0]
(1)
Type: Point DoubleFloat
```

```
p2 := point [0,1]$(Point DFLOAT)
[0.0, 1.0]
(2)
Type: Point DoubleFloat
```

```
p3 := point [0,0]$(Point DFLOAT)
[0.0, 0.0]
(3)
Type: Point DoubleFloat
```

```
p4 := point [1,0]$(Point DFLOAT)
[1.0, 0.0]
(4)
Type: Point DoubleFloat
```

```
p5 := point [1,.5]$(Point DFLOAT)
[1.0, 0.5]
(5)
Type: Point DoubleFloat
```

```
p6 := point [.5,0]$(Point DFLOAT)
[0.5, 0.0]
(6)
Type: Point DoubleFloat
```

```
p7 := point [0,0.5]$(Point DFLOAT)
[0.0, 0.5]
(7)
Type: Point DoubleFloat
```

```
p8 := point [.5,1]$(Point DFLOAT)
[0.5, 1.0]
(8)
Type: Point DoubleFloat
```

```
p9 := point [.25,.25]$(Point DFLOAT)
[0.25, 0.25] (9)
```

Type: Point DoubleFloat

```
p10 := point [.25,.75]$(Point DFLOAT)
[0.25, 0.75] (10)
```

Type: Point DoubleFloat

```
p11 := point [.75,.75]$(Point DFLOAT)
[0.75, 0.75] (11)
```

Type: Point DoubleFloat

```
p12 := point [.75,.25]$(Point DFLOAT)
[0.75, 0.25] (12)
```

Type: Point DoubleFloat

Finally, here is the list.

```
l1p := [[p1,p2], [p2,p3], [p3,p4], [p4,p1], [p5,p6],
        [p6,p7], [p7,p8], [p8,p5], [p9,p10], [p10,p11],
        [p11,p12], [p12,p9]]
[[[1.0, 1.0], [0.0, 1.0]], [[0.0, 1.0], [0.0, 0.0]], [[0.0, 0.0], [1.0, 0.0]],
 [[1.0, 0.0], [1.0, 1.0]], [[1.0, 0.5], [0.5, 0.0]], [[0.5, 0.0], [0.0, 0.5]],
 [[0.0, 0.5], [0.5, 1.0]], [[0.5, 1.0], [1.0, 0.5]],
 [[0.25, 0.25], [0.25, 0.75]], [[0.25, 0.75], [0.75, 0.75]],
 [[0.75, 0.75], [0.75, 0.25]], [[0.75, 0.25], [0.25, 0.25]]] (13)
```

Type: List List Point DoubleFloat

Now we set the point sizes for all components of the graph.

```
size1 := 6::PositiveInteger
6 (14)
```

Type: PositiveInteger

```
size2 := 8::PositiveInteger
8 (15)
```

Type: PositiveInteger

```
size3 := 10::PositiveInteger
10 (16)
```

Type: PositiveInteger

```
lsize := [size1, size1, size1, size1, size2, size2, size2,
          size2, size3, size3, size3, size3]
[6, 6, 6, 6, 8, 8, 8, 8, 10, 10, 10, 10] (17)
```

Type: List PositiveInteger

Here are the colors for the points.

```
pc1 := pastel red()
[Hue: 1 Weight: 1.0] from the Pastel palette
```

(18)

Type: Palette

```
pc2 := dim green()
[Hue: 14 Weight: 1.0] from the Dim palette
```

(19)

Type: Palette

```
pc3 := pastel yellow()
[Hue: 11 Weight: 1.0] from the Pastel palette
```

(20)

Type: Palette

```
lpc := [pc1, pc1, pc1, pc1, pc2, pc2, pc2, pc2, pc3, pc3,
        pc3, pc3]
[[Hue: 1 Weight: 1.0] from the Pastel palette,
[Hue: 1 Weight: 1.0] from the Pastel palette,
[Hue: 1 Weight: 1.0] from the Pastel palette,
[Hue: 1 Weight: 1.0] from the Pastel palette,
[Hue: 14 Weight: 1.0] from the Dim palette,
[Hue: 14 Weight: 1.0] from the Dim palette,
[Hue: 14 Weight: 1.0] from the Dim palette,
[Hue: 14 Weight: 1.0] from the Dim palette,
[Hue: 11 Weight: 1.0] from the Pastel palette,
[Hue: 11 Weight: 1.0] from the Pastel palette,
[Hue: 11 Weight: 1.0] from the Pastel palette,
[Hue: 11 Weight: 1.0] from the Pastel palette]
```

(21)

Type: List Palette

Here are the colors for the lines.

```
lc := [pastel blue(), light yellow(), dim green(), bright
       red(), light green(), dim yellow(), bright blue(), dark
       red(), pastel red(), light blue(), dim green(), light
       yellow()]
```

```
[[Hue: 22 Weight: 1.0] from the Pastel palette,
[Hue: 11 Weight: 1.0] from the Light palette,
[Hue: 14 Weight: 1.0] from the Dim palette,
[Hue: 1 Weight: 1.0] from the Bright palette,
[Hue: 14 Weight: 1.0] from the Light palette,
[Hue: 11 Weight: 1.0] from the Dim palette,
[Hue: 22 Weight: 1.0] from the Bright palette,
[Hue: 1 Weight: 1.0] from the Dark palette,
[Hue: 1 Weight: 1.0] from the Pastel palette,
[Hue: 22 Weight: 1.0] from the Light palette,
[Hue: 14 Weight: 1.0] from the Dim palette,
[Hue: 11 Weight: 1.0] from the Light palette]
```

(22)

Type: List Palette



Now the GraphImage is created according to the component specifications indicated above.

```
g := makeGraphImage(1lp,1pc,1c,1size)$GRIMAGE
Graph data being transmitted to the viewport
manager...
```

(23)

Type: GraphImage

The **makeViewport2D** function now creates a TwoDimensionalViewport for this graph according to the list of options specified within the brackets.

```
makeViewport2D(g,[title("Lines")])$VIEW2D
AXIOM2D data being transmitted to the viewport
manager...
```

TwoDimensionalViewport: "Lines" (24)

Type: TwoDimensionalViewport

This example demonstrates the use of the GraphImage functions **component** and **appendPoint** in adding points to an empty GraphImage.

```
g := graphImage()$GRIMAGE
Graph with0point lists
```

(1)

Type: GraphImage

```
p1 := point [0,0]$(Point DFLOAT)
[0.0, 0.0]
```

(2)

Type: Point DoubleFloat

```
p2 := point [.25,.25]$(Point DFLOAT)
[0.25, 0.25]
```

(3)

Type: Point DoubleFloat

```
p3 := point [.5,.5]$(Point DFLOAT)
[0.5, 0.5]
```

(4)

Type: Point DoubleFloat

```
p4 := point [.75,.75]$(Point DFLOAT)
[0.75, 0.75]
```

(5)

Type: Point DoubleFloat

```
p5 := point [1,1]$(Point DFLOAT)
[1.0, 1.0]
```

(6)

Type: Point DoubleFloat

```
component(g,p1)$GRIMAGE
```

Type: Void

```
component(g,p2)$GRIMAGE
```

Type: Void

```
appendPoint(g,p3)$GRIMAGE
```

Type: Void

```
appendPoint(g,p4)$GRIMAGE
```

Type: Void

```
appendPoint(g,p5)$GRIMAGE
```

Type: Void

```
g1 := makeGraphImage(g)$GRIMAGE
```

```
Graph data being transmitted to the viewport  
manager...
```

```
Graph with 2 point lists
```

(12)

Type: GraphImage

Here is the graph.

```
makeViewport2D(g1,[title("Graph Points")])$VIEW2D
```

```
AXIOM2D data being transmitted to the viewport  
manager...
```

```
TwoDimensionalViewport: "Graph Points"
```

---

## PART III

---

# Advanced Problem Solving and Examples



---

# Advanced Problem Solving

In this chapter we describe techniques useful in solving advanced problems with AXIOM.

## 8.1 Numeric Functions

---

AXIOM provides two basic floating-point types: `Float` and `DoubleFloat`. This section describes how to use numerical operations defined on these types and the related complex types. As we mentioned in Chapter 1, the `Float` type is a software implementation of floating-point numbers in which the exponent and the significand may have any number of digits. See ‘`Float`’ on page 427 for detailed information about this domain. The `DoubleFloat` (see ‘`DoubleFloat`’ on page 404) is usually a hardware implementation of floating point numbers, corresponding to machine double precision. The types `Complex Float` and `Complex DoubleFloat` are the corresponding software implementations of complex floating-point numbers. In this section the term *floating-point type* means any of these four types. The floating-point types implement the basic elementary functions. These include (where “\$” means `DoubleFloat`, `Float`, `Complex DoubleFloat`, or `Complex Float`):

```
exp, log: $ -> $
sin, cos, tan, cot, sec, csc: $ -> $
sin, cos, tan, cot, sec, csc: $ -> $
asin, acos, atan, acot, asec, acsc: $ -> $
sinh, cosh, tanh, coth, sech, csch: $ -> $
asinh, acosh, atanh, acoth, asech, acsch: $ -> $
pi: () -> $
sqrt: $ -> $
nthRoot: ($, Integer) -> $
**: ($, Fraction Integer) -> $
**: ($,$) -> $
```

The handling of roots depends on whether the floating-point type is real or complex: for the real floating-point types, `DoubleFloat` and `Float`, if a real root exists the one with the same sign as the radicand is returned; for the complex floating-point types, the principal value is returned. Also, for real floating-point types the inverse functions produce errors if the results are not real. This includes cases such as `asin(1.2)`, `log(-3.2)`, `sqrt(-1.1)`.

The default floating-point type is `Float` so to evaluate functions using `Float` or `Complex Float`, just use normal decimal notation.

```
exp(3.1)
22.197951281441633405
Type: Float
```

```
exp(3.1 + 4.5 * %i)
-4.6792348860969899118 - 21.699165928071731864 i
Type: Complex Float
```

To evaluate functions using DoubleFloat or Complex DoubleFloat, a declaration or conversion is required.

```
r: DFLOAT := 3.1; t: DFLOAT := 4.5; exp(r + t*i)
-4.6792348860969906 - 21.699165928071732 i
```

(3)

Type: Complex DoubleFloat

```
exp(3.1::DFLOAT + 4.5::DFLOAT * %i)
-4.6792348860969906 - 21.699165928071732 i
```

(4)

Type: Complex DoubleFloat

A number of special functions are provided by the package DoubleFloat-SpecialFunctions for the machine-precision floating-point types. The special functions provided are listed below, where **F** stands for the types DoubleFloat and Complex DoubleFloat. The real versions of the functions yield an error if the result is not real.

**Gamma:** **F** -> **F**

**Gamma(z)** is the Euler gamma function,  $\Gamma(z)$ , defined by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt.$$

**Beta:** **F** -> **F**

**Beta(u, v)** is the Euler Beta function,  $B(u, v)$ , defined by

$$B(u, v) = \int_0^1 t^{u-1} (1-t)^{v-1} dt.$$

This is related to  $\Gamma(z)$  by

$$B(u, v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}.$$

**logGamma:** **F** -> **F**

**logGamma(z)** is the natural logarithm of  $\Gamma(z)$ . This can often be computed even if  $\Gamma(z)$  cannot.

**digamma:** **F** -> **F**

**digamma(z)**, also called **psi(z)**, is the function  $\psi(z)$ , defined by

$$\psi(z) = \Gamma'(z)/\Gamma(z).$$

**polygamma:** (**NonNegativeInteger**, **F**) -> **F**

**polygamma(n, z)** is the  $n^{\text{th}}$  derivative of  $\psi(z)$ , written  $\psi^{(n)}(z)$ .

**besselJ:** (**F**, **F**) -> **F**

**besselJ(v, z)** is the Bessel function of the first kind,  $J_\nu(z)$ . This function satisfies the differential equation

$$z^2 w''(z) + z w'(z) + (z^2 - \nu^2) w(z) = 0.$$

**besselY**: (F,F) -> F

**besselY**(v,z) is the Bessel function of the second kind,  $Y_\nu(z)$ . This function satisfies the same differential equation as **besselJ**. The implementation simply uses the relation

$$Y_\nu(z) = \frac{J_\nu(z) \cos(\nu\pi) - J_{-\nu}(z)}{\sin(\nu\pi)}.$$

**besselI**: (F,F) -> F

**besselI**(v,z) is the modified Bessel function of the first kind,  $I_\nu(z)$ . This function satisfies the differential equation

$$z^2 w''(z) + zw'(z) - (z^2 + \nu^2)w(z) = 0.$$

**besselK**: (F,F) -> F

**besselK**(v,z) is the modified Bessel function of the second kind,  $K_\nu(z)$ . This function satisfies the same differential equation as **besselI**. The implementation simply uses the relation

$$K_\nu(z) = \pi \frac{I_{-\nu}(z) - I_\nu(z)}{2 \sin(\nu\pi)}.$$

**airyAi**: F -> F

**airyAi**(z) is the Airy function  $Ai(z)$ . This function satisfies the differential equation  $w''(z) - zw(z) = 0$ . The implementation simply uses the relation

$$Ai(-z) = \frac{1}{3} \sqrt{z} (J_{-1/3}(\frac{2}{3}z^{3/2}) + J_{1/3}(\frac{2}{3}z^{3/2})).$$

**airyBi**: F -> F

**airyBi**(z) is the Airy function  $Bi(z)$ . This function satisfies the same differential equation as **airyAi**. The implementation simply uses the relation

$$Bi(-z) = \frac{1}{3} \sqrt{3z} (J_{-1/3}(\frac{2}{3}z^{3/2}) - J_{1/3}(\frac{2}{3}z^{3/2})).$$

**hypergeometric0F1**: (F,F) -> F

**hypergeometric0F1**(c,z) is the hypergeometric function  ${}_0F_1(;c;z)$ .

The above special functions are defined only for small floating-point types. If you give Float arguments, they are converted to DoubleFloat by AXIOM.

`Gamma(0.5)**2`

`3.14159265358979` (5)

Type: DoubleFloat

`a := 2.1; b := 1.1; besselI(a + %i*b, b*a + 1)`

`2.4894824175473689 - 2.3658460381468345 i` (6)

Type: Complex DoubleFloat



A number of additional operations may be used to compute numerical values. These are special polynomial functions that can be evaluated for values in any commutative ring  $R$ , and in particular for values in any floating-point type. The following operations are provided by the package `OrthogonalPolynomialFunctions`:

**chebyshevT**: (`NonNegativeInteger`,  $R$ )  $\rightarrow R$

`chebyshevT(n,z)` is the  $n^{\text{th}}$  Chebyshev polynomial of the first kind,  $T_n(z)$ . These are defined by

$$\frac{1-tz}{1-2tz+t^2} = \sum_{n=0}^{\infty} T_n(z)t^n.$$

**chebyshevU**: (`NonNegativeInteger`,  $R$ )  $\rightarrow R$

`chebyshevU(n,z)` is the  $n^{\text{th}}$  Chebyshev polynomial of the second kind,  $U_n(z)$ . These are defined by

$$\frac{1}{1-2tz+t^2} = \sum_{n=0}^{\infty} U_n(z)t^n.$$

**hermiteH**: (`NonNegativeInteger`,  $R$ )  $\rightarrow R$

`hermiteH(n,z)` is the  $n^{\text{th}}$  Hermite polynomial,  $H_n(z)$ . These are defined by

$$e^{2tz-t^2} = \sum_{n=0}^{\infty} H_n(z) \frac{t^n}{n!}.$$

**laguerreL**: (`NonNegativeInteger`,  $R$ )  $\rightarrow R$

`laguerreL(n,z)` is the  $n^{\text{th}}$  Laguerre polynomial,  $L_n(z)$ . These are defined by

$$\frac{e^{-\frac{tz}{1-t}}}{1-t} = \sum_{n=0}^{\infty} L_n(z) \frac{t^n}{n!}.$$

**laguerreL**: (`NonNegativeInteger`, `NonNegativeInteger`,  $R$ )  $\rightarrow R$

`laguerreL(m,n,z)` is the associated Laguerre polynomial,  $L_n^m(z)$ . This is the  $m^{\text{th}}$  derivative of  $L_n(z)$ .

**legendreP**: (`NonNegativeInteger`,  $R$ )  $\rightarrow R$

`legendreP(n,z)` is the  $n^{\text{th}}$  Legendre polynomial,  $P_n(z)$ . These are defined by

$$\frac{1}{\sqrt{1-2tz+t^2}} = \sum_{n=0}^{\infty} P_n(z)t^n.$$

These operations require non-negative integers for the indices, but otherwise the argument can be given as desired.

$$\begin{aligned} & [\text{chebyshevT}(i, z) \text{ for } i \text{ in } 0..5] \\ & [1, z, 2z^2 - 1, 4z^3 - 3z, 8z^4 - 8z^2 + 1, 16z^5 - 20z^3 + 5z] \end{aligned} \quad (7)$$

Type: List Polynomial Integer

The expression `chebyshevT(n,z)` evaluates to the  $n^{\text{th}}$  Chebyshev polynomial of the first kind.

$$\begin{aligned} & \text{chebyshevT}(3, 5.0 + 6.0*i) \\ & -1675.0 + 918.0i \end{aligned} \quad (8)$$

Type: Complex Float

$$\begin{aligned} & \text{chebyshevT}(3, 5.0::\text{DoubleFloat}) \\ & 485.0 \end{aligned} \quad (9)$$

Type: DoubleFloat

The expression `chebyshevU(n,z)` evaluates to the  $n^{\text{th}}$  Chebyshev polynomial of the second kind.

$$\begin{aligned} & [\text{chebyshevU}(i, z) \text{ for } i \text{ in } 0..5] \\ & [1, 2z, 4z^2 - 1, 8z^3 - 4z, 16z^4 - 12z^2 + 1, 32z^5 - 32z^3 + 6z] \end{aligned} \quad (10)$$

Type: List Polynomial Integer

$$\begin{aligned} & \text{chebyshevU}(3, 0.2) \\ & -0.736 \end{aligned} \quad (11)$$

Type: Float

The expression `hermiteH(n,z)` evaluates to the  $n^{\text{th}}$  Hermite polynomial.

$$\begin{aligned} & [\text{hermiteH}(i, z) \text{ for } i \text{ in } 0..5] \\ & [1, 2z, 4z^2 - 2, 8z^3 - 12z, 16z^4 - 48z^2 + 12, \\ & 32z^5 - 160z^3 + 120z] \end{aligned} \quad (12)$$

Type: List Polynomial Integer

$$\begin{aligned} & \text{hermiteH}(100, 1.0) \\ & -0.1448706729337934088E93 \end{aligned} \quad (13)$$

Type: Float

The expression `laguerreL(n,z)` evaluates to the  $n^{\text{th}}$  Laguerre polynomial.

$$\begin{aligned} & [\text{laguerreL}(i, z) \text{ for } i \text{ in } 0..4] \\ & [1, -z + 1, z^2 - 4z + 2, -z^3 + 9z^2 - 18z + 6, \\ & z^4 - 16z^3 + 72z^2 - 96z + 24] \end{aligned} \quad (14)$$

Type: List Polynomial Integer

$$\begin{aligned} & \text{laguerreL}(4, 1.2) \\ & -13.0944 \end{aligned} \quad (15)$$

Type: Float

$$[\text{laguerreL}(j, 3, z) \text{ for } j \text{ in } 0..4]$$

$$\left[-z^3 + 9z^2 - 18z + 6, -3z^2 + 18z - 18, -6z + 18, -6, 0\right] \quad (16)$$

Type: List Polynomial Integer

$$\text{laguerreL}(1, 3, 2.1)$$

$$6.57 \quad (17)$$

Type: Float

The expression `legendreP(n,z)` evaluates to the  $n^{\text{th}}$  Legendre polynomial,

$$[\text{legendreP}(i, z) \text{ for } i \text{ in } 0..5]$$

$$\left[1, z, \frac{3}{2}z^2 - \frac{1}{2}, \frac{5}{2}z^3 - \frac{3}{2}z, \frac{35}{8}z^4 - \frac{15}{4}z^2 + \frac{3}{8}, \frac{63}{8}z^5 - \frac{35}{4}z^3 + \frac{15}{8}z\right] \quad (18)$$

Type: List Polynomial Fraction Integer

$$\text{legendreP}(3, 3.0 * i)$$

$$-72.0 i \quad (19)$$

Type: Complex Float

Finally, three number-theoretic polynomial operations may be evaluated. The following operations are provided by the package `NumberTheoreticPolynomialFunctions`.

**bernoulliB:** (`NonNegativeInteger`, `R`)  $\rightarrow$  `R`  
**bernoulliB(n,z)** is the  $n^{\text{th}}$  Bernoulli polynomial,  $B_n(z)$ . These are defined by

$$\frac{te^{zt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(z) \frac{t^n}{n!}.$$

**eulerE:** (`NonNegativeInteger`, `R`)  $\rightarrow$  `R`  
**eulerE(n,z)** is the  $n^{\text{th}}$  Euler polynomial,  $E_n(z)$ . These are defined by

$$\frac{2e^{zt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(z) \frac{t^n}{n!}.$$

**cyclotomic:** (`NonNegativeInteger`, `R`)  $\rightarrow$  `R`  
**cyclotomic(n,z)** is the  $n^{\text{th}}$  cyclotomic polynomial  $\Phi_n(z)$ . This is the polynomial whose roots are precisely the primitive  $n^{\text{th}}$  roots of unity. This polynomial has degree given by the Euler totient function  $\phi(n)$ .

The expression `bernoulliB(n,z)` evaluates to the  $n^{\text{th}}$  Bernoulli polynomial.

$$\text{bernoulliB}(3, z)$$

$$z^3 - \frac{3}{2}z^2 + \frac{1}{2}z \quad (20)$$

Type: Polynomial Fraction Integer

$$\text{bernoulliB}(3, 0.7 + 0.4 * \%i) \\ -0.138 - 0.116 i \quad (21)$$

Type: Complex Float

The expression `eulerE(n,z)` evaluates to the  $n^{\text{th}}$  Euler polynomial.

$$\text{eulerE}(3, z) \\ z^3 - \frac{3}{2} z^2 + \frac{1}{4} \quad (22)$$

Type: Polynomial Fraction Integer

$$\text{eulerE}(3, 0.7 + 0.4 * \%i) \\ -0.238 - 0.316 i \quad (23)$$

Type: Complex Float

The expression `cyclotomic(n,z)` evaluates to the  $n^{\text{th}}$  cyclotomic polynomial.

$$\text{cyclotomic}(3, z) \\ z^2 + z + 1 \quad (24)$$

Type: Polynomial Integer

$$\text{cyclotomic}(3, (-1.0 + 0.0 * \%i)^{(2/3)}) \\ 0.0 \quad (25)$$

Type: Complex Float

Drawing complex functions in AXIOM is presently somewhat awkward compared to drawing real functions. It is necessary to use the **draw** operations that operate on functions rather than expressions.

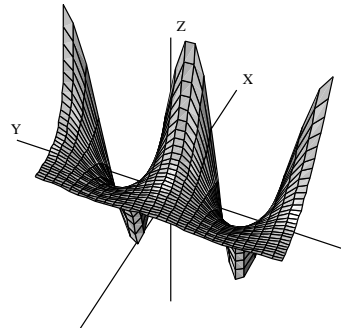
This is the complex exponential function (rotated interactively). When this is displayed in color, the height is the value of the real part of the function and the color is the imaginary part. Red indicates large negative imaginary values, green indicates imaginary values near zero and blue/violet indicates large positive imaginary values.

```
draw((x,y)+-> real exp complex(x,y), -2..2, -2*%pi..2*%pi,
      colorFunction == (x, y) +-> imag exp complex(x,y),
      title=="exp(x+%i*y)", style=="smooth")
```

Transmitting data...

ThreeDimensionalViewport: "exp(x+" (26)

Type: ThreeDimensionalViewport



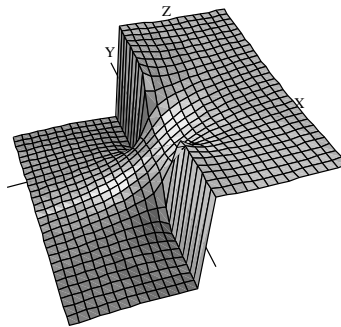
This is the complex arctangent function. Again, the height is the real part of the function value but here the color indicates the function value's phase. The position of the branch cuts are clearly visible and one can see that the function is real only for a real argument.

```
vp := draw((x,y) +-> real atan complex(x,y), -%pi..%pi,
-%pi..%pi, colorFunction==(x,y) +->argument atan
complex(x,y), title=="atan(x+%i*y)", style=="shade");
rotate(vp,-160,-45); vp
```

Transmitting data...

ThreeDimensionalViewport: "atan(x+" (27)

Type: ThreeDimensionalViewport



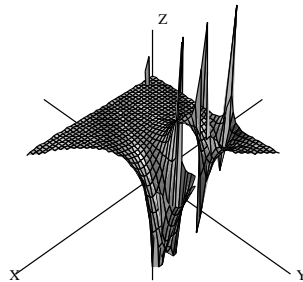
This is the complex Gamma function.

```
draw((x,y) +-> max(min(real Gamma complex(x,y),4),-4),
-%pi..%pi, -%pi..%pi, style=="shade", colorFunction
== (x,y) +-> argument Gamma complex(x,y), title ==
"Gamma(x+%i*y)", var1Steps == 50, var2Steps== 50)
```

Transmitting data...

ThreeDimensionalViewport: "Gamma(x+" (28)

Type: ThreeDimensionalViewport



This shows the real Beta function near the origin.

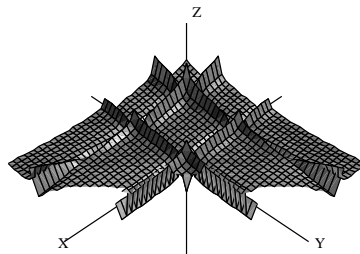
```
draw(Beta(x,y)/100, x=-1.6..1.7, y = -1.6..1.7,
     style=="shade", title=="Beta(x,y)", var1Steps==40,
     var2Steps==40)
```

```
Compiling function %A with type (DoubleFloat,
DoubleFloat) -> DoubleFloat
Transmitting data...
```

ThreeDimensionalViewport: "Beta(x,y)"

(29)

Type: ThreeDimensionalViewport



This is the Bessel function  $J_\alpha(x)$  for index  $\alpha$  in the range  $-6..4$  and argument  $x$  in the range  $2..14$ .

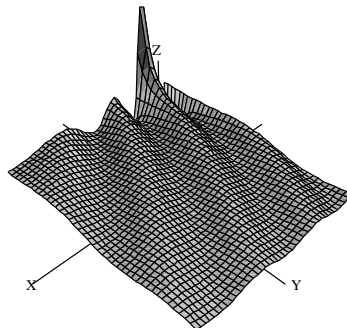
```
draw((alpha,x) +-> min(max(besselJ(alpha, x+8), -6), 6),
     -6..4, -6..6, title=="besselJ(alpha,x)", style=="shade",
     var1Steps==40, var2Steps==40)
```

```
Transmitting data...
```

ThreeDimensionalViewport: "besselJ(alpha,x)"

(30)

Type: ThreeDimensionalViewport



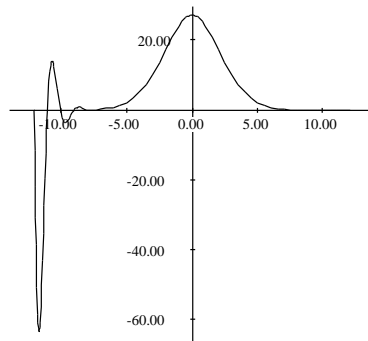
This is the modified Bessel function  $I_\alpha(x)$  evaluated for various real values of the index  $\alpha$  and fixed argument  $x = 5$ .

```
draw(besselI(alpha, 5), alpha = -12..12, unit==[5,20])
```

```
Compiling function %B with type DoubleFloat ->
DoubleFloat
Graph data being transmitted to the viewport
manager...
AXIOM2D data being transmitted to the viewport
manager...
```

TwoDimensionalViewport: "besselI(alpha,5)" (31)

Type: TwoDimensionalViewport



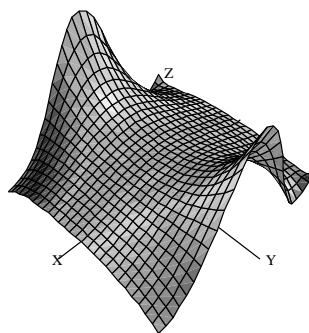
This is similar to the last example except the index  $\alpha$  takes on complex values in a  $6 \times 6$  rectangle centered on the origin.

```
draw((x,y) +-> real besselI(complex(x/20, y/20),5),
-60..60, -60..60, colorFunction == (x,y)+
> argument besselI(complex(x/20,y/20),5),
title=="besselI(x+i*y,5)", style=="shade")
```

Transmitting data...

ThreeDimensionalViewport: "besselI(x+i\*y,5)" (32)

Type: ThreeDimensionalViewport



## 8.2 Polynomial Factorization

---

The AXIOM polynomial factorization facilities are available for all polynomial types and a wide variety of coefficient domains. Here are some examples.

### 8.2.1 Integer and Rational Number Coefficients

---

Polynomials with integer coefficients can be factored.

$$\begin{aligned} v &:= (4x^3 + 2y^2 + 1)(12x^5 - x^3y + 12) \\ &- 2x^3y^3 + (24x^5 + 24)y^2 + (-4x^6 - x^3)y + 48x^8 + 12x^5 \\ &+ 48x^3 + 12 \end{aligned} \quad (1)$$

Type: Polynomial Integer

$$\begin{aligned} \text{factor } v \\ &-(x^3y - 12x^5 - 12)(2y^2 + 4x^3 + 1) \end{aligned} \quad (2)$$

Type: Factored Polynomial Integer

Also, AXIOM can factor polynomials with rational number coefficients.

$$\begin{aligned} w &:= (4x^3 + (2/3)x^2 + 1)(12x^5 - (1/2)x^3 + 12) \\ &48x^8 + 8x^7 - 2x^6 + \frac{35}{3}x^5 + \frac{95}{2}x^3 + 8x^2 + 12 \end{aligned} \quad (3)$$

Type: Polynomial Fraction Integer

$$\begin{aligned} \text{factor } w \\ &48\left(x^3 + \frac{1}{6}x^2 + \frac{1}{4}\right)\left(x^5 - \frac{1}{24}x^3 + 1\right) \end{aligned} \quad (4)$$

Type: Factored Polynomial Fraction Integer

### 8.2.2 Finite Field Coefficients

---

Polynomials with coefficients in a finite field can be also be factored.

$$\begin{aligned} u &: \text{POLY(PF(19))} := 3x^4 + 2x^2 + 15x + 18 \\ &3x^4 + 2x^2 + 15x + 18 \end{aligned} \quad (1)$$

Type: Polynomial PrimeField 19

These include the integers mod  $p$ , where  $p$  is prime, and extensions of these fields.

$$\begin{aligned} \text{factor } u \\ &3(x + 18)(x^3 + x^2 + 8x + 13) \end{aligned} \quad (2)$$

Type: Factored Polynomial PrimeField 19



Convert this to have coefficients in the finite field with  $19^3$  elements. See Section 8.11 on page 316 for more information about finite fields.

```
factor(u :: POLY FFX(PF 19,3))
```

$$3 (x + 18) (x + 5 \%C^2 + 3 \%C + 13) (x + 16 \%C^2 + 14 \%C + 13) \times (x + 17 \%C^2 + 2 \%C + 13) \quad (3)$$

Type: Factored Polynomial FiniteFieldExtension(PrimeField 19, 3)

### 8.2.3 Simple Algebraic Extension Field Coefficients

Here, `aa` and `bb` are symbolic roots of polynomials.

```
aa := rootOf(aa**2+aa+1)
```

$$aa \quad (1)$$

Type: AlgebraicNumber

```
p:=(x**3+aa**2*x+y)*(aa*x**2+aa*x+aa*y**2)**2
```

$$\begin{aligned} &(-aa - 1) y^5 + ((-aa - 1) x^3 + aa x) y^4 + ((-2 aa - 2) x^2 + \\ &(-2 aa - 2) x) y^3 + ((-2 aa - 2) x^5 + (-2 aa - 2) x^4 + 2 aa x^3 + \\ &2 aa x^2) y^2 + ((-aa - 1) x^4 + (-2 aa - 2) x^3 + (-aa - 1) x^2) y + \\ &(-aa - 1) x^7 + (-2 aa - 2) x^6 - x^5 + 2 aa x^4 + aa x^3 \end{aligned} \quad (2)$$

Type: Polynomial AlgebraicNumber

Note that the second argument to `factor` can be a list of algebraic extensions to factor over.

```
factor(p, [aa])
```

$$(-aa - 1) (y + x^3 + (-aa - 1) x) (y^2 + x^2 + x)^2 \quad (3)$$

Type: Factored Polynomial AlgebraicNumber

This factors `x**2+3` over the integers.

```
factor(x**2+3)
```

$$x^2 + 3 \quad (4)$$

Type: Factored Polynomial Integer

Factor the same polynomial over the field obtained by adjoining `aa` to the rational numbers.

```
factor(x**2+3, [aa])
```

$$(x - 2 aa - 1) (x + 2 aa + 1) \quad (5)$$

Type: Factored Polynomial AlgebraicNumber

Factor `x**6+108` over the same field.

```
factor(x**6+108, [aa])
```

$$(x^3 - 12 aa - 6) (x^3 + 12 aa + 6) \quad (6)$$

Type: Factored Polynomial AlgebraicNumber

```
bb:=rootOf(bb**3-2)
```

$$bb \quad (7)$$

Type: AlgebraicNumber

```
factor(x**6+108,[bb])
```

$$(x^2 - 3bbx + 3bb^2)(x^2 + 3bb^2)(x^2 + 3bbx + 3bb^2) \quad (8)$$

Type: Factored Polynomial AlgebraicNumber

Factor again over the field obtained by adjoining both **aa** and **bb** to the rational numbers.

```
factor(x**6+108,[aa,bb])
```

$$(x + (-2aa - 1)bb)(x + (-aa - 2)bb)(x + (-aa + 1)bb) \times (x + (aa - 1)bb)(x + (aa + 2)bb)(x + (2aa + 1)bb) \quad (9)$$

Type: Factored Polynomial AlgebraicNumber

### 8.2.4 Factoring Rational Functions

There is, instead, a specific operation **factorFraction** that separately factors the numerator and denominator and returns a fraction of the factored results.

```
factorFraction((x**2-4)/(y**2-4))
```

$$\frac{(x-2)(x+2)}{(y-2)(y+2)} \quad (1)$$

Type: Fraction Factored Polynomial Integer

You can also use **map**. This expression applies the **factor** operation to the numerator and denominator.

```
map(factor,(x**2-4)/(y**2-4))
```

$$\frac{(x-2)(x+2)}{(y-2)(y+2)} \quad (2)$$

Type: Fraction Factored Polynomial Integer

## 8.3 Manipulating Symbolic Roots of a Polynomial

In this section we show you how to work with one root or all roots of a polynomial. These roots are represented symbolically (as opposed to being numeric approximations). See Section 8.5.2 on page 284 and Section 8.5.3 on page 286 for information about solving for the roots of one or more polynomials.

### 8.3.1 Using a Single Root of a Polynomial

This creates an algebraic number **a**.

```
a := rootOf(a**4+1,a)
```

*a*

(1)

Type: Expression Integer

To find the algebraic relation that defines **a**, use **definingPolynomial**.

```
definingPolynomial a
```

$a^4 + 1$

(2)

Type: Expression Integer

You can use **a** in any further expression, including a nested **rootOf**.

```
b := rootOf(b**2-a-1,b)
```

*b*

(3)

Type: Expression Integer

Higher powers of the roots are automatically reduced during calculations.

$a + b$

$b + a$

(4)

Type: Expression Integer

```
% ** 5
```

$(10 a^3 + 11 a^2 + 2 a - 4) b + 15 a^3 + 10 a^2 + 4 a - 10$

(5)

Type: Expression Integer

The operation **zeroOf** is similar to **rootOf**, except that it may express the root using radicals in some cases.

```
rootOf(c**2+c+1,c)
```

*c*

(6)

Type: Expression Integer

```
zeroOf(d**2+d+1,d)
```

$\frac{\sqrt{-3} - 1}{2}$

(7)

Type: Expression Integer

```
rootOf(e**5-2,e)
```

*e*

(8)

Type: Expression Integer

$$\text{zeroOf}(f^{**5}-2,f)$$

$$\sqrt[5]{2}$$

(9)

Type: Expression Integer

### 8.3.2 Using All Roots of a Polynomial

Compute all the roots of  $x^{**4} + 1$ .

Use **rootsOf** to get all symbolic roots of a polynomial: **rootsOf**(p, x) returns a list of all the roots of p(x). If p(x) has a multiple root of order n, then that root appears n times in the list.

```
l := rootsOf(x**4+1,x)
```

$$[\%x0, \%x0 \%x1, -\%x0, -\%x0 \%x1]$$

(1)

Type: List Expression Integer

As a side effect, the variables %x0, %x1 and %x2 are bound to the first three roots of  $x^{**4}+1$ .

```
%x0**5
```

$$-\%x0$$

(2)

Type: Expression Integer

Although they all satisfy  $x^{**4} + 1 = 0$ , %x0, %x1, and %x2 are different algebraic numbers. To find the algebraic relation that defines each of them, use **definingPolynomial**.

```
definingPolynomial %x0
```

$$\%x0^4 + 1$$

(3)

Type: Expression Integer

```
definingPolynomial %x1
```

$$\%x1^2 + 1$$

(4)

Type: Expression Integer

```
definingPolynomial %x2
```

$$-\%x2 + \%var$$

(5)

Type: Expression Integer

We can check that the sum and product of the roots of  $x^{**4}+1$  are its trace and norm.

```
x3 := last l
```

$$-\%x0 \%x1$$

(6)

Type: Expression Integer

$$\%x0 + \%x1 + \%x2 + x3$$

$$(-\%x0 + 1) \%x1 + \%x0 + \%x2$$

(7)

Type: Expression Integer

$$\%x0 * \%x1 * \%x2 * x3$$

$$\%x2 \%x0^2$$

(8)

Type: Expression Integer

Corresponding to the pair of operations **rootOf/zeroOf** in Section 8.5.2 on page 284, there is an operation **zerosOf** that, like **rootsOf**, computes all the roots of a given polynomial, but which expresses some of them in terms of radicals.

As you see, only one implicit algebraic number was created (**%y1**), and its defining equation is this. The other three roots are expressed in radicals.

$$\text{zerosOf}(y^{**4}+1,y) \quad \left[ \frac{\sqrt{-1}+1}{\sqrt{2}}, \frac{\sqrt{-1}-1}{\sqrt{2}}, \frac{-\sqrt{-1}-1}{\sqrt{2}}, \frac{-\sqrt{-1}+1}{\sqrt{2}} \right] \quad (9)$$

Type: List Expression Integer

$$\text{definingPolynomial } \%y1 \quad \%var^2 + 1 \quad (10)$$

Type: Expression Integer

## 8.4 Computation of Eigenvalues and Eigenvectors

In this section we show you some of AXIOM's facilities for computing and manipulating eigenvalues and eigenvectors, also called characteristic values and characteristic vectors, respectively.

Let's first create a matrix with integer entries.

```
m1 := matrix [[1,2,1],[2,1,-2],[1,-2,4]]
```

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -2 \\ 1 & -2 & 4 \end{bmatrix} \quad (1)$$

Type: Matrix Integer

To get a list of the *rational* eigenvalues, use the operation **eigenvalues**.

```
leig := eigenvalues(m1)
```

$$\left[ 5, \left( \%E \mid \%E^2 - \%E - 5 \right) \right] \quad (2)$$

Type: List Union(Fraction Polynomial Integer, SuchThat(Symbol, Polynomial Integer))

Given an explicit eigenvalue, **eigenvector** computes the eigenvectors corresponding to it.

```
eigenvector(first(leig),m1)
```

$$\left[ \begin{bmatrix} 0 \\ -\frac{1}{2} \\ 1 \end{bmatrix} \right] \quad (3)$$

Type: List Matrix Fraction Polynomial Fraction Integer

The operation **eigenvectors** returns a list of pairs of values and vectors. When an eigenvalue is rational, AXIOM gives you the value explicitly; otherwise, its minimal polynomial is given, (the polynomial of lowest degree with the eigenvalues as roots), together with a parametric representation of the eigenvector using the eigenvalue. This means that if you ask AXIOM to **solve** the minimal polynomial, then you can substitute these roots into the parametric form of the corresponding eigenvectors.

You must be aware that unless an exact eigenvalue has been computed, the eigenvector may be badly in error.

```
eigenvectors(m1)
```

$$\left[ \left[ \text{eigval} = 5, \text{eigmult} = 1, \text{eigvec} = \begin{bmatrix} 0 \\ -\frac{1}{2} \\ 1 \end{bmatrix} \right] \right], \quad (4)$$

$$\left[ \left[ \text{eigval} = \left( \%F \mid \%F^2 - \%F - 5 \right), \text{eigmult} = 1, \text{eigvec} = \begin{bmatrix} \%F \\ 2 \\ 1 \end{bmatrix} \right] \right]$$

Type: List Record(eigval: Union(Fraction Polynomial Integer, SuchThat(Symbol, Polynomial Integer)), eigmult: NonNegativeInteger, eigvec: List Matrix Fraction Polynomial Integer)

Another possibility is to use the operation **radicalEigenvectors** tries to compute explicitly the eigenvectors in terms of radicals.

**radicalEigenvectors(m1)**

$$\left[ \left[ \begin{array}{l} \text{radval} = \frac{\sqrt{21} + 1}{2}, \text{radmult} = 1, \text{radvect} = \left[ \left[ \begin{array}{l} \frac{\sqrt{21}+1}{2} \\ 2 \\ 1 \end{array} \right] \right] \right], \right. \\ \left[ \begin{array}{l} \text{radval} = \frac{-\sqrt{21} + 1}{2}, \text{radmult} = 1, \text{radvect} = \left[ \left[ \begin{array}{l} \frac{-\sqrt{21}+1}{2} \\ 2 \\ 1 \end{array} \right] \right] \right], \\ \left[ \begin{array}{l} \text{radval} = 5, \text{radmult} = 1, \text{radvect} = \left[ \left[ \begin{array}{l} 0 \\ -\frac{1}{2} \\ 1 \end{array} \right] \right] \end{array} \right] \right] \end{array} \quad (5)$$

Type: List Record(radval: Expression Integer, radmult: Integer, radvect: List Matrix Expression Integer)

Alternatively, AXIOM can compute real or complex approximations to the eigenvectors and eigenvalues using the operations **realEigenvectors** or **complexEigenvectors**. They each take an additional argument  $\epsilon$  to specify the “precision” required. In the real case, this means that each approximation will be within  $\pm\epsilon$  of the actual result. In the complex case, this means that each approximation will be within  $\pm\epsilon$  of the actual result in each of the real and imaginary parts.

The precision can be specified as a Float if the results are desired in floating-point notation, or as Fraction Integer if the results are to be expressed using rational (or complex rational) numbers.

**realEigenvectors(m1, 1/1000)**

$$\left[ \left[ \begin{array}{l} \text{outval} = 5, \text{outmult} = 1, \text{outvect} = \left[ \left[ \begin{array}{l} 0 \\ -\frac{1}{2} \\ 1 \end{array} \right] \right] \right], \right. \\ \left[ \begin{array}{l} \text{outval} = \frac{5717}{2048}, \text{outmult} = 1, \text{outvect} = \left[ \left[ \begin{array}{l} \frac{5717}{2048} \\ 2 \\ 1 \end{array} \right] \right] \right], \\ \left[ \begin{array}{l} \text{outval} = -\frac{3669}{2048}, \text{outmult} = 1, \text{outvect} = \left[ \left[ \begin{array}{l} -\frac{3669}{2048} \\ 2 \\ 1 \end{array} \right] \right] \end{array} \right] \right] \end{array} \quad (6)$$

Type: List Record(outval: Fraction Integer, outmult: Integer, outvect: List Matrix Fraction Integer)

If an  $n$  by  $n$  matrix has  $n$  distinct eigenvalues (and therefore  $n$  eigenvectors) the operation **eigenMatrix** gives you a matrix of the eigenvectors.

**eigenMatrix(m1)**

$$\left[ \begin{array}{ccc} \frac{\sqrt{21}+1}{2} & \frac{-\sqrt{21}+1}{2} & 0 \\ 2 & 2 & -\frac{1}{2} \\ 1 & 1 & 1 \end{array} \right] \quad (7)$$

Type: Union(Matrix Expression Integer, ...)

$$\begin{aligned} \text{m2} &:= \text{matrix} \left[ [-5, -2], [18, 7] \right] \\ &\begin{bmatrix} -5 & -2 \\ 18 & 7 \end{bmatrix} \end{aligned} \tag{8}$$

Type: Matrix Integer

$$\begin{aligned} &\text{eigenMatrix}(\text{m2}) \\ &\text{"failed"} \end{aligned} \tag{9}$$

Type: Union("failed", ...)

If a symmetric matrix has a basis of orthonormal eigenvectors, then **orthonormalBasis** computes a list of these vectors.

$$\begin{aligned} \text{m3} &:= \text{matrix} \left[ [1, 2], [2, 1] \right] \\ &\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \end{aligned} \tag{10}$$

Type: Matrix Integer

$$\begin{aligned} &\text{orthonormalBasis}(\text{m3}) \\ &\left[ \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \right] \end{aligned} \tag{11}$$

Type: List Matrix Expression Integer



## 8.5 Solution of Linear and Polynomial Equations

### 8.5.1 Solution of Systems of Linear Equations

To solve the linear system:

$$\begin{array}{rrcr} x & + & y & + & z & = & 8 \\ 3x & - & 2y & + & z & = & 0 \\ x & + & 2y & + & 2z & = & 17 \end{array}$$

evaluate this expression.

When you solve the linear system

$$\begin{array}{rrcr} x & + & 2y & + & 3z & = & 2 \\ 2x & + & 3y & + & 4z & = & 2 \\ 3x & + & 4y & + & 5z & = & 2 \end{array}$$

with this expression you get a solution involving a parameter.

To solve the system:

$$\begin{array}{rrcr} x & + & y & + & z & = & 8 \\ 3x & - & 2y & + & z & = & 0 \\ x & + & 2y & + & 2z & = & 17 \end{array}$$

in matrix form you would evaluate this expression.

In this section we discuss the AXIOM facilities for solving systems of linear equations, finding the roots of polynomials and solving systems of polynomial equations. For a discussion of the solution of differential equations, see Section 8.10 on page 308.

You can use the operation **solve** to solve systems of linear equations.

The operation **solve** takes two arguments, the list of equations and the list of the unknowns to be solved for. A system of linear equations need not have a unique solution.

```
solve([x+y+z=8, 3*x-2*y+z=0, x+2*y+2*z=17], [x,y,z])
```

```
[[x = -1, y = 2, z = 7]]
```

 (1)

Type: List List Equation Fraction Polynomial Integer

Parameters are given as new variables starting with a percent sign and “%” and the variables are expressed in terms of the parameters. If the system has no solutions then the empty list is returned.

```
solve([x+2*y+3*z=2, 2*x+3*y+4*z=2, 3*x+4*y+5*z=2], [x,y,z])
```

```
[[x = %K - 2, y = -2 %K + 2, z = %K]]
```

 (2)

Type: List List Equation Fraction Polynomial Integer

The system can also be presented as a matrix and a vector. The matrix contains the coefficients of the linear equations and the vector contains the numbers appearing on the right-hand sides of the equations. You may input the matrix as a list of rows and the vector as a list of its elements.

```
solve([[1,1,1], [3,-2,1], [1,2,2]], [8,0,17])
```

```
[particular = [-1, 2, 7], basis = [[0, 0, 0]]]
```

 (3)

Type: Record(particular: Union(Vector Fraction Integer, "failed"), basis: List Vector Fraction Integer)

The solutions are presented as a Record with two components: the component *particular* contains a particular solution of the given system or the

item "failed" if there are no solutions, the component *basis* contains a list of vectors that are a basis for the space of solutions of the corresponding homogeneous system. If the system of linear equations does not have a unique solution, then the *basis* component contains non-trivial vectors.

This happens when you solve the linear system

$$\begin{array}{rrcr} x & + & 2y & + & 3z & = & 2 \\ 2x & + & 3y & + & 4z & = & 2 \\ 3x & + & 4y & + & 5z & = & 2 \end{array}$$

with this command.

```
solve([ [1,2,3], [2,3,4], [3,4,5] ], [2,2,2])
```

```
[particular = [-2, 2, 0], basis = [[1, -2, 1]]] (4)
```

Type: Record(particular: Union(Vector Fraction Integer, "failed"), basis: List Vector Fraction Integer)

All solutions of this system are obtained by adding the particular solution with a linear combination of the *basis* vectors.

When no solution exists then "failed" is returned as the *particular* component, as follows:

```
solve([ [1,2,3], [2,3,4], [3,4,5] ], [2,3,2])
```

```
[particular = "failed", basis = [[1, -2, 1]]] (5)
```

Type: Record(particular: Union(Vector Fraction Integer, "failed"), basis: List Vector Fraction Integer)

When you want to solve a system of homogeneous equations (that is, a system where the numbers on the right-hand sides of the equations are all zero) in the matrix form you can omit the second argument and use the **nullSpace** operation.

This computes the solutions of the following system of equations:

$$\begin{array}{rrcr} x & + & 2y & + & 3z & = & 0 \\ 2x & + & 3y & + & 4z & = & 0 \\ 3x & + & 4y & + & 5z & = & 0 \end{array}$$

The result is given as a list of vectors and these vectors form a basis for the solution space.

```
nullSpace([ [1,2,3], [2,3,4], [3,4,5] ]) (6)
```

```
[[1, -2, 1]]
```

Type: List Vector Integer

## 8.5.2 Solution of a Single Polynomial Equation

AXIOM can solve polynomial equations producing either approximate or exact solutions. Exact solutions are either members of the ground field or can be presented symbolically as roots of irreducible polynomials.

This returns the one rational root along with an irreducible polynomial describing the other solutions.

$$\text{solve}(x^3 = 8, x)$$

$$\left[ x = 2, x^2 + 2x + 4 = 0 \right] \quad (1)$$

Type: List Equation Fraction Polynomial Integer

If you want solutions expressed in terms of radicals you would use this instead.

$$\text{radicalSolve}(x^3 = 8, x)$$

$$\left[ x = -\sqrt{-3} - 1, x = \sqrt{-3} - 1, x = 2 \right] \quad (2)$$

Type: List Equation Expression Integer

The **solve** command always returns a value but **radicalSolve** returns only the solutions that it is able to express in terms of radicals.

If the polynomial equation has rational coefficients you can ask for approximations to its real roots by calling solve with a second argument that specifies the “precision”  $\epsilon$ . This means that each approximation will be within  $\pm\epsilon$  of the actual result.

Notice that the type of second argument controls the type of the result.

$$\text{solve}(x^4 - 10x^3 + 35x^2 - 50x + 25, .0001)$$

$$[x = 3.618011474609375, x = 1.381988525390625] \quad (3)$$

Type: List Equation Polynomial Float

If you give a floating-point precision you get a floating-point result; if you give the precision as a rational number you get a rational result.

$$\text{solve}(x^3 - 2, 1/1000)$$

$$\left[ x = \frac{2581}{2048} \right] \quad (4)$$

Type: List Equation Polynomial Fraction Integer

If you want approximate complex results you should use the command **complexSolve** that takes the same precision argument  $\epsilon$ .

$$\text{complexSolve}(x^3 - 2, .0001)$$

$$\begin{aligned} &[x = 1.259918212890625, \\ &x = -0.62989432795395613131 - 1.091094970703125i, \\ &x = -0.62989432795395613131 + 1.091094970703125i] \end{aligned} \quad (5)$$

Type: List Equation Polynomial Complex Float

Each approximation will be within  $\pm\epsilon$  of the actual result in each of the real and imaginary parts.

$$\text{complexSolve}(x^2 - 2\%i + 1, 1/100)$$

$$\left[ x = -\frac{13028925}{16777216} - \frac{325}{256}i, x = \frac{13028925}{16777216} + \frac{325}{256}i \right] \quad (6)$$

Type: List Equation Polynomial Complex Fraction Integer

Note that if you omit the “=” from the first argument AXIOM generates an equation by equating the first argument to zero. Also, when only one variable is present in the equation, you do not need to specify the variable to be solved for, that is, you can omit the second argument.

AXIOM can also solve equations involving rational functions. Solutions where the denominator vanishes are discarded.

`radicalSolve(1/x**3 + 1/x**2 + 1/x = 0, x)`

$$\left[ x = \frac{-\sqrt{-3} - 1}{2}, x = \frac{\sqrt{-3} - 1}{2} \right] \quad (7)$$

Type: List Equation Expression Integer

### 8.5.3 Solution of Systems of Polynomial Equations

Given a system of equations of rational functions with exact coefficients:

$$\begin{aligned} p_1(x_1, \dots, x_n) \\ \vdots \\ p_m(x_1, \dots, x_n) \end{aligned}$$

AXIOM can find numeric or symbolic solutions. The system is first split into irreducible components, then for each component, a triangular system of equations is found that reduces the problem to sequential solution of univariate polynomials resulting from substitution of partial solutions from the previous stage.

$$\begin{aligned} q_1(x_1, \dots, x_n) \\ \vdots \\ q_m(x_n) \end{aligned}$$

Symbolic solutions can be presented using “implicit” algebraic numbers defined as roots of irreducible polynomials or in terms of radicals. AXIOM can also find approximations to the real or complex roots of a system of polynomial equations to any user-specified accuracy.

The operation **solve** for systems is used in a way similar to **solve** for single equations. Instead of a polynomial equation, one has to give a list of equations and instead of a single variable to solve for, a list of variables. For solutions of single equations see Section 8.5.2 on page 284.

Use the operation **solve** if you want implicitly presented solutions.

`solve([3*x**3 + y + 1, y**2 - 4], [x, y])`

$$\left[ \left[ x = -1, y = 2 \right], \left[ x^2 - x + 1 = 0, y = 2 \right], \left[ 3x^3 - 1 = 0, y = -2 \right] \right] \quad (1)$$

Type: List List Equation Fraction Polynomial Integer

`solve([x = y**2-19, y = z**2+x+3, z = 3*x], [x, y, z])`

$$\left[ \left[ x = \frac{z}{3}, y = \frac{3z^2 + z + 9}{3}, 9z^4 + 6z^3 + 55z^2 + 15z - 90 = 0 \right] \right] \quad (2)$$

Type: List List Equation Fraction Polynomial Integer

Use **radicalSolve** if you want your solutions expressed in terms of radicals.

$$\begin{aligned} &\text{radicalSolve}([3*x**3 + y + 1, y**2 - 4], [x, y]) \\ &\left[ \left[ x = \frac{\sqrt{-3} + 1}{2}, y = 2 \right], \left[ x = \frac{-\sqrt{-3} + 1}{2}, y = 2 \right], \right. \\ &\left[ x = \frac{-\sqrt{-1} \sqrt{3} - 1}{2 \sqrt[3]{3}}, y = -2 \right], \left[ x = \frac{\sqrt{-1} \sqrt{3} - 1}{2 \sqrt[3]{3}}, y = -2 \right], \\ &\left. \left[ x = \frac{1}{\sqrt[3]{3}}, y = -2 \right], \left[ x = -1, y = 2 \right] \right] \end{aligned} \quad (3)$$

Type: List List Equation Expression Integer

To get numeric solutions you only need to give the list of equations and the precision desired. The list of variables would be redundant information since there can be no parameters for the numerical solver.

If the precision is expressed as a floating-point number you get results expressed as floats.

$$\begin{aligned} &\text{solve}([x**2*y - 1, x*y**2 - 2], .01) \\ &[[y = 1.5859375, x = 0.79296875]] \end{aligned} \quad (4)$$

Type: List List Equation Polynomial Float

To get complex numeric solutions, use the operation **complexSolve**, which takes the same arguments as in the real case.

$$\begin{aligned} &\text{complexSolve}([x**2*y - 1, x*y**2 - 2], 1/1000) \\ &\left[ \left[ y = \frac{1625}{1024}, x = \frac{1625}{2048} \right], \right. \\ &\left[ y = -\frac{435445573689}{549755813888} - \frac{1407}{1024} i, x = -\frac{435445573689}{1099511627776} - \frac{1407}{2048} i \right], \\ &\left. \left[ y = -\frac{435445573689}{549755813888} + \frac{1407}{1024} i, x = -\frac{435445573689}{1099511627776} + \frac{1407}{2048} i \right] \right] \end{aligned} \quad (5)$$

Type: List List Equation Polynomial Complex Fraction Integer

It is also possible to solve systems of equations in rational functions over the rational numbers. Note that  $[x = 0.0, a = 0.0]$  is not returned as a solution since the denominator vanishes there.

$$\begin{aligned} &\text{solve}([x**2/a = a, a = a*x], .001) \\ &[[x = 1.0, a = -1.0], [x = 1.0, a = 1.0]] \end{aligned} \quad (6)$$

Type: List List Equation Polynomial Float

When solving equations with denominators, all solutions where the denominator vanishes are discarded.

$$\begin{aligned} &\text{radicalSolve}([x**2/a + a + y**3 - 1, a*y + a + 1], [x, y]) \\ &\left[ \left[ x = -\sqrt{\frac{-a^4 + 2 a^3 + 3 a^2 + 3 a + 1}{a^2}}, y = \frac{-a - 1}{a} \right], \right. \\ &\left. \left[ x = \sqrt{\frac{-a^4 + 2 a^3 + 3 a^2 + 3 a + 1}{a^2}}, y = \frac{-a - 1}{a} \right] \right] \end{aligned} \quad (7)$$

Type: List List Equation Expression Integer

## 8.6 Limits

Issue this to compute the limit

$$\lim_{x \rightarrow 1} \frac{x^2 - 3x + 2}{x^2 - 1}.$$

The function  $\log(x)$  is only defined to the right of zero, that is, for  $x > 0$ . Thus, when computing limits of functions involving  $\log(x)$ , you probably want a “right-hand” limit.

When you do not specify “right” or “left” as the optional fourth argument, **limit** tries to compute a two-sided limit. Here the limit from the left does not exist, as AXIOM indicates when you try to take a two-sided limit.

This is what happens when we take the limit at  $y = 0$ . The answer returned by AXIOM gives both a “left-hand” and a “right-hand” limit.

Here is another example, this time using a more complicated function.

To compute a limit, you must specify a functional expression, a variable, and a limiting value for that variable. If you do not specify a direction, AXIOM attempts to compute a two-sided limit.

$$\text{limit}((x^2 - 3x + 2)/(x^2 - 1), x = 1) \\ -\frac{1}{2} \quad (1)$$

Type: Union(OrderedCompletion Fraction Polynomial Integer, ...)

Sometimes the limit when approached from the left is different from the limit from the right and, in this case, you may wish to ask for a one-sided limit. Also, if you have a function that is only defined on one side of a particular value, you can compute a one-sided limit.

$$\text{limit}(x * \log(x), x = 0, \text{"right"}) \\ 0 \quad (2)$$

Type: Union(OrderedCompletion Expression Integer, ...)

$$\text{limit}(x * \log(x), x = 0) \\ [\text{leftHandLimit} = \text{"failed"}, \text{rightHandLimit} = 0] \quad (3)$$

Type: Union(Record(leftHandLimit: Union(OrderedCompletion Expression Integer, "failed"), rightHandLimit: Union(OrderedCompletion Expression Integer, "failed")), ...)

A function can be defined on both sides of a particular value, but tend to different limits as its variable approaches that value from the left and from the right. We can construct an example of this as follows: Since  $\sqrt{y^2}$  is simply the absolute value of  $y$ , the function  $\sqrt{y^2}/y$  is simply the sign (+1 or -1) of the nonzero real number  $y$ . Therefore,  $\sqrt{y^2}/y = -1$  for  $y < 0$  and  $\sqrt{y^2}/y = +1$  for  $y > 0$ .

$$\text{limit}(\text{sqrt}(y^2)/y, y = 0) \\ [\text{leftHandLimit} = -1, \text{rightHandLimit} = 1] \quad (4)$$

Type: Union(Record(leftHandLimit: Union(OrderedCompletion Expression Integer, "failed"), rightHandLimit: Union(OrderedCompletion Expression Integer, "failed")), ...)

$$\text{limit}(\text{sqrt}(1 - \cos(t))/t, t = 0) \\ \left[ \text{leftHandLimit} = -\frac{1}{\sqrt{2}}, \text{rightHandLimit} = \frac{1}{\sqrt{2}} \right] \quad (5)$$

Type: Union(Record(leftHandLimit: Union(OrderedCompletion Expression Integer, "failed"), rightHandLimit: Union(OrderedCompletion Expression Integer, "failed")), ...)

You can compute limits at infinity by passing either  $+\infty$  or  $-\infty$  as the third argument of **limit**.

To do this, use the constants **%plusInfinity** and **%minusInfinity**.

$$\text{limit}(\text{sqrt}(3*x**2 + 1)/(5*x), x = \%plusInfinity)$$

$$\frac{\sqrt{3}}{5} \quad (6)$$

Type: Union(OrderedCompletion Expression Integer, ...)

$$\text{limit}(\text{sqrt}(3*x**2 + 1)/(5*x), x = \%minusInfinity)$$

$$-\frac{\sqrt{3}}{5} \quad (7)$$

Type: Union(OrderedCompletion Expression Integer, ...)

You can take limits of functions with parameters. As you can see, the limit is expressed in terms of the parameters.

$$\text{limit}(\text{sinh}(a*x)/\text{tan}(b*x), x = 0)$$

$$\frac{a}{b} \quad (8)$$

Type: Union(OrderedCompletion Expression Integer, ...)

When you use **limit**, you are taking the limit of a real function of a real variable.

When you compute this, AXIOM returns 0 because, as a function of a real variable,  $\sin(1/z)$  is always between -1 and 1, so  $z * \sin(1/z)$  tends to 0 as  $z$  tends to 0.

$$\text{limit}(z * \sin(1/z), z = 0)$$

$$0 \quad (9)$$

Type: Union(OrderedCompletion Expression Integer, ...)

However, as a function of a *complex* variable,  $\sin(1/z)$  is badly behaved near 0 (one says that  $\sin(1/z)$  has an *essential singularity* at  $z = 0$ ).

When viewed as a function of a complex variable,  $z * \sin(1/z)$  does not approach any limit as  $z$  tends to 0 in the complex plane. AXIOM indicates this when we call **complexLimit**.

$$\text{complexLimit}(z * \sin(1/z), z = 0)$$

$$\text{"failed"} \quad (10)$$

Type: Union("failed", ...)

You can also take complex limits at infinity, that is, limits of a function of  $z$  as  $z$  approaches infinity on the Riemann sphere. Use the symbol **%infinity** to denote “complex infinity.”

As above, to compute complex limits rather than real limits, use **complexLimit**.

$$\text{complexLimit}((2 + z)/(1 - z), z = \%infinity)$$

$$-1 \quad (11)$$

Type: OnePointCompletion Fraction Polynomial Integer

In many cases, a limit of a real function of a real variable exists when the corresponding complex limit does not. This limit exists.

$$\text{limit}(\sin(x)/x, x = \%plusInfinity)$$

$$0 \quad (12)$$

Type: Union(OrderedCompletion Expression Integer, ...)

But this limit does not.

```
complexLimit(sin(x)/x,x = %infinity)
```

```
"failed"
```

(13)

Type: Union("failed", ...)



## 8.7 Laplace Transforms

To compute the forward Laplace transform of  $F(t)$  with respect to  $t$  and express the result as  $f(s)$ , issue the command `laplace(F(t), t, s)`.

Here are some other non-trivial examples.

AXIOM can compute some forward Laplace transforms, mostly of elementary functions not involving logarithms, although some cases of special functions are handled.

$$\text{laplace}(\sin(a*t)*\cosh(a*t)-\cos(a*t)*\sinh(a*t), t, s) \\ \frac{4 a^3}{s^4 + 4 a^4} \quad (1)$$

Type: Expression Integer

$$\text{laplace}((\exp(a*t) - \exp(b*t))/t, t, s) \\ -\log(s - a) + \log(s - b) \quad (2)$$

Type: Expression Integer

$$\text{laplace}(2/t * (1 - \cos(a*t)), t, s) \\ \log(s^2 + a^2) - 2 \log(s) \quad (3)$$

Type: Expression Integer

$$\text{laplace}(\exp(-a*t) * \sin(b*t) / b^{**2}, t, s) \\ \frac{1}{b s^2 + 2 a b s + b^3 + a^2 b} \quad (4)$$

Type: Expression Integer

$$\text{laplace}((\cos(a*t) - \cos(b*t))/t, t, s) \\ \frac{\log(s^2 + b^2) - \log(s^2 + a^2)}{2} \quad (5)$$

Type: Expression Integer

AXIOM also knows about a few special functions.

$$\text{laplace}(\exp(a*t+b)*\text{Ei}(c*t), t, s) \\ \frac{e^b \log\left(\frac{s+c-a}{c}\right)}{s - a} \quad (6)$$

Type: Expression Integer

$$\text{laplace}(a*\text{Ci}(b*t) + c*\text{Si}(d*t), t, s) \\ \frac{a \log\left(\frac{s^2+b^2}{b^2}\right) + 2 c \arctan\left(\frac{d}{s}\right)}{2 s} \quad (7)$$

Type: Expression Integer

When AXIOM does not know about a particular transform, it keeps it as a formal transform in the answer.

$$\text{laplace}(\sin(a*t) - a*t*\cos(a*t) + \exp(t^{**2}), t, s) \\ \frac{(s^4 + 2 a^2 s^2 + a^4) \text{laplace}(e^{t^2}, t, s) + 2 a^3}{s^4 + 2 a^2 s^2 + a^4} \quad (8)$$

Type: Expression Integer

## 8.8 Integration

The package `FunctionSpaceIntegration` provides the top-level integration operation, **integrate**, for integrating real-valued elementary functions.

Unfortunately, antiderivatives of most functions cannot be expressed in terms of elementary functions.

Similar functions may have antiderivatives that look quite different because the form of the antiderivative depends on the sign of a constant that appears in the function.

In this case AXIOM returns a list of answers that cover all the possible cases. Here you use the answer involving the square root of  $a$  when  $a > 0$  and the answer involving the square root of  $-a$  when  $a < 0$ .

Integration is the reverse process of differentiation, that is, an *integral* of a function  $f$  with respect to a variable  $x$  is any function  $g$  such that  $D(g, x)$  is equal to  $f$ .

$$\text{integrate}(\cosh(a*x)*\sinh(a*x), x)$$

$$\frac{\sinh(ax)^2 + \cosh(ax)^2}{4a}$$

(1)

Type: Union(Expression Integer, ...)

$$\text{integrate}(\log(1 + \sqrt{a*x + b}) / x, x)$$

$$\int^x \frac{\log(\sqrt{b + \%X a} + 1)}{\%X} d\%X$$

(2)

Type: Union(Expression Integer, ...)

Given an elementary function to integrate, AXIOM returns a formal integral as above only when it can prove that the integral is not elementary and not when it cannot determine the integral. In this rare case it prints a message that it cannot determine if an elementary integral exists.

$$\text{integrate}(1/(x**2 - 2), x)$$

$$\frac{\log\left(\frac{(x^2+2)\sqrt{2}-4x}{x^2-2}\right)}{2\sqrt{2}}$$

(3)

Type: Union(Expression Integer, ...)

$$\text{integrate}(1/(x**2 + 2), x)$$

$$\frac{\arctan\left(\frac{x\sqrt{2}}{2}\right)}{\sqrt{2}}$$

(4)

Type: Union(Expression Integer, ...)

If the integrand contains parameters, then there may be several possible antiderivatives, depending on the signs of expressions of the parameters.

$$\text{integrate}(x**2 / (x**4 - a**2), x)$$

$$\left[ \frac{\log\left(\frac{(x^2+a)\sqrt{a}-2ax}{x^2-a}\right) + 2\arctan\left(\frac{x\sqrt{a}}{a}\right)}{4\sqrt{a}}, \right.$$

$$\left. \frac{\log\left(\frac{(x^2-a)\sqrt{-a}+2ax}{x^2+a}\right) - 2\arctan\left(\frac{x\sqrt{-a}}{a}\right)}{4\sqrt{-a}} \right]$$

(5)

Type: Union(List Expression Integer, ...)

If the parameters and the variables of integration can be complex numbers rather than real, then the notion of sign is not defined. In this case all the possible answers can be expressed as one complex function. To get that function, rather than a list of real functions, use **complexIntegrate**, which is provided by the package `FunctionSpaceComplexIntegration`.

This operation is used for integrating complex-valued elementary functions.

$$\text{complexIntegrate}(x^{**2} / (x^{**4} - a^{**2}), x)$$

$$\frac{\left( \sqrt{4a} \log\left(\frac{x\sqrt{-4a} + 2a}{\sqrt{-4a}}\right) - \sqrt{-4a} \log\left(\frac{x\sqrt{4a} + 2a}{\sqrt{4a}}\right) + \sqrt{-4a} \log\left(\frac{x\sqrt{4a} - 2a}{\sqrt{4a}}\right) - \sqrt{4a} \log\left(\frac{x\sqrt{-4a} - 2a}{\sqrt{-4a}}\right) \right)}{2\sqrt{-4a}\sqrt{4a}} \quad (6)$$

Type: Expression Integer

As with the real case, antiderivatives for most complex-valued functions cannot be expressed in terms of elementary functions.

$$\text{complexIntegrate}(\log(1 + \text{sqrt}(a * x + b)) / x, x)$$

$$\int x \frac{\log(\sqrt{b + \%X a} + 1)}{\%X} d\%X \quad (7)$$

Type: Expression Integer

Sometimes **integrate** can involve symbolic algebraic numbers such as those returned by **rootOf**. To see how to work with these strange generated symbols (such as `%a0`), see Section 8.3.2 on page 278.

Definite integration is the process of computing the area between the  $x$ -axis and the curve of a function  $f(x)$ . The fundamental theorem of calculus states that if  $f$  is continuous on an interval  $a..b$  and if there exists a function  $g$  that is differentiable on  $a..b$  and such that  $D(g, x)$  is equal to  $f$ , then the definite integral of  $f$  for  $x$  in the interval  $a..b$  is equal to  $g(b) - g(a)$ .

The package `RationalFunction-DefiniteIntegration` provides the top-level definite integration operation, **integrate**, for integrating real-valued rational functions.

$$\text{integrate}((x^{**4} - 3*x^{**2} + 6) / (x^{**6} - 5*x^{**4} + 5*x^{**2} + 4), x = 1..2)$$

$$\frac{2 \arctan(8) + 2 \arctan(5) + 2 \arctan(2) + 2 \arctan\left(\frac{1}{2}\right) - \pi}{2} \quad (8)$$

Type: Union(f1: OrderedCompletion Expression Integer, ...)

AXIOM checks beforehand that the function you are integrating is defined on the interval  $a..b$ , and prints an error message if it finds that this is not case, as in the following example:

```
integrate(1/(x**2-2), x = 1..2)

>> Error detected within library code:
    Pole in path of integration
    You are being returned to the top level
    of the interpreter.
```

When parameters are present in the function, the function may or may not be defined on the interval of integration.

If this is the case, AXIOM issues a warning that a pole might lie in the path of integration, and does not compute the integral.

`integrate(1/(x**2-a), x = 1..2)`

*potentialPole*

(9)

Type: Union(pole: potentialPole, ...)

If you know that you are using values of the parameter for which the function has no pole in the interval of integration, use the string "noPole" as a third argument to **integrate**:

The value here is, of course, incorrect if **sqrt(a)** is between 1 and 2.

`integrate(1/(x**2-a), x = 1..2, "noPole")`

$$\left[ \frac{-\log\left(\frac{(-4a^2-4a)\sqrt{a}+a^3+6a^2+a}{a^2-2a+1}\right) + \log\left(\frac{(-8a^2-32a)\sqrt{a}+a^3+24a^2+16a}{a^2-8a+16}\right)}{4\sqrt{a}}, \right. \\ \left. \frac{-\arctan\left(\frac{2\sqrt{-a}}{a}\right) + \arctan\left(\frac{\sqrt{-a}}{a}\right)}{\sqrt{-a}} \right] \quad (10)$$

Type: Union(f2: List OrderedCompletion Expression Integer, ...)

## 8.9 Working with Power Series

---

AXIOM has very sophisticated facilities for working with power series. Infinite series are represented by a list of the coefficients that have already been determined, together with a function for computing the additional coefficients if needed. The system command that determines how many terms of a series is displayed is `)set streams calculate`. For the purposes of this book, we have used this system command to display fewer than ten terms. Series can be created from expressions, from functions for the series coefficients, and from applications of operations on existing series. The most general function for creating a series is called **series**, although you can also use **taylor**, **laurent** and **puiseux** in situations where you know what kind of exponents are involved.

For information about solving differential equations in terms of power series, see Section 8.10.3 on page 314.

### 8.9.1 Creation of Power Series

---

This is the easiest way to create a power series. This tells AXIOM that `x` is to be treated as a power series, so functions of `x` are again power series.

```
x := series 'x
```

$$x$$

(1)

Type: UnivariatePuisseuxSeries(Expression Integer,x, 0)

We didn't say anything about the coefficients of the power series, so the coefficients are general expressions over the integers. This allows us to introduce denominators, symbolic constants, and other variables as needed.

Here the coefficients are integers (note that the coefficients are the Fibonacci numbers).

```
1/(1 - x - x**2)
```

$$1 + x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + 13x^6 + 21x^7 + O(x^8)$$

(2)

Type: UnivariatePuisseuxSeries(Expression Integer,x, 0)

This series has coefficients that are rational numbers.

```
sin(x)
```

$$x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + O(x^9)$$

(3)

Type: UnivariatePuisseuxSeries(Expression Integer,x, 0)

When you enter this expression you introduce the symbolic constants `sin(1)` and `cos(1)`.

```
sin(1 + x)
```

$$\sin(1) + \cos(1)x - \frac{\sin(1)}{2}x^2 - \frac{\cos(1)}{6}x^3 + \frac{\sin(1)}{24}x^4 + \frac{\cos(1)}{120}x^5 - \frac{\sin(1)}{720}x^6 - \frac{\cos(1)}{5040}x^7 + O(x^8)$$

(4)

Type: UnivariatePuisseuxSeries(Expression Integer,x, 0)

When you enter the expression the variable **a** appears in the resulting series expansion.

$$\sin(a * x)$$

$$a x - \frac{a^3}{6} x^3 + \frac{a^5}{120} x^5 - \frac{a^7}{5040} x^7 + O(x^9)$$

(5)

Type: UnivariatePuisseuxSeries(Expression Integer, x, 0)

You can also convert an expression into a series expansion. This expression creates the series expansion of  $1/\log(y)$  about  $y = 1$ . For details and more examples, see Section 8.9.5 on page 300.

$$\text{series}(1/\log(y), y = 1)$$

$$(y - 1)^{(-1)} + \frac{1}{2} - \frac{1}{12} (y - 1) + \frac{1}{24} (y - 1)^2 - \frac{19}{720} (y - 1)^3 +$$

$$\frac{3}{160} (y - 1)^4 - \frac{863}{60480} (y - 1)^5 + \frac{275}{24192} (y - 1)^6 + O((y - 1)^7)$$

(6)

Type: UnivariatePuisseuxSeries(Expression Integer, y, 1)

You can create power series with more general coefficients. You normally accomplish this via a type declaration (see Section 2.3 on page 103). See Section 8.9.4 on page 298 for some warnings about working with declared series.

We declare that **y** is a one-variable Taylor series (UTS is the abbreviation for UnivariateTaylorSeries) in the variable **z** with FLOAT (that is, floating-point) coefficients, centered about 0. Then, by assignment, we obtain the Taylor expansion of  $\exp(z)$  with floating-point coefficients.

$$y : \text{UTS}(\text{FLOAT}, 'z, 0) := \exp(z)$$

$$1.0 + z + 0.5 z^2 + 0.166666666666666667 z^3 +$$

$$0.04166666666666666667 z^4 + 0.00833333333333333334 z^5 +$$

$$0.00138888888888888889 z^6 + 0.0001984126984126984127 z^7 +$$

$$O(z^8)$$

(7)

Type: UnivariateTaylorSeries(Float, z, 0.0)

You can also create a power series by giving an explicit formula for its  $n^{\text{th}}$  coefficient. For details and more examples, see Section 8.9.6 on page 302.

To create a series about  $w = 0$  whose  $n^{\text{th}}$  Taylor coefficient is  $1/n!$ , you can evaluate this expression. This is the Taylor expansion of  $\exp(w)$  at  $w = 0$ .

$$\text{series}(1/\text{factorial}(n), n, w = 0)$$

$$1 + w + \frac{1}{2} w^2 + \frac{1}{6} w^3 + \frac{1}{24} w^4 + \frac{1}{120} w^5 + \frac{1}{720} w^6 + \frac{1}{5040} w^7$$

$$+ O(w^8)$$

(8)

Type: UnivariatePuisseuxSeries(Expression Integer, w, 0)

## 8.9.2 Coefficients of Power Series

You can extract any coefficient from a power series—even one that hasn't been computed yet. This is possible because in AXIOM, infinite series are represented by a list of the coefficients that have already been determined, together with a function for computing the additional coefficients. (This is known as *lazy evaluation*.) When you ask for a coefficient that hasn't yet been computed, AXIOM computes whatever additional coefficients it needs and then stores them in the representation of the power series.

Here's an example of how to extract the coefficients of a power series.

```
x := series(x)
```

$$x$$

(1)

Type: UnivariatePuisseuxSeries(Expression Integer, x, 0)

```
y := exp(x) * sin(x)
```

$$x + x^2 + \frac{1}{3} x^3 - \frac{1}{30} x^5 - \frac{1}{90} x^6 - \frac{1}{630} x^7 + O(x^9)$$

(2)

Type: UnivariatePuisseuxSeries(Expression Integer, x, 0)

This coefficient is readily available.

```
coefficient(y, 6)
```

$$-\frac{1}{90}$$

(3)

Type: Expression Integer

But let's get the fifteenth coefficient of y.

```
coefficient(y, 15)
```

$$-\frac{1}{10216206000}$$

(4)

Type: Expression Integer

If you look at y then you see that the coefficients up to order 15 have all been computed.

```
y
```

$$x + x^2 + \frac{1}{3} x^3 - \frac{1}{30} x^5 - \frac{1}{90} x^6 - \frac{1}{630} x^7 + \frac{1}{22680} x^9 + \frac{1}{113400} x^{10} + \frac{1}{1247400} x^{11} - \frac{1}{97297200} x^{13} - \frac{1}{681080400} x^{14} - \frac{1}{10216206000} x^{15} + O(x^{16})$$

(5)

Type: UnivariatePuisseuxSeries(Expression Integer, x, 0)

### 8.9.3 Power Series Arithmetic

The results of these operations are also power series.

You can manipulate power series using the usual arithmetic operations “+”, “-”, “\*”, and “/”.

```
x := series x
```

$$x$$

(1)

Type: UnivariatePuisseuxSeries(Expression Integer, x, 0)

```
(3 + x) / (1 + 7*x)
```

$$3 - 20 x + 140 x^2 - 980 x^3 + 6860 x^4 - 48020 x^5 + 336140 x^6 - 2352980 x^7 + O(x^8)$$

(2)

Type: UnivariatePuisseuxSeries(Expression Integer, x, 0)

You can also compute  $f(x) ** g(x)$ , where  $f(x)$  and  $g(x)$  are two power series.

```
base := 1 / (1 - x)
```

$$1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + O(x^8) \quad (3)$$

Type: UnivariatePuisseuxSeries(Expression Integer, x, 0)

```
expon := x * base
```

$$x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + O(x^9) \quad (4)$$

Type: UnivariatePuisseuxSeries(Expression Integer, x, 0)

```
base ** expon
```

$$1 + x^2 + \frac{3}{2}x^3 + \frac{7}{3}x^4 + \frac{43}{12}x^5 + \frac{649}{120}x^6 + \frac{241}{30}x^7 + O(x^8) \quad (5)$$

Type: UnivariatePuisseuxSeries(Expression Integer, x, 0)

### 8.9.4 Functions on Power Series

To demonstrate this, we first create the power series expansion of the rational

function  $\frac{x^2}{1 - 6x + x^2}$  about  $x = 0$ .

```
x := series 'x
```

$$x \quad (1)$$

Type: UnivariatePuisseuxSeries(Expression Integer, x, 0)

```
rat := x**2 / (1 - 6*x + x**2)
```

$$x^2 + 6x^3 + 35x^4 + 204x^5 + 1189x^6 + 6930x^7 + 40391x^8 + 235416x^9 + O(x^{10}) \quad (2)$$

Type: UnivariatePuisseuxSeries(Expression Integer, x, 0)

If you want to compute the series expansion of

$\sin\left(\frac{x^2}{1 - 6x + x^2}\right)$  you simply compute the sine of **rat**.

```
sin(rat)
```

$$x^2 + 6x^3 + 35x^4 + 204x^5 + \frac{7133}{6}x^6 + 6927x^7 + \frac{80711}{2}x^8 + 235068x^9 + O(x^{10}) \quad (3)$$

Type: UnivariatePuisseuxSeries(Expression Integer, x, 0)

**Warning:** the type of the coefficients of a power series may affect the kind of computations that you can do with that series. This can only happen when you have made a declaration to specify a series domain with a certain type of coefficient.



If you evaluate then you have declared that `y` is a one variable Taylor series (UTS is the abbreviation for `UnivariateTaylorSeries`) in the variable `y` with `FRAC INT` (that is, fractions of integer) coefficients, centered about 0.

```
y : UTS(FRAC INT, y, 0) := y
y
Type: UnivariateTaylorSeries(Fraction Integer, y, 0)
```

(4)

You can now compute certain power series in `y`, *provided* that these series have rational coefficients.

```
exp(y)
1 + y + 1/2 y^2 + 1/6 y^3 + 1/24 y^4 + 1/120 y^5 + 1/720 y^6 + 1/5040 y^7 + O(y^8)
Type: UnivariateTaylorSeries(Fraction Integer, y, 0)
```

(5)

You can get examples of such series by applying transcendental functions to series in `y` that have no constant terms.

```
tan(y**2)
y^2 + 1/3 y^6 + O(y^8)
Type: UnivariateTaylorSeries(Fraction Integer, y, 0)
```

(6)

```
cos(y + y**5)
1 - 1/2 y^2 + 1/24 y^4 - 721/720 y^6 + O(y^8)
Type: UnivariateTaylorSeries(Fraction Integer, y, 0)
```

(7)

Similarly, you can compute the logarithm of a power series with rational coefficients if the constant coefficient is 1.

```
log(1 + sin(y))
y - 1/2 y^2 + 1/6 y^3 - 1/12 y^4 + 1/24 y^5 - 1/45 y^6 + 61/5040 y^7 + O(y^8)
Type: UnivariateTaylorSeries(Fraction Integer, y, 0)
```

(8)

If you wanted to apply, say, the operation **exp** to a power series with a nonzero constant coefficient  $a_0$ , then the constant coefficient of the result would be  $e^{a_0}$ , which is *not* a rational number. Therefore, evaluating **exp**(2 + **tan**(`y`)) would generate an error message.

If you want to compute the Taylor expansion of **exp**(2 + **tan**(`y`)), you must ensure that the coefficient domain has an operation **exp** defined for it. An example of such a domain is `Expression Integer`, the type of formal functional expressions over the integers.

When working with coefficients of this type,

```
z : UTS(EXPR INT, z, 0) := z
z
Type: UnivariateTaylorSeries(Expression Integer, z, 0)
```

(9)

this presents no problems.

$$\begin{aligned} &\text{exp}(2 + \tan(z)) \\ &e^2 + e^2 z + \frac{e^2}{2} z^2 + \frac{e^2}{2} z^3 + \frac{3 e^2}{8} z^4 + \frac{37 e^2}{120} z^5 + \frac{59 e^2}{240} z^6 + \\ &\frac{137 e^2}{720} z^7 + O(z^8) \end{aligned} \quad (10)$$

Type: UnivariateTaylorSeries(Expression Integer, z, 0)

Another way to create Taylor series whose coefficients are expressions over the integers is to use **taylor** which works similarly to **series**.

This is equivalent to the previous computation, except that now we are using the variable **w** instead of **z**.

$$\begin{aligned} &\mathbf{w} := \mathbf{taylor} \ \mathbf{'w} \\ &w \end{aligned} \quad (11)$$

Type: UnivariateTaylorSeries(Expression Integer, w, 0)

$$\begin{aligned} &\text{exp}(2 + \tan(w)) \\ &e^2 + e^2 w + \frac{e^2}{2} w^2 + \frac{e^2}{2} w^3 + \frac{3 e^2}{8} w^4 + \frac{37 e^2}{120} w^5 + \frac{59 e^2}{240} w^6 + \\ &\frac{137 e^2}{720} w^7 + O(w^8) \end{aligned} \quad (12)$$

Type: UnivariateTaylorSeries(Expression Integer, w, 0)

### 8.9.5 Converting to Power Series

Evaluate this to compute the Taylor expansion of **sin x** about **x = 0**. The first argument, **sin(x)**, specifies the function whose series expansion is to be computed and the second argument, **x = 0**, specifies that the series is to be expanded in power of **(x - 0)**, that is, in power of **x**.

Here is the Taylor expansion of **sin x** about  $x = \frac{\pi}{6}$ :

The ExpressionToUnivariatePowerSeries package provides operations for computing series expansions of functions.

$$\begin{aligned} &\mathbf{taylor}(\mathbf{sin(x)}, \mathbf{x} = 0) \\ &x - \frac{1}{6} x^3 + \frac{1}{120} x^5 - \frac{1}{5040} x^7 + O(x^8) \end{aligned} \quad (1)$$

Type: UnivariateTaylorSeries(Expression Integer, x, 0)

$$\begin{aligned} &\mathbf{taylor}(\mathbf{sin(x)}, \mathbf{x} = \%pi/6) \\ &\frac{1}{2} + \frac{\sqrt{3}}{2} \left(x - \frac{\pi}{6}\right) - \frac{1}{4} \left(x - \frac{\pi}{6}\right)^2 - \frac{\sqrt{3}}{12} \left(x - \frac{\pi}{6}\right)^3 + \frac{1}{48} \left(x - \frac{\pi}{6}\right)^4 \\ &+ \frac{\sqrt{3}}{240} \left(x - \frac{\pi}{6}\right)^5 - \frac{1}{1440} \left(x - \frac{\pi}{6}\right)^6 - \frac{\sqrt{3}}{10080} \left(x - \frac{\pi}{6}\right)^7 + \\ &O\left(\left(x - \frac{\pi}{6}\right)^8\right) \end{aligned} \quad (2)$$

Type: UnivariateTaylorSeries(Expression Integer, x, pi/6)

The function to be expanded into a series may have variables other than the series variable.

For example, we may expand  $\tan(\mathbf{x}*\mathbf{y})$  as a Taylor series in  $\mathbf{x}$

$$\begin{aligned} &\text{taylor}(\tan(\mathbf{x}*\mathbf{y}), \mathbf{x} = 0) \\ &y\,x + \frac{y^3}{3}x^3 + \frac{2y^5}{15}x^5 + \frac{17y^7}{315}x^7 + O(x^8) \end{aligned} \quad (3)$$

Type: UnivariateTaylorSeries(Expression Integer, x, 0)

or as a Taylor series in  $\mathbf{y}$ .

$$\begin{aligned} &\text{taylor}(\tan(\mathbf{x}*\mathbf{y}), \mathbf{y} = 0) \\ &x\,y + \frac{x^3}{3}y^3 + \frac{2x^5}{15}y^5 + \frac{17x^7}{315}y^7 + O(y^8) \end{aligned} \quad (4)$$

Type: UnivariateTaylorSeries(Expression Integer, y, 0)

A more interesting function is  $\frac{te^{xt}}{e^t - 1}$ . When we expand this function as a Taylor series in  $\mathbf{t}$  the  $\mathbf{n}^{\text{th}}$  order coefficient is the  $\mathbf{n}^{\text{th}}$  Bernoulli polynomial divided by  $\mathbf{n}!$ .

$$\begin{aligned} &\text{bern} := \text{taylor}(t*\exp(\mathbf{x}*t)/(\exp(t) - 1), t = 0) \\ &1 + \frac{2x-1}{2}t + \frac{6x^2-6x+1}{12}t^2 + \frac{2x^3-3x^2+x}{12}t^3 + \\ &\frac{30x^4-60x^3+30x^2-1}{720}t^4 + \frac{6x^5-15x^4+10x^3-x}{720}t^5 + \\ &\frac{42x^6-126x^5+105x^4-21x^2+1}{30240}t^6 + \\ &\frac{6x^7-21x^6+21x^5-7x^3+x}{30240}t^7 + O(t^8) \end{aligned} \quad (5)$$

Type: UnivariateTaylorSeries(Expression Integer, t, 0)

Therefore, this and the next expression produce the same result.

$$\begin{aligned} &\text{factorial}(6) * \text{coefficient}(\text{bern}, 6) \\ &\frac{42x^6-126x^5+105x^4-21x^2+1}{42} \end{aligned} \quad (6)$$

Type: Expression Integer

$$\begin{aligned} &\text{bernoulliB}(6, \mathbf{x}) \\ &x^6 - 3x^5 + \frac{5}{2}x^4 - \frac{1}{2}x^2 + \frac{1}{42} \end{aligned} \quad (7)$$

Type: Polynomial Fraction Integer

Technically, a series with terms of negative degree is not considered to be a Taylor series, but, rather, a *Laurent series*. If you try to compute a Taylor series expansion of  $\frac{x}{\log x}$  at  $\mathbf{x} = 1$  via  $\text{taylor}(x/\log(x), \mathbf{x} = 1)$  you get an error message. The reason is that the function has a *pole* at  $\mathbf{x} = 1$ , meaning that its series expansion about this point has terms of negative degree. A series with finitely many terms of negative degree is called a Laurent series.

You get the desired series expansion by issuing this.

$$\begin{aligned} &\text{laurent}(x/\log(x), x = 1) \\ &(x-1)^{(-1)} + \frac{3}{2} + \frac{5}{12}(x-1) - \frac{1}{24}(x-1)^2 + \frac{11}{720}(x-1)^3 - \\ &\frac{11}{1440}(x-1)^4 + \frac{271}{60480}(x-1)^5 - \frac{13}{4480}(x-1)^6 + O((x-1)^7) \end{aligned} \quad (8)$$

Type: UnivariateLaurentSeries(Expression Integer, x, 1)

Similarly, a series with terms of fractional degree is neither a Taylor series nor a Laurent series. Such a series is called a *Puiseux series*. The expression `laurent(sqrt(sec(x)), x = 3 * %pi/2)` results in an error message because the series expansion about this point has terms of fractional degree.

However, this command produces what you want.

$$\begin{aligned} &\text{puiseux}(\sqrt{\sec(x)}, x = 3 * \%pi/2) \\ &\left(x - \frac{3\pi}{2}\right)^{(-\frac{1}{2})} + \frac{1}{12}\left(x - \frac{3\pi}{2}\right)^{\frac{3}{2}} + O\left(\left(x - \frac{3\pi}{2}\right)^{\frac{7}{2}}\right) \end{aligned} \quad (9)$$

Type: UnivariatePuiseuxSeries(Expression Integer, x, (3\*pi)/2)

Finally, consider the case of functions that do not have Puiseux expansions about certain points. An example of this is  $x^x$  about  $x = 0$ . `puiseux(x**x, x=0)` produces an error message because of the type of singularity of the function at  $x = 0$ .

The general function `series` can be used in this case. Notice that the series returned is not, strictly speaking, a power series because of the  $\log(x)$  in the expansion.

$$\begin{aligned} &\text{series}(x**x, x=0) \\ &1 + \log(x)x + \frac{\log(x)^2}{2}x^2 + \frac{\log(x)^3}{6}x^3 + \frac{\log(x)^4}{24}x^4 + \frac{\log(x)^5}{120}x^5 \\ &+ \frac{\log(x)^6}{720}x^6 + \frac{\log(x)^7}{5040}x^7 + O(x^8) \end{aligned} \quad (10)$$

Type: GeneralUnivariatePowerSeries(Expression Integer, x, 0)

The operation `series` returns the most general type of infinite series. The user who is not interested in distinguishing between various types of infinite series may wish to use this operation exclusively.

### 8.9.6 Power Series from Formulas

The `GenerateUnivariatePowerSeries` package enables you to create power series from explicit formulas for their  $n^{\text{th}}$  coefficients. In what follows, we construct series expansions for certain transcendental functions by giving formulas for their coefficients. You can also compute such series expansions directly simply by specifying the function and the point about which

the series is to be expanded. See Section 8.9.5 on page 300 for more information.

Consider the Taylor expansion of  $e^x$  about  $x = 0$ :

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots \\ &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \end{aligned}$$

The  $n^{\text{th}}$  Taylor coefficient is  $1/n!$ .

This is how you create this series in AXIOM.

```
series(n +-> 1/factorial(n), x = 0)
1 + x + 1/2 x^2 + 1/6 x^3 + 1/24 x^4 + 1/120 x^5 + 1/720 x^6 + 1/5040 x^7 + O(x^8) (1)
Type: UnivariatePuisseuxSeries(Expression Integer, x, 0)
```

The first argument specifies a formula for the  $n^{\text{th}}$  coefficient by giving a function that maps  $n$  to  $1/n!$ . The second argument specifies that the series is to be expanded in powers of  $(x - 0)$ , that is, in powers of  $x$ . Since we did not specify an initial degree, the first term in the series was the term of degree 0 (the constant term). Note that the formula was given as an anonymous function. These are discussed in Section 6.17 on page 218.

Consider the Taylor expansion of  $\log x$  about  $x = 1$ :

$$\begin{aligned} \log(x) &= (x - 1) - \frac{(x - 1)^2}{2} + \frac{(x - 1)^3}{3} - \cdots \\ &= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x - 1)^n}{n} \end{aligned}$$

If you were to evaluate the expression `series(n +-> (-1)**(n-1) / n, x = 1)` you would get an error message because AXIOM would try to calculate a term of degree 0 and therefore divide by 0.

Instead, evaluate this. The third argument, `1..`, indicates that only terms of degree  $n = 1, \dots$  are to be computed.

```
series(n +-> (-1)**(n-1)/n, x = 1, 1..)
(x - 1) - 1/2 (x - 1)^2 + 1/3 (x - 1)^3 - 1/4 (x - 1)^4 + 1/5 (x - 1)^5 -
1/6 (x - 1)^6 + 1/7 (x - 1)^7 - 1/8 (x - 1)^8 + O((x - 1)^9)
Type: UnivariatePuisseuxSeries(Expression Integer, x, 1) (2)
```

Next consider the Taylor expansion of an odd function, say,  $\sin(x)$ :

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

Here every other coefficient is zero and we would like to give an explicit formula only for the odd Taylor coefficients.

This is one way to do it. The third argument, `1..`, specifies that the first term to be computed is the term of degree 1. The fourth argument, `2`, specifies that we increment by 2 to find the degrees of subsequent terms, that is, the next term is of degree `1 + 2`, the next of degree `1 + 2 + 2`, etc.

$$\text{series}(n \rightarrow (-1)^{((n-1)/2)} / \text{factorial}(n), x = 0, 1.., 2)$$

$$x - \frac{1}{6} x^3 + \frac{1}{120} x^5 - \frac{1}{5040} x^7 + O(x^9) \quad (3)$$

Type: UnivariatePuisseuxSeries(Expression Integer, x, 0)

The initial degree and the increment do not have to be integers. For example, this expression produces a series expansion of  $\sin(x^{1/3})$ .

$$\text{series}(n \rightarrow (-1)^{((3*n-1)/2)} / \text{factorial}(3*n), x = 0, 1/3.., 2/3)$$

$$x^{1/3} - \frac{1}{6} x + \frac{1}{120} x^{5/3} - \frac{1}{5040} x^{7/3} + O(x^3) \quad (4)$$

Type: UnivariatePuisseuxSeries(Expression Integer, x, 0)

While the increment must be positive, the initial degree may be negative. This yields the Laurent expansion of  $\csc(x)$  at  $x = 0$ .

$$\text{cscx} := \text{series}(n \rightarrow (-1)^{((n-1)/2)} * 2 * (2^{**n}-1) * \text{bernoulli}(\text{numer}(n+1)) / \text{factorial}(n+1), x=0, -1.., 2)$$

$$x^{(-1)} + \frac{1}{6} x + \frac{7}{360} x^3 + \frac{31}{15120} x^5 + O(x^7) \quad (5)$$

Type: UnivariatePuisseuxSeries(Expression Integer, x, 0)

Of course, the reciprocal of this power series is the Taylor expansion of  $\sin(x)$ .

$$1/\text{cscx}$$

$$x - \frac{1}{6} x^3 + \frac{1}{120} x^5 - \frac{1}{5040} x^7 + O(x^9) \quad (6)$$

Type: UnivariatePuisseuxSeries(Expression Integer, x, 0)

As a final example, here is the Taylor expansion of  $\arcsin(x)$  about  $x = 0$ .

$$\text{asinx} := \text{series}(n \rightarrow \text{binomial}(n-1, (n-1)/2) / (n * 2^{**n-1}), x=0, 1.., 2)$$

$$x + \frac{1}{6} x^3 + \frac{3}{40} x^5 + \frac{5}{112} x^7 + O(x^9) \quad (7)$$

Type: UnivariatePuisseuxSeries(Expression Integer, x, 0)

When we compute the  $\sin$  of this series, we get  $x$  (in the sense that all higher terms computed so far are zero).

$$\sin(\text{asinx})$$

$$x + O(x^9) \quad (8)$$

Type: UnivariatePuisseuxSeries(Expression Integer, x, 0)

As we discussed in Section 8.9.5 on page 300, you can also use the operations **taylor**, **laurent** and **puisseux** instead of **series** if you know ahead of time what kind of exponents a series has. You can't go wrong using **series**, though.

## 8.9.7 Substituting Numerical Values in Power Series

First you create the desired Taylor expansion.

```
f := taylor(exp(x))
```

$$1 + x + \frac{1}{2} x^2 + \frac{1}{6} x^3 + \frac{1}{24} x^4 + \frac{1}{120} x^5 + \frac{1}{720} x^6 + \frac{1}{5040} x^7 + O(x^8) \quad (1)$$

Type: UnivariateTaylorSeries(Expression Integer, x, 0)

Then you evaluate the series at the value 1.0. The result is a sequence of the partial sums.

```
eval(f, 1.0)
```

$$[1.0, 2.0, 2.5, 2.6666666666666667, 2.7083333333333333, 2.7166666666666667, 2.7180555555555556, \dots] \quad (2)$$

Type: Stream Expression Float

## 8.9.8 Example: Bernoulli Polynomials and Sums of Powers

You can compute the sum of the first ten fourth powers by evaluating this. This creates a list whose entries are  $m^4$  as  $m$  ranges from 1 to 10, and then computes the sum of the entries of that list.

```
reduce(+,[m**4 for m in 1..10])
```

$$25333 \quad (1)$$

Type: PositiveInteger

You can also compute a formula for the sum of the first  $k$  fourth powers, where  $k$  is an unspecified positive integer.

```
sum4 := sum(m**4, m = 1..k)
```

$$\frac{6 k^5 + 15 k^4 + 10 k^3 - k}{30} \quad (2)$$

Type: Fraction Polynomial Integer

This formula is valid for any positive integer  $k$ . For instance, if we replace  $k$  by 10, we obtain the number we computed earlier.

```
eval(sum4, k = 10)
```

$$25333 \quad (3)$$

Type: Fraction Polynomial Integer

You can compute a formula for the sum of the first  $k$   $n^{\text{th}}$  powers in a similar fashion. Just replace the 4 in the definition of **sum4** by any expression not involving  $k$ . AXIOM computes these formulas using Bernoulli polynomials; we use the rest of this section to describe this method.

First consider this function of  $t$  and  $x$ .

```
f := t*exp(x*t) / (exp(t) - 1)
```

$$\frac{t e^{(t x)}}{e^t - 1} \quad (4)$$

Type: Expression Integer

Since the expressions involved get quite large, we tell AXIOM to show us only terms of degree up to 5.

If we look at the Taylor expansion of  $f(x, t)$  about  $t = 0$ , we see that the coefficients of the powers of  $t$  are polynomials in  $x$ .

```
)set streams calculate 5
```

```
ff := taylor(f, t = 0)
```

$$1 + \frac{2x-1}{2}t + \frac{6x^2-6x+1}{12}t^2 + \frac{2x^3-3x^2+x}{12}t^3 +$$

$$\frac{30x^4-60x^3+30x^2-1}{720}t^4 + \frac{6x^5-15x^4+10x^3-x}{720}t^5 + O(t^6)$$

Type: UnivariateTaylorSeries(Expression Integer, t, 0)

In fact, the  $n^{\text{th}}$  coefficient in this series is essentially the  $n^{\text{th}}$  Bernoulli polynomial: the  $n^{\text{th}}$  coefficient of the series is  $\frac{1}{n!}B_n(x)$ , where  $B_n(x)$  is the  $n^{\text{th}}$  Bernoulli polynomial. Thus, to obtain the  $n^{\text{th}}$  Bernoulli polynomial, we multiply the  $n^{\text{th}}$  coefficient of the series **ff** by  $n!$ .

For example, the sixth Bernoulli polynomial is this.

```
factorial(6) * coefficient(ff, 6)
```

$$\frac{42x^6 - 126x^5 + 105x^4 - 21x^2 + 1}{42}$$

Type: Expression Integer

We derive some properties of the function  $f(x, t)$ . First we compute  $f(x+1, t) - f(x, t)$ .

```
g := eval(f, x = x + 1) - f
```

$$\frac{te^{(t(x+1))} - te^{(tx)}}{e^t - 1}$$

Type: Expression Integer

If we normalize **g**, we see that it has a particularly simple form.

```
normalize(g)
```

$$te^{(tx)}$$

Type: Expression Integer

From this it follows that the  $n^{\text{th}}$  coefficient in the Taylor expansion of  $g(x, t)$  at  $t = 0$  is  $\frac{1}{(n-1)!}x^{n-1}$ .

If you want to check this, evaluate the next expression.

```
taylor(g, t = 0)
```

$$t + x t^2 + \frac{x^2}{2} t^3 + \frac{x^3}{6} t^4 + \frac{x^4}{24} t^5 + O(t^6)$$

Type: UnivariateTaylorSeries(Expression Integer, t, 0)

However, since  $g(x, t) = f(x+1, t) - f(x, t)$ , it follows that the  $n^{\text{th}}$  coefficient is  $\frac{1}{n!}(B_n(x+1) - B_n(x))$ . Equating coefficients, we see that  $\frac{1}{(n-1)!}x^{n-1} = \frac{1}{n!}(B_n(x+1) - B_n(x))$  and, therefore,  $x^{n-1} = \frac{1}{n}(B_n(x+1) - B_n(x))$ . Let's apply this formula repeatedly, letting  $x$  vary between two integers **a** and **b**, with  $a < b$ :



$$\begin{aligned}
a^{n-1} &= \frac{1}{n}(B_n(a+1) - B_n(a)) \\
(a+1)^{n-1} &= \frac{1}{n}(B_n(a+2) - B_n(a+1)) \\
(a+2)^{n-1} &= \frac{1}{n}(B_n(a+3) - B_n(a+2)) \\
&\vdots \\
(b-1)^{n-1} &= \frac{1}{n}(B_n(b) - B_n(b-1)) \\
b^{n-1} &= \frac{1}{n}(B_n(b+1) - B_n(b))
\end{aligned}$$

When we add these equations we find that the sum of the left-hand sides is  $\sum_{m=a}^b m^{n-1}$ , the sum of the  $(n-1)^{\text{st}}$  powers from  $a$  to  $b$ . The sum of the right-hand sides is a “telescoping series.” After cancellation, the sum is simply  $\frac{1}{n}(B_n(b+1) - B_n(a))$ .

Replacing  $n$  by  $n + 1$ , we have shown that

$$\sum_{m=a}^b m^n = \frac{1}{n+1} (B_{n+1}(b+1) - B_{n+1}(a)).$$

Let’s use this to obtain the formula for the sum of fourth powers.

First we obtain the Bernoulli polynomial  $B_5$ .

`B5 := factorial(5) * coefficient(ff,5)`

$$\frac{6x^5 - 15x^4 + 10x^3 - x}{6} \tag{10}$$

Type: Expression Integer

To find the sum of the first  $k$  4th powers, we multiply  $1/5$  by  $B_5(k+1) - B_5(1)$ .

`1/5 * (eval(B5, x = k + 1) - eval(B5, x = 1))`

$$\frac{6k^5 + 15k^4 + 10k^3 - k}{30} \tag{11}$$

Type: Expression Integer

This is the same formula that we obtained via `sum(m**4, m = 1..k)`.

`sum4`

$$\frac{6k^5 + 15k^4 + 10k^3 - k}{30} \tag{12}$$

Type: Fraction Polynomial Integer

At this point you may want to do the same computation, but with an exponent other than 4. For example, you might try to find a formula for the sum of the first  $k$  20th powers.

## 8.10 Solution of Differential Equations

---

In this section we discuss AXIOM's facilities for solving differential equations in closed-form and in series.

AXIOM provides facilities for closed-form solution of single differential equations of the following kinds:

- linear ordinary differential equations, and
- non-linear first order ordinary differential equations when integrating factors can be found just by integration.

For a discussion of the solution of systems of linear and polynomial equations, see Section 8.5 on page 283.

### 8.10.1 Closed-Form Solutions of Linear Differential Equations

---

A *differential equation* is an equation involving an unknown *function* and one or more of its derivatives. The equation is called *ordinary* if derivatives with respect to only one dependent variable appear in the equation (it is called *partial* otherwise). The package ElementaryFunctionODESolver provides the top-level operation **solve** for finding closed-form solutions of ordinary differential equations.

To solve a differential equation, you must first create an operator for the unknown function.

We let  $y$  be the unknown function in terms of  $x$ .

```
y := operator 'y
y
(1)
```

Type: BasicOperator

You then type the equation using **D** to create the derivatives of the unknown function  $y(x)$  where  $x$  is any symbol you choose (the so-called *dependent variable*).

This is how you enter the equation  $y'' + y' + y = 0$ .

```
deq := D(y x, x, 2) + D(y x, x) + y x = 0
y''(x) + y'(x) + y(x) = 0
(2)
```

Type: Equation Expression Integer

The simplest way to invoke the **solve** command is with three arguments.

- the differential equation,
- the operator representing the unknown function,
- the dependent variable.

So, to solve the above equation, we enter this.

```
solve(deq, y, x)
```

$$\left[ \text{particular} = 0, \text{basis} = \left[ \cos\left(\frac{x\sqrt{3}}{2}\right) e^{(-\frac{x}{2})}, e^{(-\frac{x}{2})} \sin\left(\frac{x\sqrt{3}}{2}\right) \right] \right] \quad (3)$$

Type: Union(Record(particular: Expression Integer, basis: List Expression Integer), ...)

Since linear ordinary differential equations have infinitely many solutions, **solve** returns a *particular solution*  $f_p$  and a basis  $f_1, \dots, f_n$  for the solutions of the corresponding homogeneous equation. Any expression of the form  $f_p + c_1 f_1 + \dots c_n f_n$  where the  $c_i$  do not involve the dependent variable is also a solution. This is similar to what you get when you solve systems of linear algebraic equations.

A way to select a unique solution is to specify *initial conditions*: choose a value **a** for the dependent variable and specify the values of the unknown function and its derivatives at **a**. If the number of initial conditions is equal to the order of the equation, then the solution is unique (if it exists in closed form!) and **solve** tries to find it. To specify initial conditions to **solve**, use an Equation of the form  $x = a$  for the third parameter instead of the dependent variable, and add a fourth parameter consisting of the list of values  $y(a), y'(a), \dots$ .

To find the solution of  $y'' + y = 0$  satisfying  $y(0) = y'(0) = 1$ , do this.

```
deq := D(y x, x, 2) + y x
y''(x) + y(x) \quad (4)
```

Type: Expression Integer

You can omit the  $= 0$  when you enter the equation to be solved.

```
solve(deq, y, x = 0, [1, 1])
sin(x) + cos(x) \quad (5)
```

Type: Union(Expression Integer, ...)

AXIOM is not limited to linear differential equations with constant coefficients. It can also find solutions when the coefficients are rational or algebraic functions of the dependent variable. Furthermore, AXIOM is not limited by the order of the equation.

AXIOM can solve the following third order equations with polynomial coefficients.

```
deq := x**3 * D(y x, x, 3) + x**2 * D(y x, x, 2) - 2 * x *
D(y x, x) + 2 * y x = 2 * x**4
x^3 y'''(x) + x^2 y''(x) - 2 x y'(x) + 2 y(x) = 2 x^4 \quad (6)
```

Type: Equation Expression Integer

`solve(deq, y, x)`

$$\left[ \begin{array}{l} \text{particular} = \frac{x^5 - 10x^3 + 20x^2 + 4}{15x}, \\ \text{basis} = \left[ \frac{2x^3 - 3x^2 + 1}{x}, \frac{x^3 - 1}{x}, \frac{x^3 - 3x^2 - 1}{x} \right] \end{array} \right] \quad (7)$$

Type: Union(Record(particular: Expression Integer, basis: List Expression Integer), ...)

Here we are solving a homogeneous equation.

$$\text{deq} := (x^{**9} + x^{**3}) * D(y \ x, \ x, \ 3) + 18 * x^{**8} * D(y \ x, \ x, \ 2) - 90 * x * D(y \ x, \ x) - 30 * (11 * x^{**6} - 3) * y \ x$$

$$(x^9 + x^3) y'''(x) + 18x^8 y''(x) - 90x y'(x) + (-330x^6 + 90) y(x) \quad (8)$$

Type: Expression Integer

`solve(deq, y, x)`

$$\left[ \begin{array}{l} \text{particular} = 0, \text{basis} = \left[ \frac{x}{x^6 + 1}, \frac{x e^{(-\sqrt{91} \log(x))}}{x^6 + 1}, \frac{x e^{(\sqrt{91} \log(x))}}{x^6 + 1} \right] \end{array} \right] \quad (9)$$

Type: Union(Record(particular: Expression Integer, basis: List Expression Integer), ...)

On the other hand, and in contrast with the operation **integrate**, it can happen that AXIOM finds no solution and that some closed-form solution still exists. While it is mathematically complicated to describe exactly when the solutions are guaranteed to be found, the following statements are correct and form good guidelines for linear ordinary differential equations:

- If the coefficients are constants, AXIOM finds a complete basis of solutions (i.e, all solutions).
- If the coefficients are rational functions in the dependent variable, AXIOM at least finds all solutions that do not involve algebraic functions.

Note that this last statement does not mean that AXIOM does not find the solutions that are algebraic functions. It means that it is not guaranteed that the algebraic function solutions will be found.

This is an example where all the algebraic solutions are found.

$$\text{deq} := (x^{**2} + 1) * D(y \ x, \ x, \ 2) + 3 * x * D(y \ x, \ x) + y \ x = 0$$

$$(x^2 + 1) y''(x) + 3x y'(x) + y(x) = 0 \quad (10)$$

Type: Equation Expression Integer

```
solve(deq, y, x)
```

$$\left[ \text{particular} = 0, \text{basis} = \left[ \frac{1}{\sqrt{x^2 + 1}}, \frac{\log(\sqrt{x^2 + 1} - x)}{\sqrt{x^2 + 1}} \right] \right] \quad (11)$$

Type: Union(Record(particular: Expression Integer, basis: List Expression Integer), ...)

## 8.10.2 Closed-Form Solutions of Non-Linear Differential Equations

Using the notation  $m(x, y) + n(x, y) y' = 0$ , we have  $m = -y$  and  $n = x + y \log y$ .

```
m := -y
```

```
-y
```

(1)

Type: Polynomial Integer

```
n := x + y * log y
```

```
y log(y) + x
```

(2)

Type: Expression Integer

We first check for exactness, that is, does  $dm/dy = dn/dx$ ?

```
D(m, y) - D(n, x)
```

```
-2
```

(3)

Type: Expression Integer

This is not zero, so the equation is not exact. Therefore we must look for an integrating factor: a function  $\mu(x, y)$  such that  $d(\mu m)/dy = d(\mu n)/dx$ . Normally, we first search for  $\mu(x, y)$  depending only on  $x$  or only on  $y$ .

Let's search for such a  $\mu(x)$  first.

```
mu := operator 'mu
```

```
mu
```

(4)

Type: BasicOperator

```
a := D(mu(x) * m, y) - D(mu(x) * n, x)
```

```
(-y log(y) - x) mu'(x) - 2 mu(x)
```

(5)

Type: Expression Integer

If the above is zero for a function  $\mu$  that does *not* depend on  $y$ , then  $\mu(x)$  is an integrating factor.

```
solve(a = 0, mu, x)
```

$$\left[ \text{particular} = 0, \text{basis} = \left[ \frac{1}{y^2 \log(y)^2 + 2 x y \log(y) + x^2} \right] \right] \quad (6)$$

Type: Union(Record(particular: Expression Integer, basis: List Expression Integer), ...)

The solution depends on  $y$ , so there is no integrating factor that depends on  $x$  only.

Let's look for one that depends on  $y$  only.

```
b := D(mu(y) * m, y) - D(mu(y) * n, x)
```

$$-y \mu'(y) - 2 \mu(y) \quad (7)$$

Type: Expression Integer

```
sb := solve(b = 0, mu, y)
```

$$\left[ \text{particular} = 0, \text{basis} = \left[ \frac{1}{y^2} \right] \right] \quad (8)$$

Type: Union(Record(particular: Expression Integer, basis: List Expression Integer), ...)

We've found one!

The above  $\mu(y)$  is an integrating factor. We must multiply our initial equation (that is,  $m$  and  $n$ ) by the integrating factor.

```
intFactor := sb.basis.1
```

$$\frac{1}{y^2} \quad (9)$$

Type: Expression Integer

```
m := intFactor * m
```

$$-\frac{1}{y} \quad (10)$$

Type: Expression Integer

```
n := intFactor * n
```

$$\frac{y \log(y) + x}{y^2} \quad (11)$$

Type: Expression Integer

Let's check for exactness.

```
D(m, y) - D(n, x)
```

$$0 \quad (12)$$

Type: Expression Integer

We must solve the exact equation, that is, find a function  $s(x,y)$  such that  $ds/dx = m$  and  $ds/dy = n$ .

We start by writing  $s(x, y) = h(y) + \text{integrate}(m, x)$  where  $h(y)$  is an unknown function of  $y$ . This guarantees that  $ds/dx = m$ .

```
h := operator 'h
h
Type: BasicOperator
```

```
sol := h y + integrate(m, x)
y h(y) - x
y
Type: Expression Integer
```

All we want is to find  $h(y)$  such that  $ds/dy = n$ .

```
dsol := D(sol, y)
y^2 h'(y) + x
y^2
Type: Expression Integer
```

```
nsol := solve(dsol = n, h, y)
[particular = log(y)^2/2, basis = [1]]
Type: Union(Record(particular: Expression Integer, basis: List Expression Integer, ...)
```

The above particular solution is the  $h(y)$  we want, so we just replace  $h(y)$  by it in the implicit solution.

```
eval(sol, h y = nsol.particular)
y log(y)^2 - 2 x
2 y
Type: Expression Integer
```

A first integral of the initial equation is obtained by setting this result equal to an arbitrary constant.

Now that we've seen how to solve the equation "by hand," we show you how to do it with the **solve** operation.

First define  $y$  to be an operator.

```
y := operator 'y
y
Type: BasicOperator
```

Next we create the differential equation.

```
deq := D(y x, x) = y(x) / (x + y(x) * log y x)
y'(x) = y(x) / (y(x) log(y(x)) + x)
Type: Equation Expression Integer
```

Finally, we solve it.

```
solve(deq, y, x)
y(x) log(y(x))^2 - 2 x
2 y(x)
Type: Union(Expression Integer, ...)
```

### 8.10.3 Power Series Solutions of Differential Equations

Since the coefficients of some solutions are quite large, we reset the default to compute only seven terms.

We first tell AXIOM that the symbol 'y denotes a new operator.

Enter the differential equation using y like any system function.

Solve it around  $x = 0$  with the initial conditions  $y(0) = 1$ ,  $y'(0) = y''(0) = 0$ .

We tell AXIOM that x is also an operator.

Enter the two equations forming our system.

The command to solve differential equations in power series around a particular initial point with specific initial conditions is called **seriesSolve**. It can take a variety of parameters, so we illustrate its use with some examples.

```
)set streams calculate 7
```

You can solve a single nonlinear equation of any order. For example, we solve  $y''' = \sin(y'') * \exp(y) + \cos(x)$  subject to  $y(0) = 1$ ,  $y'(0) = 0$ ,  $y''(0) = 0$ .

```
y := operator 'y
y
```

Type: BasicOperator

```
eq := D(y(x), x, 3) - sin(D(y(x), x, 2))*exp(y(x)) =
      cos(x)
```

```
y'''(x) - ey(x) sin(y''(x)) = cos(x)
```

Type: Equation Expression Integer

```
seriesSolve(eq, y, x = 0, [1, 0, 0])
```

Compiling function %CJ with type List

```
UnivariateTaylorSeries(Expression Integer,x,0) ->
UnivariateTaylorSeries(Expression Integer,x,0)
```

$$1 + \frac{1}{6}x^3 + \frac{e}{24}x^4 + \frac{e^2 - 1}{120}x^5 + \frac{e^3 - 2e}{720}x^6 + \frac{e^4 - 8e^2 + 4e + 1}{5040}x^7 + O(x^8) \quad (3)$$

Type: UnivariateTaylorSeries(Expression Integer, x, 0)

You can also solve a system of nonlinear first order equations. For example, we solve a system that has **tan(t)** and **sec(t)** as solutions.

```
x := operator 'x
```

Compiled code for %CJ has been cleared.

```
x
```

Type: BasicOperator

```
eq1 := D(x(t), t) = 1 + x(t)**2
```

```
x'(t) = x(t)2 + 1
```

Type: Equation Expression Integer



$$\text{eq2} := D(y(t), t) = x(t) * y(t)$$

$$y'(t) = x(t) y(t) \quad (6)$$

Type: Equation Expression Integer

Solve the system around  $t = 0$  with the initial conditions  $x(0) = 0$  and  $y(0) = 1$ . Notice that since we give the unknowns in the order  $[x, y]$ , the answer is a list of two series in the order `[series for x(t), series for y(t)]`.

```
seriesSolve([eq2, eq1], [x, y], t = 0, [y(0) = 1, x(0) = 0])
```

```
Compiling function %CL with type List
```

```
UnivariateTaylorSeries(Expression Integer,t,0) ->
```

```
UnivariateTaylorSeries(Expression Integer,t,0)
```

```
Compiling function %CM with type List
```

```
UnivariateTaylorSeries(Expression Integer,t,0) ->
```

```
UnivariateTaylorSeries(Expression Integer,t,0)
```

$$\left[ t + \frac{1}{3} t^3 + \frac{2}{15} t^5 + \frac{17}{315} t^7 + O(t^8), \right. \quad (7)$$

$$\left. 1 + \frac{1}{2} t^2 + \frac{5}{24} t^4 + \frac{61}{720} t^6 + O(t^8) \right]$$

Type: List UnivariateTaylorSeries(Expression Integer, t, 0)

The order in which we give the equations and the initial conditions has no effect on the order of the solution.

## 8.11 Finite Fields

A *finite field* (also called a *Galois field*) is a finite algebraic structure where one can add, multiply and divide under the same laws (for example, commutativity, associativity or distributivity) as apply to the rational, real or complex numbers. Unlike those three fields, for any finite field there exists a positive prime integer  $p$ , called the **characteristic**, such that  $p x = 0$  for any element  $x$  in the finite field. In fact, the number of elements in a finite field is a power of the characteristic and for each prime  $p$  and positive integer  $n$  there exists exactly one finite field with  $p^n$  elements, up to isomorphism.<sup>1</sup>

When  $n = 1$ , the field has  $p$  elements and is called a *prime field*, discussed in the next section. There are several ways of implementing extensions of finite fields, and AXIOM provides quite a bit of freedom to allow you to choose the one that is best for your application. Moreover, we provide operations for converting among the different representations of extensions and different extensions of a single field. Finally, note that you usually need to package-call operations from finite fields if the operations do not take as an argument an object of the field. See Section 2.9 on page 119 for more information on package-calling.

### 8.11.1 Modular Arithmetic and Prime Fields

Let  $n$  be a positive integer. It is well known that you can get the same result if you perform addition, subtraction or multiplication of integers and then take the remainder on dividing by  $n$  as if you had first done such remaindering on the operands, performed the arithmetic and then (if necessary) done remaindering again. This allows us to speak of arithmetic *modulo*  $n$  or, more simply *mod*  $n$ .

In AXIOM, you use IntegerMod to do such arithmetic.

```
(a,b) : IntegerMod 12
```

Type: Void

```
(a, b) := (16, 7)
```

```
7
```

(2)

Type: IntegerMod 12

```
[a - b, a * b]
```

```
[9, 4]
```

(3)

Type: List IntegerMod 12

---

<sup>1</sup>For more information about the algebraic structure and properties of finite fields, see, for example, S. Lang, *Algebra*, Second Edition, New York: Addison-Wesley Publishing Company, Inc., 1984, ISBN 0 201 05487 6; or R. Lidl, H. Niederreiter, *Finite Fields*, Encyclopedia of Mathematics and Its Applications, Vol. 20, Cambridge: Cambridge Univ. Press, 1983, ISBN 0 521 30240 4.

If  $n$  is not prime, there is only a limited notion of reciprocals and division.

`a / b`

There are 11 exposed and 12 unexposed library operations named `/` having 2 argument(s) but none was determined to be applicable. Use HyperDoc Browse, or issue `)display op /` to learn more about the available operations. Perhaps package-calling the operation or using coercions on the arguments will allow you to apply the operation.

Cannot find a definition or applicable library operation named `/` with argument type(s)  
`IntegerMod 12`  
`IntegerMod 12`

Perhaps you should use `"@"` to indicate the required return type, or `"$"` to specify which version of the function you need.

`recip a`

`"failed"` (4)  
Type: Union("failed", ...)

Here 7 and 12 are relatively prime, so 7 has a multiplicative inverse modulo 12.

`recip b`

7 (5)  
Type: Union(IntegerMod 12, ...)

If we take  $n$  to be a prime number  $p$ , then taking inverses and, therefore, division are generally defined.

Use `PrimeField` instead of `IntegerMod` for  $n$  prime.

`c : PrimeField 11 := 8`

8 (6)  
Type: PrimeField 11

`inv c`

7 (7)  
Type: PrimeField 11

You can also use `1/c` and `c**(-1)` for the inverse of  $c$ .

`9/c`

8 (8)  
Type: PrimeField 11

`PrimeField` (abbreviation `PF`) checks if its argument is prime when you try to use an operation from it. If you know the argument is prime (particularly if it is large), `InnerPrimeField` (abbreviation `IPF`) assumes the argument has already been verified to be prime. If you do use a number that is not

prime, you will eventually get an error message, most likely a division by zero message. For computer science applications, the most important finite fields are PrimeField 2 and its extensions.

In the following examples, we work with the finite field with  $p = 101$  elements.

```
GF101 := PF 101
PrimeField 101
```

(9)  
Type: Domain

Like many domains in AXIOM, finite fields provide an operation for returning a random element of the domain.

```
x := random()$GF101
50
```

(10)  
Type: PrimeField 101

```
y : GF101 := 37
37
```

(11)  
Type: PrimeField 101

```
z := x/y
15
```

(12)  
Type: PrimeField 101

```
z * y - x
0
```

(13)  
Type: PrimeField 101

The element 2 is a *primitive element* of this field,

```
pe := primitiveElement()$GF101
2
```

(14)  
Type: PrimeField 101

in the sense that its powers enumerate all nonzero elements.

```
[pe**i for i in 0..99]
[1, 2, 4, 8, 16, 32, 64, 27, 54, 7, 14, 28, 56, 11, 22, 44, 88, 75,
49, 98, 95, 89, 77, 53, 5, 10, 20, 40, 80, 59, 17, 34, 68, 35, 70,
39, 78, 55, 9, 18, 36, 72, 43, 86, 71, 41, 82, 63, 25, 50, 100, 99,
97, 93, 85, 69, 37, 74, 47, 94, 87, 73, 45, 90, 79, 57, 13, 26, 52,
3, 6, 12, 24, 48, 96, 91, 81, 61, 21, 42, 84, 67, 33, 66, 31, 62, 23,
46, 92, 83, 65, 29, 58, 15, 30, 60, 19, 38, 76, 51]
```

(15)  
Type: List PrimeField 101

If every nonzero element is a power of a primitive element, how do you determine what the exponent is? Use **discreteLog**.

```
ex := discreteLog(y)
56
```

(16)  
Type: PositivelInteger

```
pe ** ex
37
```

(17)

Type: PrimeField 101

The **order** of a nonzero element  $x$  is the smallest positive integer  $t$  such  $x^t = 1$ .

```
order y
25
```

(18)

Type: PositiveInteger

The order of a primitive element is the defining  $p - 1$ .

```
order pe
100
```

(19)

Type: PositiveInteger

### 8.11.2 Extensions of Finite Fields

---

When you want to work with an extension of a finite field in AXIOM, you have three choices to make:

1. Do you want to generate an extension of the prime field (for example, PrimeField 2) or an extension of a given field?
2. Do you want to use a representation that is particularly efficient for multiplication, exponentiation and addition but uses a lot of computer memory (a representation that models the cyclic group structure of the multiplicative group of the field extension and uses a Zech logarithm table), one that uses a normal basis for the vector space structure of the field extension, or one that performs arithmetic modulo an irreducible polynomial? The cyclic group representation is only usable up to “medium” (relative to your machine’s performance) sized fields. If the field is large and the normal basis is relatively simple, the normal basis representation is more efficient for exponentiation than the irreducible polynomial representation.
3. Do you want to provide a polynomial explicitly, a root of which “generates” the extension in one of the three senses in (2), or do you wish to have the polynomial generated for you?

This illustrates one of the most important features of AXIOM: you can choose exactly the right data-type and representation to suit your application best.

We first tell you what domain constructors to use for each case above, and then give some examples.

Constructors that automatically generate extensions of the prime field:

```
FiniteField
FiniteFieldCyclicGroup
FiniteFieldNormalBasis
```

Constructors that generate extensions of an arbitrary field:

- FiniteFieldExtension
- FiniteFieldExtensionByPolynomial
- FiniteFieldCyclicGroupExtension
- FiniteFieldCyclicGroupExtensionByPolynomial
- FiniteFieldNormalBasisExtension
- FiniteFieldNormalBasisExtensionByPolynomial

Constructors that use a cyclic group representation:

- FiniteFieldCyclicGroup
- FiniteFieldCyclicGroupExtension
- FiniteFieldCyclicGroupExtensionByPolynomial

Constructors that use a normal basis representation:

- FiniteFieldNormalBasis
- FiniteFieldNormalBasisExtension
- FiniteFieldNormalBasisExtensionByPolynomial

Constructors that use an irreducible modulus polynomial representation:

- FiniteField
- FiniteFieldExtension
- FiniteFieldExtensionByPolynomial

Constructors that generate a polynomial for you:

- FiniteField
- FiniteFieldExtension
- FiniteFieldCyclicGroup
- FiniteFieldCyclicGroupExtension
- FiniteFieldNormalBasis
- FiniteFieldNormalBasisExtension

Constructors for which you provide a polynomial:

- FiniteFieldExtensionByPolynomial
- FiniteFieldCyclicGroupExtensionByPolynomial
- FiniteFieldNormalBasisExtensionByPolynomial

These constructors are discussed in the following sections where we collect together descriptions of extension fields that have the same underlying representation.<sup>2</sup>

If you don't really care about all this detail, just use `FiniteField`. As your knowledge of your application and its AXIOM implementation grows, you can come back and choose an alternative constructor that may improve the efficiency of your code. Note that the exported operations are almost

---

<sup>2</sup>For more information on the implementation aspects of finite fields, see J. Grabmeier, A. Scheerhorn, *Finite Fields in AXIOM*, Technical Report, IBM Heidelberg Scientific Center, 1992.

the same for all constructors of finite field extensions and include the operations exported by PrimeField.

### 8.11.3 Irreducible Modulus Polynomial Representations

AXIOM uses the prime field PrimeField(p), here PrimeField 2, and it chooses an irreducible polynomial of degree  $n$ , here 12, over the ground field.

All finite field extension constructors discussed in this section use a representation that performs arithmetic with univariate (one-variable) polynomials modulo an irreducible polynomial. This polynomial may be given explicitly by you or automatically generated. The ground field may be the prime field or one you specify. See Section 8.11.2 on page 319 for general information about finite field extensions.

For FiniteField (abbreviation FF) you provide a prime number  $p$  and an extension degree  $n$ . This degree can be 1.

```
GF4096 := FF(2,12);
```

(1)  
Type: Domain

The objects in the generated field extension are polynomials of degree at most  $n - 1$  with coefficients in the prime field. The polynomial indeterminate is automatically chosen by AXIOM and is typically something like %A or %D. These (strange) variables are *only* for output display; there are several ways to construct elements of this field.

The operation **index** enumerates the elements of the field extension and accepts as argument the integers from 1 to  $p^n$ .

The expression **index(p)** always gives the indeterminate.

```
a := index(2)$GF4096
```

(2)  
Type: FiniteField(2, 12)

You can build polynomials in  $a$  and calculate in GF4096.

```
b := a**12 - a**5 + a
```

(3)  
Type: FiniteField(2, 12)

```
b ** 1000
```

(4)  
Type: FiniteField(2, 12)

```
c := a/b
```

(5)  
Type: FiniteField(2, 12)

Among the available operations are **norm** and **trace**.

$$\text{norm } c$$

$$1 \tag{6}$$

Type: PrimeField 2

$$\text{trace } c$$

$$0 \tag{7}$$

Type: PrimeField 2

Since any nonzero element is a power of a primitive element, how do we discover what the exponent is?

The operation **discreteLog** calculates the exponent and, if it is called with only one argument, always refers to the primitive element returned by **primitiveElement**.

$$\text{dL} := \text{discreteLog } a$$

$$1729 \tag{8}$$

Type: PositiveInteger

$$g^{**} \text{ dL}$$

$$g^{1729} \tag{9}$$

Type: Polynomial Integer

FiniteFieldExtension (abbreviation FFX) is similar to FiniteField except that the ground-field for FiniteFieldExtension is arbitrary and chosen by you.

In case you select the prime field as ground field, there is essentially no difference between the constructed two finite field extensions.

$$\text{GF16} := \text{FF}(2, 4);$$

$$\tag{10}$$

Type: Domain

$$\text{GF4096} := \text{FFX}(\text{GF16}, 3);$$

$$\tag{11}$$

Type: Domain

$$r := (\text{random}() \$ \text{GF4096})^{**} 20$$

$$\%CO \%CP^2 + 1 \tag{12}$$

Type: FiniteFieldExtension(FiniteField(2, 4), 3)

$$\text{norm}(r)$$

$$\%CO^2 + \%CO + 1 \tag{13}$$

Type: FiniteField(2, 4)

FiniteFieldExtensionByPolynomial (abbreviation FFP) is similar to FiniteField and FiniteFieldExtension but is more general.



```
GF4 := FF(2,2);
```

(14)

Type: Domain

```
f :=
  nextIrreduciblePoly(random(6)$FFPOLY(GF4))$FFPOLY(GF4)
?^6 + ?^5 + %CQ ?^4 + %CQ + 1
```

(15)

Type: Union(SparseUnivariatePolynomial FiniteField(2, 2), ...)

For FFP you choose both the ground field and the irreducible polynomial used in the representation. The degree of the extension is the degree of the polynomial.

```
GF4096 := FFP(GF4,f);
```

(16)

Type: Domain

```
discreteLog random()$GF4096
3387
```

(17)

Type: PositiveInteger

#### 8.11.4 Cyclic Group Representations

---

In every finite field there exist elements whose powers are all the nonzero elements of the field. Such an element is called a *primitive element*.

In FiniteFieldCyclicGroup (abbreviation FFCG) the nonzero elements are represented by the powers of a fixed primitive element of the field (that is, a generator of its cyclic multiplicative group). Multiplication (and hence exponentiation) using this representation is easy. To do addition, we consider our primitive element as the root of a primitive polynomial (an irreducible polynomial whose roots are all primitive). See Section 8.11.7 on page 329 for examples of how to compute such a polynomial.

To use FiniteFieldCyclicGroup you provide a prime number and an extension degree.

```
GF81 := FFCG(3,4);
```

(1)

Type: Domain

AXIOM uses the prime field, here PrimeField 3, as the ground field and it chooses a primitive polynomial of degree  $n$ , here 4, over the prime field.

```
a := primitiveElement()$GF81
%CS1
```

(2)

Type: FiniteFieldCyclicGroup(3, 4)

You can calculate in GF81.

```
b := a**12 - a**5 + a
%CS72
```

(3)

Type: FiniteFieldCyclicGroup(3, 4)

In this representation of finite fields the discrete logarithm of an element can be seen directly in its output form.

```
b
%CS72
Type: FiniteFieldCyclicGroup(3, 4)
```

```
discreteLog b
72
Type: PositiveInteger
```

FiniteFieldCyclicGroupExtension (abbreviation FFCGX) is similar to FiniteFieldCyclicGroup except that the ground field for FiniteFieldCyclicGroupExtension is arbitrary and chosen by you. In case you select the prime field as ground field, there is essentially no difference between the constructed two finite field extensions.

```
GF9 := FF(3, 2);
Type: Domain
```

```
GF729 := FFCGX(GF9, 3);
Type: Domain
```

```
r := (random()$GF729) ** 20
%CU420
Type: FiniteFieldCyclicGroupExtension(FiniteField(3, 2), 3)
```

```
trace(r)
0
Type: FiniteField(3, 2)
```

FiniteFieldCyclicGroupExtensionByPolynomial (abbreviation FFCGP) is similar to FiniteFieldCyclicGroup and FiniteFieldCyclicGroupExtension but is more general. For FiniteFieldCyclicGroupExtensionByPolynomial you choose both the ground field and the irreducible polynomial used in the representation. The degree of the extension is the degree of the polynomial.

```
GF3 := PrimeField 3;
Type: Domain
```

We use a utility operation to generate an irreducible primitive polynomial (see Section 8.11.7 on page 329). The polynomial has one variable that is “anonymous”: it displays as a question mark.

```
f := createPrimitivePoly(4)$FFPOLY(GF3)
?^4+? + 2
```

(11)

Type: SparseUnivariatePolynomial PrimeField 3

```
GF81 := FFCGP(GF3,f);
```

(12)

Type: Domain

Let’s look at a random element from this field.

```
random()$GF81
%CS13
```

(13)

Type: FiniteFieldCyclicGroupExtensionByPolynomial(PrimeField 3, ?\*\*4+?+2)

### 8.11.5 Normal Basis Representations

---

Let  $K$  be a finite extension of degree  $n$  of the finite field  $F$  and let  $F$  have  $q$  elements. An element  $x$  of  $K$  is said to be *normal* over  $F$  if the elements

$$1, x^q, x^{q^2}, \dots, x^{q^{n-1}}$$

form a basis of  $K$  as a vector space over  $F$ . Such a basis is called a *normal basis*.<sup>3</sup>

If  $x$  is normal over  $F$ , its minimal polynomial is also said to be *normal* over  $F$ . There exist normal bases for all finite extensions of arbitrary finite fields.

In `FiniteFieldNormalBasis` (abbreviation `FFNB`), the elements of the finite field are represented by coordinate vectors with respect to a normal basis.

You provide a prime  $p$  and an extension degree  $n$ .

```
K := FFNB(3,8)
FiniteFieldNormalBasis (3,8)
```

(1)

Type: Domain

AXIOM uses the prime field `PrimeField(p)`, here `PrimeField 3`, and it chooses a normal polynomial of degree  $n$ , here 8, over the ground field. The remainder class of the indeterminate is used as the normal element. The polynomial indeterminate is automatically chosen by AXIOM and is typically something like `%A` or `%D`. These (strange) variables are only for output display; there are several ways to construct elements of this field. The output of the basis elements is something like `%Aqi`.

---

<sup>3</sup>This agrees with the general definition of a normal basis because the  $n$  distinct powers of the automorphism  $x \mapsto x^q$  constitute the Galois group of  $K/F$ .

```
a := normalElement()$K
%CV
```

Type: FiniteFieldNormalBasis(3, 8)

You can calculate in  $K$  using  $a$ .

```
b := a**12 - a**5 + a
2 %CV^7 + %CV^5 + %CV^4
```

Type: FiniteFieldNormalBasis(3, 8)

FiniteFieldNormalBasisExtension (abbreviation FFNBX) is similar to FiniteFieldNormalBasis except that the groundfield for FiniteFieldNormalBasisExtension is arbitrary and chosen by you. In case you select the prime field as ground field, there is essentially no difference between the constructed two finite field extensions.

```
GF9 := FFNB(3, 2);
```

Type: Domain

```
GF729 := FFNBX(GF9, 3);
```

Type: Domain

```
r := random()$GF729
(2 %CW^q + %CW) %CX^q^2 + %CW %CX^q
```

Type: FiniteFieldNormalBasisExtension(FiniteFieldNormalBasis(3, 2), 3)

```
r + r**3 + r**9 + r**27
%CW %CX^q^2 + 2 %CW^q %CX^q + %CW %CX
```

Type: FiniteFieldNormalBasisExtension(FiniteFieldNormalBasis(3, 2), 3)

FiniteFieldNormalBasisExtensionByPolynomial (abbreviation FFNBP) is similar to FiniteFieldNormalBasis and FiniteFieldNormalBasisExtension but is more general. For FiniteFieldNormalBasisExtensionByPolynomial you choose both the ground field and the irreducible polynomial used in the representation. The degree of the extension is the degree of the polynomial.

```
GF3 := PrimeField 3;
```

Type: Domain

We use a utility operation to generate an irreducible normal polynomial (see Section 8.11.7 on page 329). The polynomial has one variable that is “anonymous”: it displays as a question mark.

```
f := createNormalPoly(4)$FFPOLY(GF3)
?4 + 2 ?3 + 2
```

(9)

Type: SparseUnivariatePolynomial PrimeField 3

```
GF81 := FFNBP(GF3,f);
```

(10)

Type: Domain

Let’s look at a random element from this field.

```
r := random()$GF81
2 %CYq
```

(11)

```
Type: FiniteFieldNormalBasisExtensionByPolynomial(PrimeField 3,
? **4+2*? **3+2)
r * r **3 * r **9 * r **27
2 %CYq3 + 2 %CYq2 + 2 %CYq + 2 %CY
```

(12)

```
Type: FiniteFieldNormalBasisExtensionByPolynomial(PrimeField 3,
? **4+2*? **3+2)
norm r
2
```

(13)

Type: PrimeField 3

## 8.11.6 Conversion Operations for Finite Fields

---

Let  $K$  be a finite field.

```
K := PrimeField 3
PrimeField 3
```

(1)

Type: Domain

An extension field  $K_m$  of degree  $m$  over  $K$  is a subfield of an extension field  $K_n$  of degree  $n$  over  $K$  if and only if  $m$  divides  $n$ .

$$\begin{array}{c} K_n \\ | \\ K_m \\ | \\ K \end{array} \iff m|n$$

FiniteFieldHomomorphisms provides conversion operations between different

extensions of one fixed finite ground field and between different representations of these finite fields.

Let's choose  $m$  and  $n$ ,

```
(m,n) := (4,8)
```

(2)

Type: PositivelInteger

build the field extensions,

```
Km := FiniteFieldExtension(K,m)
FiniteFieldExtension (PrimeField 3, 4)
```

(3)

Type: Domain

and pick two random elements from the smaller field.

```
Kn := FiniteFieldExtension(K,n)
FiniteFieldExtension (PrimeField 3, 8)
```

(4)

Type: Domain

```
a1 := random()$Km
2 %CZ3 + 2 %CZ2 + 2 %CZ
```

(5)

Type: FiniteFieldExtension(PrimeField 3, 4)

```
b1 := random()$Km
%CZ3 + %CZ2 + 2 %CZ + 1
```

(6)

Type: FiniteFieldExtension(PrimeField 3, 4)

Since  $m$  divides  $n$ ,  $K_m$  is a subfield of  $K_n$ .

```
a2 := a1 :: Kn
2 %DA6
```

(7)

Type: FiniteFieldExtension(PrimeField 3, 8)

Therefore we can convert the elements of  $K_m$  into elements of  $K_n$ .

```
b2 := b1 :: Kn
2 %DA6 + 2 %DA4 + %DA2 + 1
```

(8)

Type: FiniteFieldExtension(PrimeField 3, 8)

To check this, let's do some arithmetic.

```
a1+b1 - ((a2+b2) :: Km)
0
```

(9)

Type: FiniteFieldExtension(PrimeField 3, 4)

```
a1*b1 - ((a2*b2) :: Km)
0
```

(10)

Type: FiniteFieldExtension(PrimeField 3, 4)

There are also conversions available for the situation, when  $K_m$  and  $K_n$  are represented in different ways (see Section 8.11.2 on page 319). For example let's choose  $K_m$  where the representation is 0 plus the cyclic multiplicative group and  $K_n$  with a normal basis representation.

```
Km := FFCGX(K,m)
FiniteFieldCyclicGroupExtension (PrimeField 3, 4) (11)
```

Type: Domain

```
Kn := FFNBX(K,n)
FiniteFieldNormalBasisExtension (PrimeField 3, 8) (12)
```

Type: Domain

```
(a1,b1) := (random()$Km,random()$Km)
%CS8 (13)
```

Type: FiniteFieldCyclicGroupExtension(PrimeField 3, 4)

```
a2 := a1 :: Kn
2 %DBq6 + 2 %DBq2 (14)
```

Type: FiniteFieldNormalBasisExtension(PrimeField 3, 8)

```
b2 := b1 :: Kn
%DBq7 + %DBq6 + 2 %DBq4 + %DBq3 + %DBq2 + 2 %DB (15)
```

Type: FiniteFieldNormalBasisExtension(PrimeField 3, 8)

Check the arithmetic again.

```
a1+b1 - ((a2+b2) :: Km)
0 (16)
```

Type: FiniteFieldCyclicGroupExtension(PrimeField 3, 4)

```
a1*b1 - ((a2*b2) :: Km)
0 (17)
```

Type: FiniteFieldCyclicGroupExtension(PrimeField 3, 4)

### 8.11.7 Utility Operations for Finite Fields

FiniteFieldPolynomialPackage (abbreviation FFPOLY) provides operations for generating, counting and testing polynomials over finite fields. Let's start with a couple of definitions:

- A polynomial is *primitive* if its roots are primitive elements in an extension of the coefficient field of degree equal to the degree of the polynomial.
- A polynomial is *normal* over its coefficient field if its roots are linearly independent elements in an extension of the coefficient field of degree equal to the degree of the polynomial.

In what follows, many of the generated polynomials have one “anonymous” variable. This indeterminate is displayed as a question mark (“?”).

To fix ideas, let's use the field with five elements for the first few examples.

```
GF5 := PF 5;
```

(1)

Type: Domain

You can generate irreducible polynomials of any (positive) degree (within the storage capabilities of the computer and your ability to wait) by using **createIrreduciblePoly**.

```
f := createIrreduciblePoly(8)$FFPOLY(GF5)
```

```
?^8 + ?^4 + 2
```

(2)

Type: SparseUnivariatePolynomial PrimeField 5

Does this polynomial have other important properties? Use **primitive?** to test whether it is a primitive polynomial.

```
primitive?(f)$FFPOLY(GF5)
```

```
false
```

(3)

Type: Boolean

Use **normal?** to test whether it is a normal polynomial.

```
normal?(f)$FFPOLY(GF5)
```

```
false
```

(4)

Type: Boolean

Note that this is actually a trivial case, because a normal polynomial of degree  $n$  must have a nonzero term of degree  $n - 1$ . We will refer back to this later.

To get a primitive polynomial of degree 8 just issue this.

```
p := createPrimitivePoly(8)$FFPOLY(GF5)
```

```
?^8 + ?^3 + ?^2 + ? + 2
```

(5)

Type: SparseUnivariatePolynomial PrimeField 5

```
primitive?(p)$FFPOLY(GF5)
```

```
true
```

(6)

Type: Boolean

This polynomial is not normal,

```
normal?(p)$FFPOLY(GF5)
```

```
false
```

(7)

Type: Boolean

but if you want a normal one simply write this.

```
n := createNormalPoly(8)$FFPOLY(GF5)
```

```
?^8 + 4 ?^7 + ?^3 + 1
```

(8)

Type: SparseUnivariatePolynomial PrimeField 5

This polynomial is not primitive!

```
primitive?(n)$FFPOLY(GF5)
```

```
false
```

(9)

Type: Boolean

This could have been seen directly, as the constant term is 1 here, which is not a primitive element up to the factor  $(-1)$  raised to the degree of the



polynomial.<sup>4</sup>

What about polynomials that are both primitive and normal? The existence of such a polynomial is by no means obvious.<sup>5</sup>

If you really need one use either **createPrimitiveNormalPoly** or **createNormalPrimitivePoly**.

```
createPrimitiveNormalPoly(8)$FFPOLY(GF5)
```

$$x^8 + 4x^7 + 2x^5 + 2 \quad (10)$$

Type: SparseUnivariatePolynomial PrimeField 5

If you want to obtain additional polynomials of the various types above as given by the **create...** operations above, you can use the **next...** operations. For instance, **nextIrreduciblePoly** yields the next monic irreducible polynomial with the same degree as the input polynomial. By “next” we mean “next in a natural order using the terms and coefficients.” This will become more clear in the following examples.

This is the field with five elements.

```
GF5 := PF 5;
```

$$(11)$$

Type: Domain

Our first example irreducible polynomial, say of degree 3, must be “greater” than this.

```
h := monomial(1,8)$SUP(GF5)
```

$$x^8 \quad (12)$$

Type: SparseUnivariatePolynomial PrimeField 5

You can generate it by doing this.

```
nh := nextIrreduciblePoly(h)$FFPOLY(GF5)
```

$$x^8 + 2 \quad (13)$$

Type: Union(SparseUnivariatePolynomial PrimeField 5, ...)

Notice that this polynomial is not the same as the one **createIrreduciblePoly**.

```
createIrreduciblePoly(3)$FFPOLY(GF5)
```

$$x^3 + x + 1 \quad (14)$$

Type: SparseUnivariatePolynomial PrimeField 5

You can step through all irreducible polynomials of degree 8 over the field with 5 elements by repeatedly issuing this.

```
nh := nextIrreduciblePoly(nh)$FFPOLY(GF5)
```

$$x^8 + 3 \quad (15)$$

Type: Union(SparseUnivariatePolynomial PrimeField 5, ...)

You could also ask for the total number of these.

```
numberOfIrreduciblePoly(5)$FFPOLY(GF5)
```

$$624 \quad (16)$$

Type: PositiveInteger

<sup>4</sup>Cf. Lidl, R. & Niederreiter, H., *Finite Fields*, Encycl. of Math. 20, (Addison-Wesley, 1983), p.90, Th. 3.18.

<sup>5</sup>The existence of such polynomials is proved in Lenstra, H. W. & Schoof, R. J., *Primitive Normal Bases for Finite Fields*, Math. Comp. 48, 1987, pp. 217-231.

We hope that “natural order” on polynomials is now clear: first we compare the number of monomials of two polynomials (“more” is “greater”); then, if necessary, the degrees of these monomials (lexicographically), and lastly their coefficients (also lexicographically, and using the operation **lookup** if our field is not a prime field). Also note that we make both polynomials monic before looking at the coefficients: multiplying either polynomial by a nonzero constant produces the same result.

The package `FiniteFieldPolynomialPackage` also provides similar operations for primitive and normal polynomials. With the exception of the number of primitive normal polynomials; we’re not aware of any known formula for this.

Take these,

```
numberOfPrimitivePoly(3)$FFPOLY(GF5)
20
Type: PositivelInteger
```

(17)

```
m := monomial(1,1)$SUP(GF5)
?
Type: SparseUnivariatePolynomial PrimeField 5
```

(18)

```
f := m**3 + 4*m**2 + m + 2
?^3 + 4 ?^2 + ? + 2
Type: SparseUnivariatePolynomial PrimeField 5
```

(19)

and then we have:

```
f1 := nextPrimitivePoly(f)$FFPOLY(GF5)
?^3 + 4 ?^2 + 4 ? + 2
Type: Union(SparseUnivariatePolynomial PrimeField 5, ...)
```

(20)

What happened?

```
nextPrimitivePoly(f1)$FFPOLY(GF5)
?^3 + 2 ?^2 + 3
Type: Union(SparseUnivariatePolynomial PrimeField 5, ...)
```

(21)

Well, for the ordering used in **nextPrimitivePoly** we use as first criterion a comparison of the constant terms of the polynomials. Analogously, in **nextNormalPoly** we first compare the monomials of degree 1 less than the degree of the polynomials (which is nonzero, by an earlier remark).

```
f := m**3 + m**2 + 4*m + 1
?^3 + ?^2 + 4 ? + 1
Type: SparseUnivariatePolynomial PrimeField 5
```

(22)

```
f1 := nextNormalPoly(f)$FFPOLY(GF5)
?^3 + ?^2 + 4 ? + 3
Type: Union(SparseUnivariatePolynomial PrimeField 5, ...)
```

(23)

```
nextNormalPoly(f1)$FFPOLY(GF5)
?3 + 2 ?2 + 1
```

(24)

Type: Union(SparseUnivariatePolynomial PrimeField 5, ...)

We don't have to restrict ourselves to prime fields.

Let's consider, say, a field with 16 elements.

```
GF16 := FFX(FFX(PF 2, 2), 2);
```

(25)

Type: Domain

We can apply any of the operations described above.

```
createIrreduciblePoly(5)$FFPOLY(GF16)
?5 + %DD
```

(26)

Type: SparseUnivariatePolynomial  
FiniteFieldExtension(FiniteFieldExtension(PrimeField 2, 2), 2)

AXIOM also provides operations for producing random polynomials of a given degree

```
random(5)$FFPOLY(GF16)
?5 + (%CQ %DD + 1) ?4 + ((%CQ + 1) %DD + 1) ?3 +
(%CQ + 1) %DD ?2 + (%DD + 1) ? + (%CQ + 1) %DD + %CQ
```

(27)

Type: SparseUnivariatePolynomial  
FiniteFieldExtension(FiniteFieldExtension(PrimeField 2, 2), 2)

or with degree between two given bounds.

```
random(3, 9)$FFPOLY(GF16)
?8 + (%CQ %DD + %CQ) ?7 + (%DD + %CQ + 1) ?6 +
((%CQ + 1) %DD + 1) ?5 + ((%CQ + 1) %DD + %CQ) ?4 +
(%CQ + 1) %DD ?3 + ((%CQ + 1) %DD + %CQ) ?2 + %DD ?
```

(28)

Type: SparseUnivariatePolynomial  
FiniteFieldExtension(FiniteFieldExtension(PrimeField 2, 2), 2)

**FiniteFieldPolynomialPackage2** (abbreviation **FFPOLY2**) exports an operation **rootOfIrreduciblePoly** for finding one root of an irreducible polynomial **f** in an extension field of the coefficient field. The degree of the extension has to be a multiple of the degree of **f**. It is not checked whether **f** actually is irreducible.

To illustrate this operation, we fix a ground field **GF**

```
GF2 := PrimeField 2;
```

(29)

Type: Domain

and then an extension field.

```
F := FFX(GF2, 12)
FiniteFieldExtension (PrimeField 2, 12)
```

(30)

Type: Domain

We construct an irreducible polynomial over  $\mathbf{GF}2$ .

```
f := createIrreduciblePoly(6)$FFPOLY(GF2)
?^6+?+1
```

(31)

Type: SparseUnivariatePolynomial PrimeField 2

We compute a root of  $f$ .

```
root := rootOfIrreduciblePoly(f)$FFPOLY2(F,GF2)
%CN11 + %CN8 + %CN7 + %CN5 + %CN + 1
```

(32)

Type: FiniteFieldExtension(PrimeField 2, 12)

## 8.12 Primary Decomposition of Ideals

---

First consider the ideal generated by  $x^2 + y^2 - 1$  (which defines a circle in the  $(x, y)$ -plane) and the ideal generated by  $x^2 - y^2$  (corresponding to the straight lines  $x = y$  and  $x = -y$ ).

We find the equations defining the intersection of the two loci. This correspond to the sum of the associated ideals.

We can check if the locus contains only a finite number of points, that is, if the ideal is zero-dimensional.

AXIOM provides a facility for the primary decomposition of polynomial ideals over fields of characteristic zero. The algorithm works in essentially two steps:

1. the problem is solved for 0-dimensional ideals by “generic” projection on the last coordinate
2. a “reduction process” uses localization and ideal quotients to reduce the general case to the 0-dimensional one.

The AXIOM constructor `PolynomialIdeals` represents ideals with coefficients in any field and supports the basic ideal operations, including intersection, sum and quotient. `IdealDecompositionPackage` contains the specific operations for the primary decomposition and the computation of the radical of an ideal with polynomial coefficients in a field of characteristic 0 with an effective algorithm for factoring polynomials.

The following examples illustrate the capabilities of this facility.

```
(n,m) : List DMP([x,y],FRAC INT)
```

Type: Void

```
m := [x**2+y**2-1]
```

$$\left[ x^2 + y^2 - 1 \right] \quad (2)$$

Type: List DistributedMultivariatePolynomial([x, y], Fraction Integer)

```
n := [x**2-y**2]
```

$$\left[ x^2 - y^2 \right] \quad (3)$$

Type: List DistributedMultivariatePolynomial([x, y], Fraction Integer)

```
id := ideal m + ideal n
```

$$\left[ x^2 - \frac{1}{2}, y^2 - \frac{1}{2} \right] \quad (4)$$

Type: PolynomialIdeals(Fraction Integer, DirectProduct(2, NonNegativeInteger), OrderedVariableList [x, y], DistributedMultivariatePolynomial([x, y], Fraction Integer))

```
zeroDim? id
```

```
true \quad (5)
```

Type: Boolean

```
zeroDim?(ideal m)
false
```

Type: Boolean

```
dimension ideal m
1
```

Type: PositiveInteger

We can find polynomial relations among the generators (**f** and **g** are the parametric equations of the knot).

```
(f,g):DMP([x,y],FRAC INT)
```

Type: Void

```
f := x**2-1
x2 - 1
```

Type: DistributedMultivariatePolynomial([x, y], Fraction Integer)

```
g := x*(x**2-1)
x3 - x
```

Type: DistributedMultivariatePolynomial([x, y], Fraction Integer)

```
relationsIdeal [f,g]
[-%DF2 +%DE3 +%DE2] | [%DE = x2 - 1, %DF = x3 - x]
```

Type: SuchThat(List Polynomial Fraction Integer, List Equation Polynomial Fraction Integer)

We can compute the primary decomposition of an ideal.

```
l: List DMP([x,y,z],FRAC INT)
```

Type: Void

```
l:=[x**2+2*y**2,x*z**2-y*z,z**2-4]
[x2 + 2 y2, x z2 - y z, z2 - 4]
```

Type: List DistributedMultivariatePolynomial([x, y, z], Fraction Integer)

```
ld:=primaryDecomp ideal l
[[x + 1/2 y, y2, z + 2], [x - 1/2 y, y2, z - 2]]
```

Type: List PolynomialIdeals(Fraction Integer, DirectProduct(3, NonNegativeInteger), OrderedVariableList [x, y, z], DistributedMultivariatePolynomial([x, y, z], Fraction Integer))

We can intersect back.

```
reduce(intersect,ld)
```

$$\left[ x - \frac{1}{4} y z, y^2, z^2 - 4 \right] \quad (15)$$

```
Type: PolynomialIdeals(Fraction Integer, DirectProduct(3, NonNegativeInteger),
OrderedVariableList [x, y, z], DistributedMultivariatePolynomial([x, y, z],
Fraction Integer))
```

We can compute the radical of every primary component.

```
reduce(intersect,[radical ld.i for i in 1..2])
```

$$\left[ x, y, z^2 - 4 \right] \quad (16)$$

```
Type: PolynomialIdeals(Fraction Integer, DirectProduct(3, NonNegativeInteger),
OrderedVariableList [x, y, z], DistributedMultivariatePolynomial([x, y, z],
Fraction Integer))
```

Their intersection is equal to the radical of the ideal of 1.

```
radical ideal 1
```

$$\left[ x, y, z^2 - 4 \right] \quad (17)$$

```
Type: PolynomialIdeals(Fraction Integer, DirectProduct(3, NonNegativeInteger),
OrderedVariableList [x, y, z], DistributedMultivariatePolynomial([x, y, z],
Fraction Integer))
```

## 8.13 Computation of Galois Groups

As a sample use of AXIOM's algebraic number facilities, we compute the Galois group of the polynomial  $p(x) = x^5 - 5x + 12$ .

```
p := x**5 - 5*x + 12
```

$$x^5 - 5x + 12 \quad (1)$$

Type: Polynomial Integer

We would like to construct a polynomial  $f(x)$  such that the splitting field of  $p(x)$  is generated by one root of  $f(x)$ . First we construct a polynomial  $r = r(x)$  such that one root of  $r(x)$  generates the field generated by two roots of the polynomial  $p(x)$ . (As it will turn out, the field generated by two roots of  $p(x)$  is, in fact, the splitting field of  $p(x)$ .)

From the proof of the primitive element theorem we know that if  $a$  and  $b$  are algebraic numbers, then the field  $\mathbf{Q}(a, b)$  is equal to  $\mathbf{Q}(a + kb)$  for an appropriately chosen integer  $k$ . In our case, we construct the minimal polynomial of  $a_i - a_j$ , where  $a_i$  and  $a_j$  are two roots of  $p(x)$ . We construct this polynomial using **resultant**. The main result we need is the following: If  $f(x)$  is a polynomial with roots  $a_1 \dots a_m$  and  $g(x)$  is a polynomial with roots  $b_1 \dots b_n$ , then the polynomial  $h(x) = \text{resultant}(f(y), g(x-y), y)$  is a polynomial of degree  $m*n$  with roots  $a_i + b_j, i = 1 \dots m, j = 1 \dots n$ .

For  $f(x)$  we use the polynomial  $p(x)$ . For  $g(x)$  we use the polynomial  $-p(-x)$ . Thus, the polynomial we first construct is  $\text{resultant}(p(y), -p(y-x), y)$ .

```
q := resultant(eval(p,x,y), -eval(p,x,y-x), y)
```

$$x^{25} - 50x^{21} - 2375x^{17} + 90000x^{15} - 5000x^{13} + 2700000x^{11} + 250000x^9 + 18000000x^7 + 64000000x^5 \quad (2)$$

Type: Polynomial Integer

The roots of  $q(x)$  are  $a_i - a_j, i \leq 1, j \leq 5$ . Of course, there are five pairs  $(i, j)$  with  $i = j$ , so 0 is a 5-fold root of  $q(x)$ .

Let's get rid of this factor.

```
q1 := exquo(q, x**5)
```

$$x^{20} - 50x^{16} - 2375x^{12} + 90000x^{10} - 5000x^8 + 2700000x^6 + 250000x^4 + 18000000x^2 + 64000000 \quad (3)$$

Type: Union(Polynomial Integer, ...)

Factor the polynomial  $q1$ .

```
factoredQ := factor q1
```

$$\left( x^{10} - 10x^8 - 75x^6 + 1500x^4 - 5500x^2 + 16000 \right) \times \left( x^{10} + 10x^8 + 125x^6 + 500x^4 + 2500x^2 + 4000 \right) \quad (4)$$

Type: Factored Polynomial Integer

We see that  $q1$  has two irreducible factors, each of degree 10. (The fact that the polynomial  $q1$  has two factors of degree 10 is enough to show that the Galois group of  $p(x)$  is the dihedral group of order 10.<sup>6</sup> Note that

<sup>6</sup>See McKay, Soicher, Computing Galois Groups over the Rationals, Journal of



the type of `factoredQ` is FR POLY INT, that is, Factored Polynomial Integer. This is a special data type for recording factorizations of polynomials with integer coefficients (see ‘Factored’ on page 414).

We can access the individual factors using the operation `nthFactor`.

```
r := nthFactor(factoredQ,1)
```

$$x^{10} - 10 x^8 - 75 x^6 + 1500 x^4 - 5500 x^2 + 16000 \quad (5)$$

Type: Polynomial Integer

Consider the polynomial  $r = r(x)$ . This is the minimal polynomial of the difference of two roots of  $p(x)$ . Thus, the splitting field of  $p(x)$  contains a subfield of degree 10. We show that this subfield is, in fact, the splitting field of  $p(x)$  by showing that  $p(x)$  factors completely over this field.

First we create a symbolic root of the polynomial  $r(x)$ . (We replaced `x` by `b` in the polynomial `r` so that our symbolic root would be printed as `b`.)

```
beta:AN := rootOf(eval(r,x,b))
```

$$b \quad (6)$$

Type: AlgebraicNumber

We next tell AXIOM to view  $p(x)$  as a univariate polynomial in `x` with algebraic number coefficients. This is accomplished with this type declaration.

```
p := p::UP(x,INT)::UP(x,AN)
```

$$x^5 - 5 x + 12 \quad (7)$$

Type: UnivariatePolynomial(x, AlgebraicNumber)

---

Number Theory 20, 273-281 (1983). We do not assume the results of this paper, however, and we continue with the computation.

Factor  $p(x)$  over the field  $\mathbf{Q}(\beta)$ .  
(This computation will take  
some time!)

`algFactors := factor(p,[beta])`

$$\left( x + \frac{\begin{pmatrix} -85 b^9 - 116 b^8 + 780 b^7 + 2640 b^6 + 14895 b^5 - 8820 b^4 \\ -127050 b^3 - 327000 b^2 - 405200 b + 2062400 \end{pmatrix}}{1339200} \right) \cdot$$

$$\left( x + \frac{-17 b^8 + 156 b^6 + 2979 b^4 - 25410 b^2 - 14080}{66960} \right) \cdot$$

$$\left( x + \frac{143 b^8 - 2100 b^6 - 10485 b^4 + 290550 b^2 - 334800 b - 960800}{669600} \right) \cdot \quad (8)$$

$$\left( x + \frac{143 b^8 - 2100 b^6 - 10485 b^4 + 290550 b^2 + 334800 b - 960800}{669600} \right) \cdot$$

$$\left( x + \frac{\begin{pmatrix} 85 b^9 - 116 b^8 - 780 b^7 + 2640 b^6 - 14895 b^5 - 8820 b^4 \\ +127050 b^3 - 327000 b^2 + 405200 b + 2062400 \end{pmatrix}}{1339200} \right)$$

Type: Factored UnivariatePolynomial(x, AlgebraicNumber)

When factoring over number fields, it is important to specify the field over which the polynomial is to be factored, as polynomials have different factorizations over different fields. When you use the operation **factor**, the field over which the polynomial is factored is the field generated by

1. the algebraic numbers that appear in the coefficients of the polynomial, and
2. the algebraic numbers that appear in a list passed as an optional second argument of the operation.

In our case, the coefficients of **p** are all rational integers and only **beta** appears in the list, so the field is simply  $\mathbf{Q}(\beta)$ .

It was necessary to give the list **[beta]** as a second argument of the operation because otherwise the polynomial would have been factored over the field generated by its coefficients, namely the rational numbers.

`factor(p)`

$$x^5 - 5 x + 12 \quad (9)$$

Type: Factored UnivariatePolynomial(x, AlgebraicNumber)

We have shown that the splitting field of  $p(x)$  has degree 10. Since the symmetric group of degree 5 has only one transitive subgroup of order 10, we know that the Galois group of  $p(x)$  must be this group, the dihedral group of order 10. Rather than stop here, we explicitly compute the action of the Galois group on the roots of  $p(x)$ .

First we assign the roots of  $p(x)$  as the values of five variables.

We can obtain an individual root by negating the constant coefficient of one of the factors of  $p(x)$ .

$$\text{factor1} := \text{nthFactor}(\text{algFactors}, 1)$$

$$x + \frac{\begin{pmatrix} -85 b^9 - 116 b^8 + 780 b^7 + 2640 b^6 + 14895 b^5 - 8820 b^4 \\ -127050 b^3 - 327000 b^2 - 405200 b + 2062400 \end{pmatrix}}{1339200} \quad (10)$$

Type: UnivariatePolynomial(x, AlgebraicNumber)

$$\text{root1} := -\text{coefficient}(\text{factor1}, 0)$$

$$\frac{\begin{pmatrix} 85 b^9 + 116 b^8 - 780 b^7 - 2640 b^6 - 14895 b^5 + 8820 b^4 \\ +127050 b^3 + 327000 b^2 + 405200 b - 2062400 \end{pmatrix}}{1339200} \quad (11)$$

Type: AlgebraicNumber

We can obtain a list of all the roots in this way.

$$\text{roots} := [-\text{coefficient}(\text{nthFactor}(\text{algFactors}, i), 0) \text{ for } i \text{ in } 1..5]$$

$$\left[ \frac{\begin{pmatrix} 85 b^9 + 116 b^8 - 780 b^7 - 2640 b^6 - 14895 b^5 + 8820 b^4 \\ +127050 b^3 + 327000 b^2 + 405200 b - 2062400 \end{pmatrix}}{1339200}, \right.$$

$$\frac{17 b^8 - 156 b^6 - 2979 b^4 + 25410 b^2 + 14080}{66960},$$

$$\frac{-143 b^8 + 2100 b^6 + 10485 b^4 - 290550 b^2 + 334800 b + 960800}{669600}, \quad (12)$$

$$\frac{-143 b^8 + 2100 b^6 + 10485 b^4 - 290550 b^2 - 334800 b + 960800}{669600},$$

$$\left. \frac{\begin{pmatrix} -85 b^9 + 116 b^8 + 780 b^7 - 2640 b^6 + 14895 b^5 + 8820 b^4 \\ -127050 b^3 + 327000 b^2 - 405200 b - 2062400 \end{pmatrix}}{1339200} \right]$$

Type: List AlgebraicNumber

The expression

`- coefficient(nthFactor(algFactors, i), 0)}`

is the  $i^{\text{th}}$  root of  $p(x)$  and the elements of `roots` are the  $i^{\text{th}}$  roots of  $p(x)$  as  $i$  ranges from 1 to 5.

Assign the roots as the values of the variables  $a_1, \dots, a_5$ .

$$\begin{aligned} & (a_1, a_2, a_3, a_4, a_5) := \\ & \quad (\text{roots.1}, \text{roots.2}, \text{roots.3}, \text{roots.4}, \text{roots.5}) \\ & \quad \left( \frac{-85 b^9 + 116 b^8 + 780 b^7 - 2640 b^6 + 14895 b^5 + 8820 b^4}{-127050 b^3 + 327000 b^2 - 405200 b - 2062400} \right) \end{aligned} \quad (13)$$

Type: AlgebraicNumber

Next we express the roots of  $r(x)$  as polynomials in  $\beta$ . We could obtain these roots by calling the operation **factor**: **factor**( $r$ , [ $\beta$ ]) factors  $r(x)$  over  $\mathbf{Q}(\beta)$ . However, this is a lengthy computation and we can obtain the roots of  $r(x)$  as differences of the roots  $a_1, \dots, a_5$  of  $p(x)$ . Only ten of these differences are roots of  $r(x)$  and the other ten are roots of the other irreducible factor of  $q_1$ . We can determine if a given value is a root of  $r(x)$  by evaluating  $r(x)$  at that particular value. (Of course, the order in which factors are returned by the operation **factor** is unimportant and may change with different implementations of the operation. Therefore, we cannot predict in advance which differences are roots of  $r(x)$  and which are not.)

Let's look at four examples (two are roots of  $r(x)$  and two are not).

$$\begin{aligned} & \text{eval}(r, x, a_1 - a_2) \\ & 0 \end{aligned} \quad (14)$$

Type: Polynomial AlgebraicNumber

$$\begin{aligned} & \text{eval}(r, x, a_1 - a_3) \\ & \left( \frac{47905 b^9 + 66920 b^8 - 536100 b^7 - 980400 b^6 - 3345075 b^5 - 5787000 b^4 + 75572250 b^3 + 161688000 b^2 - 184600000 b - 710912000}{4464} \right) \end{aligned} \quad (15)$$

Type: Polynomial AlgebraicNumber

$$\begin{aligned} & \text{eval}(r, x, a_1 - a_4) \\ & 0 \end{aligned} \quad (16)$$

Type: Polynomial AlgebraicNumber

$$\begin{aligned} & \text{eval}(r, x, a_1 - a_5) \\ & \frac{405 b^8 + 3450 b^6 - 19875 b^4 - 198000 b^2 - 588000}{31} \end{aligned} \quad (17)$$

Type: Polynomial AlgebraicNumber

Take one of the differences that was a root of  $r(x)$  and assign it to the variable  $bb$ .

For example, if `eval(r,x,a1 - a4)` returned 0, you would enter this.

$$\text{bb} := \text{a1} - \text{a4} \quad \frac{\left( 85 b^9 + 402 b^8 - 780 b^7 - 6840 b^6 - 14895 b^5 - 12150 b^4 + 127050 b^3 + 908100 b^2 + 1074800 b - 3984000 \right)}{1339200} \quad (18)$$

Type: AlgebraicNumber

Of course, if the difference is, in fact, equal to the root `beta`, you should choose another root of  $r(x)$ .

Automorphisms of the splitting field are given by mapping a generator of the field, namely `beta`, to other roots of its minimal polynomial. Let's see what happens when `beta` is mapped to `bb`.

We compute the images of the roots `a1, ..., a5` under this automorphism:

$$\text{aa1} := \text{subst}(\text{a1}, \text{beta} = \text{bb}) \quad \frac{-143 b^8 + 2100 b^6 + 10485 b^4 - 290550 b^2 + 334800 b + 960800}{669600} \quad (19)$$

Type: AlgebraicNumber

$$\text{aa2} := \text{subst}(\text{a2}, \text{beta} = \text{bb}) \quad \frac{\left( -85 b^9 + 116 b^8 + 780 b^7 - 2640 b^6 + 14895 b^5 + 8820 b^4 - 127050 b^3 + 327000 b^2 - 405200 b - 2062400 \right)}{1339200} \quad (20)$$

Type: AlgebraicNumber

$$\text{aa3} := \text{subst}(\text{a3}, \text{beta} = \text{bb}) \quad \frac{\left( 85 b^9 + 116 b^8 - 780 b^7 - 2640 b^6 - 14895 b^5 + 8820 b^4 + 127050 b^3 + 327000 b^2 + 405200 b - 2062400 \right)}{1339200} \quad (21)$$

Type: AlgebraicNumber

$$\text{aa4} := \text{subst}(\text{a4}, \text{beta} = \text{bb}) \quad \frac{-143 b^8 + 2100 b^6 + 10485 b^4 - 290550 b^2 - 334800 b + 960800}{669600} \quad (22)$$

Type: AlgebraicNumber

$$\text{aa5} := \text{subst}(\text{a5}, \text{beta} = \text{bb}) \quad \frac{17 b^8 - 156 b^6 - 2979 b^4 + 25410 b^2 + 14080}{66960} \quad (23)$$

Type: AlgebraicNumber

Of course, the values `aa1, ..., aa5` are simply a permutation of the values `a1, ..., a5`.

Let's find the value of **aa1**  
(execute as many of the  
following five commands as  
necessary).

```
(aa1 = a1) :: Boolean
false
```

(24)  
Type: Boolean

```
(aa1 = a2) :: Boolean
false
```

(25)  
Type: Boolean

```
(aa1 = a3) :: Boolean
true
```

(26)  
Type: Boolean

```
(aa1 = a4) :: Boolean
false
```

(27)  
Type: Boolean

```
(aa1 = a5) :: Boolean
false
```

(28)  
Type: Boolean

Proceeding in this fashion, you can find the values of **aa2**, ..., **aa5**.<sup>7</sup> You have represented the automorphism **beta**  $\rightarrow$  **bb** as a permutation of the roots **a1**, ..., **a5**. If you wish, you can repeat this computation for all the roots of  $r(x)$  and represent the Galois group of  $p(x)$  as a subgroup of the symmetric group on five letters.

Here are two other problems that you may attack in a similar fashion:

1. Show that the Galois group of  $p(x) = x^4 + 2x^3 - 2x^2 - 3x + 1$  is the dihedral group of order eight. (The splitting field of this polynomial is the Hilbert class field of the quadratic field  $\mathbf{Q}(\sqrt{145})$ .)
2. Show that the Galois group of  $p(x) = x^6 + 108$  has order 6 and is isomorphic to  $S_3$ , the symmetric group on three letters. (The splitting field of this polynomial is the splitting field of  $x^3 - 2$ .)

---

<sup>7</sup>Here you should use the Clef line editor. See Section 1.1.1 on page 45 for more information about Clef.

## 8.14 Non-Associative Algebras and Modelling Genetic Laws

---

Many algebraic structures of mathematics and AXIOM have a multiplication operation “ $*$ ” that satisfies the associativity law  $a*(b*c) = (a*b)*c$  for all  $a, b$  and  $c$ . The octonions (see ‘Octonion’ on page 511) are a well known exception. There are many other interesting non-associative structures, such as the class of Lie algebras.<sup>8</sup> Lie algebras can be used, for example, to analyse Lie symmetry algebras of partial differential equations. In this section we show a different application of non-associative algebras, the modelling of genetic laws.

The AXIOM library contains several constructors for creating non-associative structures, ranging from the categories `Monad`, `NonAssociativeRng`, and `FramedNonAssociativeAlgebra`, to the domains `AlgebraGivenByStructuralConstants` and `GenericNonAssociativeAlgebra`. Furthermore, the package `AlgebraPackage` provides operations for analysing the structure of such algebras.<sup>9</sup>

Mendel’s genetic laws are often written in a form like

$$Aa \times Aa = \frac{1}{4}AA + \frac{1}{2}Aa + \frac{1}{4}aa.$$

The implementation of general algebras in AXIOM allows us to use this as the definition for multiplication in an algebra. Hence, it is possible to study questions of genetic inheritance using AXIOM. To demonstrate this more precisely, we discuss one example from a monograph of A. Wörz-Busekros, where you can also find a general setting of this theory.<sup>10</sup>

We assume that there is an infinitely large random mating population. Random mating of two gametes  $a_i$  and  $a_j$  gives zygotes  $a_i a_j$ , which produce new gametes. In classical Mendelian segregation we have  $a_i a_j = \frac{1}{2}a_i + \frac{1}{2}a_j$ . In general, we have

$$a_i a_j = \sum_{k=1}^n \gamma_{i,j}^k a_k.$$

The segregation rates  $\gamma_{i,j}$  are the structural constants of an  $n$ -dimensional algebra. This is provided in AXIOM by the constructor `AlgebraGivenByStructuralConstants` (abbreviation `ALGSC`).

Consider two coupled autosomal loci with alleles  $A, a, B$ , and  $b$ , building four different gametes  $a_1 = AB, a_2 = Ab, a_3 = aB$ , and  $a_4 = ab$ . The

---

<sup>8</sup>Two AXIOM implementations of Lie algebras are `LieSquareMatrix` and `FreeNilpotentLie`.

<sup>9</sup>The interested reader can learn more about these aspects of the AXIOM library from the paper “Computations in Algebras of Finite Rank,” by Johannes Grabmeier and Robert Wisbauer, Technical Report, IBM Heidelberg Scientific Center, 1992.

<sup>10</sup>Wörz-Busekros, A., *Algebras in Genetics*, Springer Lectures Notes in Biomathematics 36, Berlin e.a. (1980). In particular, see example 1.3.

zygotes  $a_i a_j$  produce gametes  $a_i$  and  $a_j$  with classical Mendelian segregation. Zygote  $a_1 a_4$  undergoes transition to  $a_2 a_3$  and vice versa with probability  $0 \leq \theta \leq \frac{1}{2}$ .

Define a list  $[(\gamma_{i,j}^k) 1 \leq k \leq 4]$  of four four-by-four matrices giving the segregation rates. We use the value  $1/10$  for  $\theta$ .

```
segregationRates : List SquareMatrix(4,FRAC INT) :=
  [matrix [ [1, 1/2, 1/2, 9/20], [1/2, 0, 1/20, 0], [1/2,
    1/20, 0, 0], [9/20, 0, 0, 0] ], matrix [ [0, 1/2, 0,
    1/20], [1/2, 1, 9/20, 1/2], [0, 9/20, 0, 0], [1/20,
    1/2, 0, 0] ], matrix [ [0, 0, 1/2, 1/20], [0, 0, 9/20,
    0], [1/2, 9/20, 1, 1/2], [1/20, 0, 1/2, 0] ], matrix [
    [0, 0, 0, 9/20], [0, 0, 1/20, 1/2], [0, 1/20, 0, 1/2],
    [9/20, 1/2, 1/2, 1] ] ]
```

$$\left[ \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} & \frac{9}{20} \\ \frac{1}{2} & 0 & \frac{1}{20} & 0 \\ \frac{1}{2} & \frac{1}{20} & 0 & 0 \\ \frac{9}{20} & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{20} \\ \frac{1}{2} & 1 & \frac{9}{20} & \frac{1}{2} \\ 0 & \frac{9}{20} & 0 & 0 \\ \frac{1}{20} & \frac{1}{2} & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{20} \\ 0 & 0 & \frac{9}{20} & 0 \\ \frac{1}{2} & \frac{9}{20} & 1 & \frac{1}{2} \\ \frac{1}{20} & 0 & \frac{1}{2} & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & \frac{9}{20} \\ 0 & 0 & \frac{1}{20} & \frac{1}{2} \\ 0 & \frac{1}{20} & 0 & \frac{1}{2} \\ \frac{9}{20} & \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix} \right],$$

(1)

Type: List SquareMatrix(4, Fraction Integer)

Choose the appropriate symbols for the basis of gametes,

```
gametes := ['AB','Ab','aB','ab']
```

$$[AB, Ab, aB, ab]$$

(2)

Type: List OrderedVariableList [AB, Ab, aB, ab]

Define the algebra.

```
A := ALGSC(FRAC INT, 4, gametes, segregationRates);
```

(3)

Type: Domain

What are the probabilities for zygote  $a_1 a_4$  to produce the different gametes?

```
a := basis($A); a.1*a.4
```

$$\frac{9}{20} ab + \frac{1}{20} aB + \frac{1}{20} Ab + \frac{9}{20} AB$$

(4)

Type: AlgebraGivenByStructuralConstants(Fraction Integer, 4, [AB, Ab, aB, ab], [MATRIX, MATRIX, MATRIX, MATRIX])

Elements in this algebra whose coefficients sum to one play a distinguished role. They represent a population with the distribution of gametes reflected by the coefficients with respect to the basis of gametes.

Random mating of different populations  $x$  and  $y$  is described by their product  $x * y$ .



This product is commutative only if the gametes are not sex-dependent, as in our example.

```
commutative?()$A
algebra is commutative
true
```

(5)

Type: Boolean

In general, it is not associative.

```
associative?()$A
algebra is not associative
false
```

(6)

Type: Boolean

Random mating within a population  $x$  is described by  $x * x$ . The next generation is  $(x * x) * (x * x)$ .

Use decimal numbers to compare the distributions more easily.

```
x : ALGSC(DECIMAL, 4, gametes, segregationRates) :=
  convert [3/10, 1/5, 1/10, 2/5]
```

$0.4 \text{ } ab + 0.1 \text{ } aB + 0.2 \text{ } Ab + 0.3 \text{ } AB$  (7)

Type: AlgebraGivenByStructuralConstants(DecimalExpansion, 4, [AB, Ab, aB, ab], [MATRIX, MATRIX, MATRIX, MATRIX])

To compute directly the gametic distribution in the fifth generation, we use **plenaryPower**.

```
plenaryPower(x, 5)
```

$0.36561 \text{ } ab + 0.13439 \text{ } aB + 0.23439 \text{ } Ab + 0.26561 \text{ } AB$  (8)

Type: AlgebraGivenByStructuralConstants(DecimalExpansion, 4, [AB, Ab, aB, ab], [MATRIX, MATRIX, MATRIX, MATRIX])

We now ask two questions: Does this distribution converge to an equilibrium state? What are the distributions that are stable?

This is an invariant of the algebra and it is used to answer the first question. The new indeterminates describe a symbolic distribution.

```
q := leftRankPolynomial()$GCNAALG(FRAC INT, 4, gametes,
  segregationRates) :: UP(Y, POLY FRAC INT)
```

$Y^3 + \left( -\frac{29}{20} \%x4 - \frac{29}{20} \%x3 - \frac{29}{20} \%x2 - \frac{29}{20} \%x1 \right) Y^2 +$

$\left( \frac{9}{20} \%x4^2 + \left( \frac{9}{10} \%x3 + \frac{9}{10} \%x2 + \frac{9}{10} \%x1 \right) \%x4 + \right.$   
 $\left. \frac{9}{20} \%x3^2 + \left( \frac{9}{10} \%x2 + \frac{9}{10} \%x1 \right) \%x3 + \right.$   
 $\left. \frac{9}{20} \%x2^2 + \frac{9}{10} \%x1 \%x2 + \frac{9}{20} \%x1^2 \right) Y$  (9)

Type: UnivariatePolynomial(Y, Polynomial Fraction Integer)

Because the coefficient  $\frac{9}{20}$  has absolute value less than 1, all distributions do converge, by a theorem of this theory.

$$\text{factor}(q :: \text{POLY FRAC INT})$$

$$(Y - \frac{1}{10}x_4 - \frac{1}{10}x_3 - \frac{1}{10}x_2 - \frac{1}{10}x_1) \times$$

$$\left( Y - \frac{9}{20}x_4 - \frac{9}{20}x_3 - \frac{9}{20}x_2 - \frac{9}{20}x_1 \right) Y \quad (10)$$

Type: Factored Polynomial Fraction Integer

The second question is answered by searching for idempotents in the algebra.

$$\text{cI} := \text{conditionsForIdempotents}() \$ \text{GCNAALG}(\text{FRAC INT}, 4, \text{gametes}, \text{segregationRates})$$

$$\left[ \frac{9}{10}x_1x_4 + \left( \frac{1}{10}x_2 + x_1 \right) x_3 + x_1x_2 + x_1^2 - x_1, \right.$$

$$\left( x_2 + \frac{1}{10}x_1 \right) x_4 + \frac{9}{10}x_2x_3 + x_2^2 + (x_1 - 1)x_2,$$

$$\left( x_3 + \frac{1}{10}x_1 \right) x_4 + x_3^2 + \left( \frac{9}{10}x_2 + x_1 - 1 \right) x_3,$$

$$\left. x_4^2 + \left( x_3 + x_2 + \frac{9}{10}x_1 - 1 \right) x_4 + \frac{1}{10}x_2x_3 \right] \quad (11)$$

Type: List Polynomial Fraction Integer

Solve these equations and look at the first solution.

$$\text{gbs} := \text{groebnerFactorize cI}; \text{gbs.1}$$

$$\left[ x_4 + x_3 + x_2 + x_1 - 1, \right.$$

$$\left. (x_2 + x_1) x_3 + x_1x_2 + x_1^2 - x_1 \right] \quad (12)$$

Type: List Polynomial Fraction Integer

Further analysis using the package PolynomialIdeals shows that there is a two-dimensional variety of equilibrium states and all other solutions are contained in it.

Choose one equilibrium state by setting two indeterminates to concrete values.

$$\text{sol} := \text{solve concat}(\text{gbs.1}, [x_1 - 1/10, x_2 - 1/10])$$

$$\left[ \left[ x_4 = \frac{2}{5}, x_3 = \frac{2}{5}, x_2 = \frac{1}{10}, x_1 = \frac{1}{10} \right] \right] \quad (13)$$

Type: List List Equation Fraction Polynomial Integer

$$\text{e : A} := \text{represents reverse (map(rhs, sol.1))} :: \text{List FRAC INT})$$

$$\frac{2}{5}ab + \frac{2}{5}aB + \frac{1}{10}Ab + \frac{1}{10}AB \quad (14)$$

Type: AlgebraGivenByStructuralConstants(Fraction Integer, 4, [AB, Ab, aB, ab], [MATRIX, MATRIX, MATRIX, MATRIX])

Verify the result.

$$e * e - e$$

$$0 \quad (15)$$

Type: AlgebraGivenByStructuralConstants(Fraction Integer, 4, [AB, Ab, aB, ab],  
[MATRIX, MATRIX, MATRIX, MATRIX])



---

# Some Examples of Domains and Packages

In this chapter we show examples of many of the most commonly used AXIOM domains and packages. The sections are organized by constructor names.

## 9.1 AssociationList

The AssociationList constructor provides a general structure for associative storage. This type provides association lists in which data objects can be saved according to keys of any type. For a given association list, specific types must be chosen for the keys and entries. You can think of the representation of an association list as a list of records with key and entry fields.

Association lists are a form of table and so most of the operations available for Table are also available for AssociationList. They can also be viewed as lists and can be manipulated accordingly.

This is a Record type with age and gender fields.

```
Data := Record(monthsOld : Integer, gender : String)
Record (monthsOld : Integer , gender : String )
```

(1)  
Type: Domain

In this expression, `al` is declared to be an association list whose keys are strings and whose entries are the above records.

```
al : AssociationList(String,Data)
```

Type: Void

The **table** operation is used to create an empty association list.

```
al := table()
table()
```

(3)  
Type: AssociationList(String, Record(monthsOld: Integer, gender: String))

You can use assignment syntax to add things to the association list.

```
al."bob" := [407,"male"]$Data
[monthsOld = 407, gender = "male"]
```

(4)  
Type: Record(monthsOld: Integer, gender: String)

```
al."judith" := [366,"female"]$Data
[monthsOld = 366, gender = "female"]
```

(5)  
Type: Record(monthsOld: Integer, gender: String)

```
al."katie" := [24,"female"]$Data
[monthsOld = 24, gender = "female"]
```

(6)  
Type: Record(monthsOld: Integer, gender: String)

Perhaps we should have included a species field.

```
al."smokie" := [200,"female"]$Data
[monthsOld = 200, gender = "female"]
```

(7)  
Type: Record(monthsOld: Integer, gender: String)

Now look at what is in the association list. Note that the last-added (key, entry) pair is at the beginning of the list.

```
a1
table("smokie" = [monthsOld = 200, gender = "female"],
      "katie" = [monthsOld = 24, gender = "female"],
      "judith" = [monthsOld = 366, gender = "female"],
      "bob" = [monthsOld = 407, gender = "male"])
(8)
```

Type: AssociationList(String, Record(monthsOld: Integer, gender: String))

You can reset the entry for an existing key.

```
a1."katie" := [23,"female"]$Data
[monthsOld = 23, gender = "female"]
(9)
```

Type: Record(monthsOld: Integer, gender: String)

Use **delete!** to destructively remove an element of the association list. Use **delete** to return a copy of the association list with the element deleted. The second argument is the index of the element to delete.

```
delete!(a1,1)
table("katie" = [monthsOld = 23, gender = "female"],
      "judith" = [monthsOld = 366, gender = "female"],
      "bob" = [monthsOld = 407, gender = "male"])
(10)
```

Type: AssociationList(String, Record(monthsOld: Integer, gender: String))

For more information about tables, see ‘Table’ on page 585. For more information about lists, see ‘List’ on page 489. Issue the system command `)show AssociationList` to display the full list of operations defined by AssociationList.

## 9.2 BalancedBinary- Tree

---

BalancedBinaryTrees(*S*) is the domain of balanced binary trees with elements of type *S* at the nodes. A binary tree is either **empty** or else consists of a **node** having a **value** and two branches, each branch a binary tree. A balanced binary tree is one that is balanced with respect its leaves. One with  $2^k$  leaves is perfectly “balanced”: the tree has minimum depth, and the **left** and **right** branch of every interior node is identical in shape.

Balanced binary trees are useful in algebraic computation for so-called “divide-and-conquer” algorithms. Conceptually, the data for a problem is initially placed at the root of the tree. The original data is then split into two subproblems, one for each subtree. And so on. Eventually, the problem is solved at the leaves of the tree. A solution to the original problem is obtained by some mechanism that can reassemble the pieces. In fact, an implementation of the Chinese Remainder Algorithm using balanced binary trees was first proposed by David Y. Y. Yun at the IBM T. J. Watson Research Center in Yorktown Heights, New York, in 1978. It served as the prototype for polymorphic algorithms in AXIOM.

In what follows, rather than perform a series of computations with a single expression, the expression is reduced modulo a number of integer primes, a computation is done with modular arithmetic for each prime, and the Chinese Remainder Algorithm is used to obtain the answer to the original problem. We illustrate this principle with the computation of  $12^2 = 144$ .

A list of moduli.

```
1m := [3,5,7,11]
[3, 5, 7, 11]
(1)
Type: List PositiveInteger
```

The expression `modTree(n, 1m)` creates a balanced binary tree with leaf values  $n \bmod m$  for each modulus *m* in *1m*.

```
modTree(12,1m)
[0, 2, 5, 1]
(2)
Type: List Integer
```

Operation `modTree` does this using operations on balanced binary trees. We trace its steps. Create a balanced binary tree *t* of zeros with four leaves.

```
t := balancedBinaryTree(#1m, 0)
[[0, 0, 0], 0, [0, 0, 0]]
(3)
Type: BalancedBinaryTree NonNegativeInteger
```

The leaves of the tree are set to the individual moduli.

```
setleaves!(t,1m)
[[3, 0, 5], 0, [7, 0, 11]]
(4)
Type: BalancedBinaryTree NonNegativeInteger
```

Use `mapUp!` to do a bottom-up traversal of *t*, setting each interior node to the product of the values at the nodes of its children.

```
mapUp!(t,*)
1155
(5)
Type: PositiveInteger
```



<p>The value at the node of every subtree is the product of the moduli of the leaves of the subtree.</p>	<pre>t</pre> <pre>[[3, 15, 5], 1155, [7, 77, 11]]</pre>	<p>(6)</p> <p>Type: <code>BalancedBinaryTree NonNegativeInteger</code></p>
<p>Operation <b>mapDown!</b>(<i>t</i>,<i>a</i>,<i>fn</i>) replaces the value <i>v</i> at each node of <i>t</i> by <i>fn</i>(<i>a</i>,<i>v</i>).</p>	<pre>mapDown! (t, 12, _rem)</pre> <pre>[[0, 12, 2], 12, [5, 12, 1]]</pre>	<p>(7)</p> <p>Type: <code>BalancedBinaryTree NonNegativeInteger</code></p>
<p>The operation <b>leaves</b> returns the leaves of the resulting tree. In this case, it returns the list of <i>12 mod m</i> for each modulus <i>m</i>.</p>	<pre>leaves %</pre> <pre>[0, 2, 5, 1]</pre>	<p>(8)</p> <p>Type: <code>List NonNegativeInteger</code></p>
<p>Compute the square of the images of <i>12</i> modulo each <i>m</i>.</p>	<pre>squares := [x**2 rem m for x in % for m in 1m]</pre> <pre>[0, 4, 4, 1]</pre>	<p>(9)</p> <p>Type: <code>List NonNegativeInteger</code></p>
<p>Call the Chinese Remainder Algorithm to get the answer for <i>12<sup>5</sup></i>.</p>	<pre>chineseRemainder(%, 1m)</pre> <pre>144</pre>	<p>(10)</p> <p>Type: <code>PositiveInteger</code></p>

## 9.3 BasicOperator

A basic operator is an object that can be symbolically applied to a list of arguments from a set, the result being a kernel over that set or an expression. In addition to this section, please see ‘Expression’ on page 410 and ‘Kernel’ on page 457 for additional information and examples.

You create an object of type BasicOperator by using the **operator** operation. This first form of this operation has one argument and it must be a symbol. The symbol should be quoted in case the name has been used as an identifier to which a value has been assigned.

A frequent application of BasicOperator is the creation of an operator to represent the unknown function when solving a differential equation.

Let  $y$  be the unknown function in terms of  $x$ .

```
y := operator 'y
y
```

Type: BasicOperator

This is how you enter the equation  $y'' + y' + y = 0$ .

```
deq := D(y x, x, 2) + D(y x, x) + y x = 0
y''(x) + y'(x) + y(x) = 0
```

Type: Equation Expression Integer

To solve the above equation, enter this.

```
solve(deq, y, x)
[particular = 0, basis = [cos(x*sqrt(3)/2)*e^(-x/2), e^(-x/2)*sin(x*sqrt(3)/2)]] (3)
```

Type: Union(Record(particular: Expression Integer, basis: List Expression Integer), ...)

See Section 8.10 on page 308 for this kind of use of BasicOperator.

Use the single argument form of **operator** (as above) when you intend to use the operator to create functional expressions with an arbitrary number of arguments

*Nary* means an arbitrary number of arguments can be used in the functional expressions.

```
nary? y
true
```

Type: Boolean

```
unary? y
false
```

Type: Boolean

Use the two-argument form when you want to restrict the number of arguments in the functional expressions created with the operator.

This operator can only be used to create functional expressions with one argument.

```
opOne := operator('opOne, 1)
opOne
```

(6)  
Type: BasicOperator

```
nary? opOne
false
```

(7)  
Type: Boolean

```
unary? opOne
true
```

(8)  
Type: Boolean

Use **arity** to learn the number of arguments that can be used. It returns "**false**" if the operator is nary.

```
arity opOne
1
```

(9)  
Type: Union(NonNegativeInteger, ...)

Use **name** to learn the name of an operator.

```
name opOne
opOne
```

(10)  
Type: Symbol

Use **is?** to learn if an operator has a particular name.

```
is?(opOne, 'z2)
false
```

(11)  
Type: Boolean

You can also use a string as the name to be tested against.

```
is?(opOne, "opOne")
true
```

(12)  
Type: Boolean

You can attached named properties to an operator. These are rarely used at the top-level of the AXIOM interactive environment but are used with AXIOM library source code.

By default, an operator has no properties.

```
properties y
table()
```

(13)  
Type: AssociationList(String, None)

The interface for setting and getting properties is somewhat awkward because the property values are stored as values of type None.

Attach a property by using **setProperty**.

```
setProperty(y, "use", "unknown function" :: None )
y
```

(14)  
Type: BasicOperator

	<pre>properties y table("use" = NONE )</pre>	(15)
		Type: AssociationList(String, None)
We <i>know</i> the property value has type String.	<pre>property(y, "use") :: None pretend String "unknown function"</pre>	(16)
		Type: String
Use <b>deleteProperty!</b> to destructively remove a property.	<pre>deleteProperty!(y, "use") y</pre>	(17)
		Type: BasicOperator
	<pre>properties y table()</pre>	(18)
		Type: AssociationList(String, None)

## 9.4 BinaryExpansion

All rational numbers have repeating binary expansions. Operations to access the individual bits of a binary expansion can be obtained by converting the value to RadixExpansion(2). More examples of expansions are available in ‘DecimalExpansion’ on page 401, ‘HexadecimalExpansion’ on page 444, and ‘RadixExpansion’ on page 537.

The expansion (of type BinaryExpansion) of a rational number is returned by the **binary** operation.

```
r := binary(22/7)
```

$$11.\overline{001} \quad (1)$$

Type: BinaryExpansion

Arithmetic is exact.

```
r + binary(6/7)
```

$$100 \quad (2)$$

Type: BinaryExpansion

The period of the expansion can be short or long ...

```
[binary(1/i) for i in 102..106]
```

$$\begin{aligned} &[0.\overline{000000101}, \\ &0.\overline{000000100111110001000101100101111001110010010101001}, \\ &0.\overline{000000100111011}, \\ &0.\overline{000000100111}, \\ &0.\overline{00000010011010100100001110011111011001010110111100011}] \end{aligned} \quad (3)$$

Type: List BinaryExpansion

or very long.

```
binary(1/1007)
```

$$\begin{aligned} &0.\overline{0000000001000001000101001001011110000011111100001011} \\ &\overline{111100101100011111010001001110010011001100011001001010101} \\ &\overline{111011010011000000001100001100111101110001101000101111010} \\ &\overline{010001111011000010101110111001110101011100110010100101110} \\ &\overline{000000111000111100100000010010010011011100101010011101000} \\ &\overline{110111011010111000100100000110010110110000001011001011111} \\ &\overline{00010100000101010101101011000001101101110100101011111101} \\ &\overline{011101010011001000010100110110001001100010001000010000110} \\ &\overline{00111010011110001} \end{aligned} \quad (4)$$

Type: BinaryExpansion

These numbers are bona fide algebraic objects.

```
p := binary(1/4)*x**2 + binary(2/3)*x + binary(4/9)
```

$$0.01 x^2 + 0.\overline{10} x + 0.\overline{011100} \quad (5)$$

Type: Polynomial BinaryExpansion

```
q := D(p, x)
```

$$0.1 x + 0.\overline{10} \quad (6)$$

Type: Polynomial BinaryExpansion

$g := \gcd(p, q)$

$x + 1.\overline{01}$

(7)

Type: Polynomial BinaryExpansion

## 9.5 BinarySearchTree

Define a list of values to be placed across the tree. The resulting tree has 8 at the root; all other elements are in the left subtree.

A convenient way to create a binary search tree is to apply the operation **binarySearchTree** to a list of elements.

Another approach is to first create an empty binary search tree of integers.

Insert the value 8. This establishes 8 as the root of the binary search tree. Values inserted later that are less than 8 get stored in the **left** subtree, others in the **right** subtree.

Insert the value 3. This number becomes the root of the **left** subtree of **t1**. For optimal retrieval, it is thus important to insert the middle elements first.

We go back to the original tree **t**. The leaves of the binary search tree are those which have empty **left** and **right** subtrees.

The operation **split(k,t)** returns a *record* containing the two subtrees: one with all elements “less” than **k**, another with elements “greater” than **k**.

Define **insertRoot** to insert new elements by creating a new node.

The new node puts the inserted value between its “less” tree and “greater” tree.

BinarySearchTree(R) is the domain of binary trees with elements of type **R**, ordered across the nodes of the tree. A non-empty binary search tree has a value of type **R**, and **right** and **left** binary search subtrees. If a subtree is empty, it is displayed as a period (“.”).

```
lv := [8,3,5,4,6,2,1,5,7]
[8, 3, 5, 4, 6, 2, 1, 5, 7]
```

Type: List PositiveInteger

```
t := binarySearchTree lv
[[[1, 2, .], 3, [4, 5, [5, 6, 7]]], 8, .]
```

Type: BinarySearchTree PositiveInteger

```
emptybst := empty()$BSTREE(INT)
[]
```

Type: BinarySearchTree Integer

```
t1 := insert!(8,emptybst)
8
```

Type: BinarySearchTree Integer

```
insert!(3,t1)
[3, 8, .]
```

Type: BinarySearchTree Integer

```
leaves t
[1, 4, 5, 7]
```

Type: List PositiveInteger

```
split(3,t)
[less = [1, 2, .], greater = [[., 3, [4, 5, [5, 6, 7]]], 8, .]]
```

Type: Record(less: BinarySearchTree PositiveInteger, greater: BinarySearchTree PositiveInteger)

```
insertRoot: (INT,BSTREE INT) -> BSTREE INT
```

Type: Void

```
insertRoot(x, t) ==
  a := split(x, t)
  node(a.less, x, a.greater)
```

Type: Void

Function **buildFromRoot**  
builds a binary search tree from  
a list of elements **ls** and the  
empty tree **emptybst**.

```
buildFromRoot ls == reduce(insertRoot,ls,emptybst)
```

Type: Void

Apply this to the reverse of the  
list **lv**.

```
rt := buildFromRoot reverse lv
```

```
Compiling function buildFromRoot with type List
  PositiveInteger -> BinarySearchTree Integer
Compiling function insertRoot with type (Integer,
  BinarySearchTree Integer) -> BinarySearchTree
  Integer
```

```
[[[1, 2, .], 3, [4, 5, [5, 6, 7]]], 8, .] (11)
```

Type: BinarySearchTree Integer

Have AXIOM check that these  
are equal.

```
(t = rt)@Boolean
```

```
true
```

(12)

Type: Boolean



## 9.6 CardinalNumber

The `CardinalNumber` domain can be used for values indicating the cardinality of sets, both finite and infinite. For example, the **dimension** operation in the category `VectorSpace` returns a cardinal number.

The non-negative integers have a natural construction as cardinals

$$0 = \#\{\ }, 1 = \{0\}, 2 = \{0, 1\}, \dots, n = \{i \mid 0 \leq i < n\}.$$

The fact that 0 acts as a zero for the multiplication of cardinals is equivalent to the axiom of choice.

Cardinal numbers can be created by conversion from non-negative integers.

```
c0 := 0 :: CardinalNumber
0
Type: CardinalNumber
```

```
c1 := 1 :: CardinalNumber
1
Type: CardinalNumber
```

```
c2 := 2 :: CardinalNumber
2
Type: CardinalNumber
```

```
c3 := 3 :: CardinalNumber
3
Type: CardinalNumber
```

They can also be obtained as the named cardinal `Aleph(n)`.

```
A0 := Aleph 0
Aleph(0)
Type: CardinalNumber
```

```
A1 := Aleph 1
Aleph(1)
Type: CardinalNumber
```

The **finite?** operation tests whether a value is a finite cardinal, that is, a non-negative integer.

```
finite? c2
true
Type: Boolean
```

```
finite? A0
false
Type: Boolean
```

Similarly, the **countable?** operation determines whether a value is a countable cardinal, that is, finite or **Aleph(0)**.

```
countable? c2
true
```

(9)

Type: Boolean

```
countable? A0
true
```

(10)

Type: Boolean

```
countable? A1
false
```

(11)

Type: Boolean

Arithmetic operations are defined on cardinal numbers as follows: If  $x = \#X$  and  $y = \#Y$  then

$x+y$	$= \#(X+Y)$	cardinality of the disjoint union
$x-y$	$= \#(X-Y)$	cardinality of the relative complement
$x*y$	$= \#(X*Y)$	cardinality of the Cartesian product
$x**y$	$= \#(X**Y)$	cardinality of the set of maps from $Y$ to $X$

Here are some arithmetic examples.

```
[c2 + c2, c2 + A1]
[4, Aleph(1)]
```

(12)

Type: List CardinalNumber

```
[c0*c2, c1*c2, c2*c2, c0*A1, c1*A1, c2*A1, A0*A1]
[0, 2, 4, 0, Aleph(1), Aleph(1), Aleph(1)]
```

(13)

Type: List CardinalNumber

```
[c2**c0, c2**c1, c2**c2, A1**c0, A1**c1, A1**c2]
[1, 2, 4, 1, Aleph(1), Aleph(1)]
```

(14)

Type: List CardinalNumber

Subtraction is a partial operation: it is not defined when subtracting a larger cardinal from a smaller one, nor when subtracting two equal infinite cardinals.

```
[c2-c1, c2-c2, c2-c3, A1-c2, A1-A0, A1-A1]
[1, 0, "failed", Aleph(1), Aleph(1), "failed"]
```

(15)

Type: List Union(CardinalNumber, "failed")

The generalized continuum hypothesis asserts that

$2^{**\text{Aleph } i} = \text{Aleph}(i+1)$

and is independent of the axioms of set theory.<sup>1</sup>

---

<sup>1</sup>Goedel, *The consistency of the continuum hypothesis*, Ann. Math. Studies, Princeton Univ. Press, 1940.

The CardinalNumber domain provides an operation to assert whether the hypothesis is to be assumed.

$$\begin{aligned} &\text{generalizedContinuumHypothesisAssumed true} \\ &\text{true} \end{aligned} \tag{16}$$

Type: Boolean

When the generalized continuum hypothesis is assumed, exponentiation to a transfinite power is allowed.

$$\begin{aligned} &[c0^{**}A0, c1^{**}A0, c2^{**}A0, A0^{**}A0, A0^{**}A1, A1^{**}A0, A1^{**}A1] \\ &[0, 1, \text{Aleph}(1), \text{Aleph}(1), \text{Aleph}(2), \text{Aleph}(1), \text{Aleph}(2)] \end{aligned} \tag{17}$$

Type: List CardinalNumber

Three commonly encountered cardinal numbers are

$$\begin{aligned} a &= \#\mathbf{Z} && \text{countable infinity} \\ c &= \#\mathbf{R} && \text{the continuum} \\ f &= \#\{g|g : [0, 1] \rightarrow \mathbf{R}\} \end{aligned}$$

In this domain, these values are obtained under the generalized continuum hypothesis in this way.

$$\begin{aligned} a &:= \text{Aleph } 0 \\ \text{Aleph } (0) \end{aligned} \tag{18}$$

Type: CardinalNumber

$$\begin{aligned} c &:= 2^{**}a \\ \text{Aleph } (1) \end{aligned} \tag{19}$$

Type: CardinalNumber

$$\begin{aligned} f &:= 2^{**}c \\ \text{Aleph } (2) \end{aligned} \tag{20}$$

Type: CardinalNumber

## 9.7 CartesianTensor

`CartesianTensor(i0,dim,R)` provides Cartesian tensors with components belonging to a commutative ring  $R$ . Tensors can be described as a generalization of vectors and matrices. This gives a concise *tensor algebra* for multilinear objects supported by the `CartesianTensor` domain. You can form the inner or outer product of any two tensors and you can add or subtract tensors with the same number of components. Additionally, various forms of traces and transpositions are useful.

The `CartesianTensor` constructor allows you to specify the minimum index for subscripting. In what follows we discuss in detail how to manipulate tensors.

Here we construct the domain of Cartesian tensors of dimension 2 over the integers, with indices starting at 1.

```
CT := CARTEN(i0 := 1, 2, Integer)
CartesianTensor (1, 2, Integer )
```

(1)  
Type: Domain

### Forming tensors

Scalars can be converted to tensors of rank zero.

```
t0: CT := 8
8
```

(2)  
Type: CartesianTensor(1, 2, Integer)

```
rank t0
0
```

(3)  
Type: NonNegativeInteger

Vectors (mathematical direct products, rather than one dimensional array structures) can be converted to tensors of rank one.

```
v: DirectProduct(2, Integer) := directProduct [3,4]
[3, 4]
```

(4)  
Type: DirectProduct(2, Integer)

```
Tv: CT := v
[3, 4]
```

(5)  
Type: CartesianTensor(1, 2, Integer)

Matrices can be converted to tensors of rank two.

```
m: SquareMatrix(2, Integer) := matrix [[1,2],[4,5]]
[ 1  2 ]
[ 4  5 ]
```

(6)  
Type: SquareMatrix(2, Integer)

```
Tm: CT := m
[ 1  2 ]
[ 4  5 ]
```

(7)  
Type: CartesianTensor(1, 2, Integer)

```
n: SquareMatrix(2, Integer) := matrix [[2,3],[0,1]]
```

$$\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} \quad (8)$$

Type: SquareMatrix(2, Integer)

```
Tn: CT := n
```

$$\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} \quad (9)$$

Type: CartesianTensor(1, 2, Integer)

In general, a tensor of rank  $k$  can be formed by making a list of rank  $k-1$  tensors or, alternatively, a  $k$ -deep nested list of lists.

```
t1: CT := [2, 3]
```

$$[2, 3] \quad (10)$$

Type: CartesianTensor(1, 2, Integer)

```
rank t1
```

$$1 \quad (11)$$

Type: PositiveInteger

```
t2: CT := [t1, t1]
```

$$\begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix} \quad (12)$$

Type: CartesianTensor(1, 2, Integer)

```
t3: CT := [t2, t2]
```

$$\left[ \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix} \right] \quad (13)$$

Type: CartesianTensor(1, 2, Integer)

```
tt: CT := [t3, t3]; tt := [tt, tt]
```

$$\left[ \left[ \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix} \right], \left[ \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix} \right] \right] \quad (14)$$

Type: CartesianTensor(1, 2, Integer)

```
rank tt
```

$$5 \quad (15)$$

Type: PositiveInteger

## Multiplication

Given two tensors of rank  $k_1$  and  $k_2$ , the outer **product** forms a new tensor of rank  $k_1+k_2$ .

Here  
 $T_{mn}(i, j, k, l) = T_m(i, j) T_n(k, l).$

$$\mathbf{Tmn} := \text{product}(\mathbf{Tm}, \mathbf{Tn})$$

$$\begin{bmatrix} \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 4 & 6 \\ 0 & 2 \end{bmatrix} \\ \begin{bmatrix} 8 & 12 \\ 0 & 4 \end{bmatrix} & \begin{bmatrix} 10 & 15 \\ 0 & 5 \end{bmatrix} \end{bmatrix}$$
(16)

Type: CartesianTensor(1, 2, Integer)

The inner product (**contract**) forms a tensor of rank  $k_1+k_2-2$ . This product generalizes the vector dot product and matrix-vector product by summing component products along two indices.

Here we sum along the second index of  $T_m$  and the first index of  $T_v$ . Here  
 $T_{mv} = \sum_{j=1}^{\dim} T_m(i, j) T_v(j)$

$$\mathbf{Tmv} := \text{contract}(\mathbf{Tm}, 2, \mathbf{Tv}, 1)$$

$$[11, 32]$$
(17)

Type: CartesianTensor(1, 2, Integer)

The multiplication operator “ $*$ ” is scalar multiplication or an inner product depending on the ranks of the arguments.

If either argument is rank zero it is treated as scalar multiplication. Otherwise,  $\mathbf{a}*\mathbf{b}$  is the inner product summing the last index of  $\mathbf{a}$  with the first index of  $\mathbf{b}$ .

$$\mathbf{Tm}*\mathbf{Tv}$$

$$[11, 32]$$
(18)

Type: CartesianTensor(1, 2, Integer)

This definition is consistent with the inner product on matrices and vectors.

$$\mathbf{Tmv} = \mathbf{m} * \mathbf{v}$$

$$[11, 32] = [11, 32]$$
(19)

Type: Equation CartesianTensor(1, 2, Integer)

## Selecting Components

For tensors of low rank (that is, four or less), components can be selected by applying the tensor to its indices.

$$\mathbf{t0}()$$

$$8$$
(20)

Type: PositiveInteger

$$\mathbf{t1}(1+1)$$

$$3$$
(21)

Type: PositiveInteger

$$\mathbf{t2}(2, 1)$$

$$2$$
(22)

Type: PositiveInteger

$$\mathbf{t3}(2, 1, 2)$$

$$3$$
(23)

Type: PositiveInteger

$$\begin{aligned} & \text{Tmn}(2, 1, 2, 1) \\ & 0 \end{aligned} \tag{24}$$

Type: NonNegativeInteger

A general indexing mechanism is provided for a list of indices.

$$\begin{aligned} & \text{t0}[] \\ & 8 \end{aligned} \tag{25}$$

Type: PositiveInteger

$$\begin{aligned} & \text{t1}[2] \\ & 3 \end{aligned} \tag{26}$$

Type: PositiveInteger

$$\begin{aligned} & \text{t2}[2, 1] \\ & 2 \end{aligned} \tag{27}$$

Type: PositiveInteger

The general mechanism works for tensors of arbitrary rank, but is somewhat less efficient since the intermediate index list must be created.

$$\begin{aligned} & \text{t3}[2, 1, 2] \\ & 3 \end{aligned} \tag{28}$$

Type: PositiveInteger

$$\begin{aligned} & \text{Tmn}[2, 1, 2, 1] \\ & 0 \end{aligned} \tag{29}$$

Type: NonNegativeInteger

## Contraction

A “contraction” between two tensors is an inner product, as we have seen above. You can also contract a pair of indices of a single tensor. This corresponds to a “trace” in linear algebra. The expression `contract(t,k1,k2)` forms a new tensor by summing the diagonal given by indices in position `k1` and `k2`.

This is the tensor given by  $xT_{mn} = \sum_{k=1}^{\dim} T_{mn}(k, k, i, j)$ .

$$\begin{aligned} & \text{cTmn} := \text{contract}(\text{Tmn}, 1, 2) \\ & \begin{bmatrix} 12 & 18 \\ 0 & 6 \end{bmatrix} \end{aligned} \tag{30}$$

Type: CartesianTensor(1, 2, Integer)

Since `Tmn` is the outer product of matrix `m` and matrix `n`, the above is equivalent to this.

$$\begin{aligned} & \text{trace}(\text{m}) * \text{n} \\ & \begin{bmatrix} 12 & 18 \\ 0 & 6 \end{bmatrix} \end{aligned} \tag{31}$$

Type: SquareMatrix(2, Integer)

In this and the next few examples, we show all possible contractions of **Tmn** and their matrix algebra equivalents.

$$\text{contract}(\text{Tmn}, 1, 2) = \text{trace}(\mathbf{m}) * \mathbf{n}$$

$$\begin{bmatrix} 12 & 18 \\ 0 & 6 \end{bmatrix} = \begin{bmatrix} 12 & 18 \\ 0 & 6 \end{bmatrix} \quad (32)$$

Type: Equation CartesianTensor(1, 2, Integer)

$$\text{contract}(\text{Tmn}, 1, 3) = \text{transpose}(\mathbf{m}) * \mathbf{n}$$

$$\begin{bmatrix} 2 & 7 \\ 4 & 11 \end{bmatrix} = \begin{bmatrix} 2 & 7 \\ 4 & 11 \end{bmatrix} \quad (33)$$

Type: Equation CartesianTensor(1, 2, Integer)

$$\text{contract}(\text{Tmn}, 1, 4) = \text{transpose}(\mathbf{m}) * \text{transpose}(\mathbf{n})$$

$$\begin{bmatrix} 14 & 4 \\ 19 & 5 \end{bmatrix} = \begin{bmatrix} 14 & 4 \\ 19 & 5 \end{bmatrix} \quad (34)$$

Type: Equation CartesianTensor(1, 2, Integer)

$$\text{contract}(\text{Tmn}, 2, 3) = \mathbf{m} * \mathbf{n}$$

$$\begin{bmatrix} 2 & 5 \\ 8 & 17 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 8 & 17 \end{bmatrix} \quad (35)$$

Type: Equation CartesianTensor(1, 2, Integer)

$$\text{contract}(\text{Tmn}, 2, 4) = \mathbf{m} * \text{transpose}(\mathbf{n})$$

$$\begin{bmatrix} 8 & 2 \\ 23 & 5 \end{bmatrix} = \begin{bmatrix} 8 & 2 \\ 23 & 5 \end{bmatrix} \quad (36)$$

Type: Equation CartesianTensor(1, 2, Integer)

$$\text{contract}(\text{Tmn}, 3, 4) = \text{trace}(\mathbf{n}) * \mathbf{m}$$

$$\begin{bmatrix} 3 & 6 \\ 12 & 15 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 12 & 15 \end{bmatrix} \quad (37)$$

Type: Equation CartesianTensor(1, 2, Integer)

## Transpositions

You can exchange any desired pair of indices using the **transpose** operation.

Here the indices in positions one and three are exchanged, that is,  $tT_{mn}(i, j, k, l) = T_{mn}(k, j, i, l)$ .

$$t\text{Tmn} := \text{transpose}(\text{Tmn}, 1, 3)$$

$$\begin{bmatrix} \begin{bmatrix} 2 & 3 \\ 8 & 12 \end{bmatrix} & \begin{bmatrix} 4 & 6 \\ 10 & 15 \end{bmatrix} \\ \begin{bmatrix} 0 & 1 \\ 0 & 4 \end{bmatrix} & \begin{bmatrix} 0 & 2 \\ 0 & 5 \end{bmatrix} \end{bmatrix} \quad (38)$$

Type: CartesianTensor(1, 2, Integer)



If no indices are specified, the first and last index are exchanged.

$$\text{transpose Tmn} \quad \left[ \begin{array}{cc} \begin{bmatrix} 2 & 8 \\ 0 & 0 \\ 3 & 12 \\ 1 & 4 \end{bmatrix} & \begin{bmatrix} 4 & 10 \\ 0 & 0 \\ 6 & 15 \\ 2 & 5 \end{bmatrix} \end{array} \right] \quad (39)$$

Type: CartesianTensor(1, 2, Integer)

This is consistent with the matrix transpose.

$$\text{transpose Tm} = \text{transpose m} \quad \left[ \begin{array}{cc} \begin{bmatrix} 1 & 4 \\ 2 & 5 \end{bmatrix} & = \begin{bmatrix} 1 & 4 \\ 2 & 5 \end{bmatrix} \end{array} \right] \quad (40)$$

Type: Equation CartesianTensor(1, 2, Integer)

If a more complicated reordering of the indices is required, then the **reindex** operation can be used. This operation allows the indices to be arbitrarily permuted.

This defines  $rT_{mn}(i, j, k, l) = T_{mn}(i, l, j, k)$ .

$$\text{rTmn} := \text{reindex}(\text{Tmn}, [1, 4, 2, 3]) \quad \left[ \begin{array}{cc} \begin{bmatrix} 2 & 0 \\ 4 & 0 \\ 8 & 0 \\ 10 & 0 \end{bmatrix} & \begin{bmatrix} 3 & 1 \\ 6 & 2 \\ 12 & 4 \\ 15 & 5 \end{bmatrix} \end{array} \right] \quad (41)$$

Type: CartesianTensor(1, 2, Integer)

## Arithmetic

Tensors of equal rank can be added or subtracted so arithmetic expressions can be used to produce new tensors.

$$\text{tt} := \text{transpose}(\text{Tm}) * \text{Tn} - \text{Tn} * \text{transpose}(\text{Tm}) \quad \left[ \begin{array}{cc} \begin{bmatrix} -6 & -16 \\ 2 & 6 \end{bmatrix} \end{array} \right] \quad (42)$$

Type: CartesianTensor(1, 2, Integer)

$$\text{Tv} * (\text{tt} + \text{Tn}) \quad [-4, -11] \quad (43)$$

Type: CartesianTensor(1, 2, Integer)

$$\text{reindex}(\text{product}(\text{Tn}, \text{Tn}), [4, 3, 2, 1]) + 3 * \text{Tn} * \text{product}(\text{Tm}, \text{Tm}) \quad \left[ \begin{array}{cc} \begin{bmatrix} 46 & 84 \\ 174 & 212 \\ 18 & 24 \\ 57 & 63 \end{bmatrix} & \begin{bmatrix} 57 & 114 \\ 228 & 285 \\ 17 & 30 \\ 63 & 76 \end{bmatrix} \end{array} \right] \quad (44)$$

Type: CartesianTensor(1, 2, Integer)

## Specific Tensors

Two specific tensors have properties which depend only on the dimension.

The Kronecker delta satisfies

$$\delta(i, j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

`delta: CT := kroneckerDelta()`

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(45)

Type: CartesianTensor(1, 2, Integer)

This can be used to reindex via contraction.

`contract(Tmn, 2, delta, 1) = reindex(Tmn, [1,3,4,2])`

$$\begin{bmatrix} \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix} \\ \begin{bmatrix} 8 & 10 \\ 12 & 15 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 4 & 5 \end{bmatrix} \end{bmatrix} =$$

(46)

$$\begin{bmatrix} \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix} \\ \begin{bmatrix} 8 & 10 \\ 12 & 15 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 4 & 5 \end{bmatrix} \end{bmatrix}$$

Type: Equation CartesianTensor(1, 2, Integer)

The Levi Civita symbol determines the sign of a permutation of indices.

`epsilon:CT := leviCivitaSymbol()`

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

(47)

Type: CartesianTensor(1, 2, Integer)

Here we have:

$$\epsilon(i_1, \dots, i_{\dim}) = \begin{cases} +1 & \text{if } i_1, \dots, i_{\dim} \text{ is an even permutation of } i_0, \dots, i_0 + \dim - 1 \\ -1 & \text{if } i_1, \dots, i_{\dim} \text{ is an odd permutation of } i_0, \dots, i_0 + \dim - 1 \\ 0 & \text{if } i_1, \dots, i_{\dim} \text{ is not a permutation of } i_0, \dots, i_0 + \dim - 1 \end{cases}$$

This property can be used to form determinants.

`contract(epsilon*Tm*epsilon, 1,2) = 2 * determinant m`

$$-6 = -6$$

(48)

Type: Equation CartesianTensor(1, 2, Integer)

Properties of the CartesianTensor domain

`GradedModule(R,E)` denotes “E-graded R-module”, that is, a collection of R-modules indexed by an abelian monoid **E**. An element **g** of **G[s]** for some specific **s** in **E** is said to be an element of **G** with **degree s**. Sums are defined in each module **G[s]** so two elements of **G** can be added if they have the same degree. Morphisms can be defined and composed by degree to give the mathematical category of graded modules.

`GradedAlgebra(R,E)` denotes “E-graded R-algebra.” A graded algebra is a graded module together with a degree preserving R-bilinear map, called the **product**.

$$\text{degree}(\text{product}(a,b)) = \text{degree}(a) + \text{degree}(b)$$

$$\begin{aligned} \text{product}(r*a,b) &= \text{product}(a,r*b) = r*\text{product}(a,b) \\ \text{product}(a_1+a_2,b) &= \text{product}(a_1,b) + \text{product}(a_2,b) \\ \text{product}(a,b_1+b_2) &= \text{product}(a,b_1) + \text{product}(a,b_2) \\ \text{product}(a,\text{product}(b,c)) &= \text{product}(\text{product}(a,b),c) \end{aligned}$$

The domain `CartesianTensor(i0, dim, R)` belongs to the category `GradedAlgebra(R, NonNegativeInteger)`. The non-negative integer **degree** is the tensor rank and the graded algebra **product** is the tensor outer product. The graded module addition captures the notion that only tensors of equal rank can be added.

If  $V$  is a vector space of dimension `dim` over  $R$ , then the tensor module  $T[k](V)$  is defined as

$$\begin{aligned} T[0](V) &= R \\ T[k](V) &= T[k-1](V) * V \end{aligned}$$

where “ $*$ ” denotes the  $R$ -module tensor **product**. `CartesianTensor(i0,dim,R)` is the graded algebra in which the degree  $k$  module is  $T[k](V)$ .

## Tensor Calculus

It should be noted here that often tensors are used in the context of tensor-valued manifold maps. This leads to the notion of covariant and contravariant bases with tensor component functions transforming in specific ways under a change of coordinates on the manifold. This is no more directly supported by the `CartesianTensor` domain than it is by the `Vector` domain. However, it is possible to have the components implicitly represent component maps by choosing a polynomial or expression type for the components. In this case, it is up to the user to satisfy any constraints which arise on the basis of this interpretation.

## 9.8 Character

The members of the domain `Character` are values representing letters, numerals and other text elements. For more information on related topics, see ‘`CharacterClass`’ on page 376 and ‘`String`’ on page 577.

Characters can be obtained using `String` notation.

```
chars := [char "a", char "A", char "X", char "8", char
          "+"]
[a, A, X, 8, +]
(1)
Type: List Character
```

Certain characters are available by name. This is the blank character.

```
space()
(2)
Type: Character
```

This is the quote that is used in strings.

```
quote()
"
(3)
Type: Character
```

This is the escape character that allows quotes and other characters within strings.

```
escape()
-
(4)
Type: Character
```

Characters are represented as integers in a machine-dependent way. The integer value can be obtained using the **ord** operation. It is always true that `char(ord c) = c` and `ord(char i) = i`, provided that `i` is in the range `0..size()$Character-1`.

```
[ord c for c in chars]
[97, 65, 88, 56, 43]
(5)
Type: List Integer
```

The **lowerCase** operation converts an upper case letter to the corresponding lower case letter. If the argument is not an upper case letter, then it is returned unchanged.

```
[upperCase c for c in chars]
[A, A, X, 8, +]
(6)
Type: List Character
```

Likewise, the **upperCase** operation converts lower case letters to upper case.

```
[lowerCase c for c in chars]
[a, a, x, 8, +]
(7)
Type: List Character
```

A number of tests are available to determine whether characters belong to certain families.

```
[alphabetic? c for c in chars]
[true, true, true, false, false]
(8)
Type: List Boolean
```

```
[upperCase? c for c in chars]
[false, true, true, false, false]
(9)
Type: List Boolean
```

```
[lowerCase? c for c in chars]
```

```
[true, false, false, false, false]
```

(10)

Type: List Boolean

```
[digit? c for c in chars]
```

```
[false, false, false, true, false]
```

(11)

Type: List Boolean

```
[hexDigit? c for c in chars]
```

```
[true, true, false, true, false]
```

(12)

Type: List Boolean

```
[alphanumeric? c for c in chars]
```

```
[true, true, true, true, false]
```

(13)

Type: List Boolean

## 9.9 CharacterClass

Character classes can be created by giving either a string or a list of characters.

The CharacterClass domain allows classes of characters to be defined and manipulated efficiently.

```
c11 := charClass [char "a", char "e", char "i", char "o",  
  char "u", char "y"]  
"aeiouy" (1)
```

Type: CharacterClass

```
c12 := charClass "bcdfghjklmnpqrstvwxyz"  
"bcdfghjklmnpqrstvwxyz" (2)
```

Type: CharacterClass

A number of character classes are predefined for convenience.

```
digit()  
"0123456789" (3)
```

Type: CharacterClass

```
hexDigit()  
"0123456789ABCDEFabcdef" (4)
```

Type: CharacterClass

```
upperCase()  
"ABCDEFGHIJKLMNOPQRSTUVWXYZ" (5)
```

Type: CharacterClass

```
lowerCase()  
"abcdefghijklmnopqrstuvwxyz" (6)
```

Type: CharacterClass

```
alphabetic()  
"ABCDEFGHIJKLMNOPQRSTUVWXYZabcdefghijklmnopqrstuvwxyz" (7)
```

Type: CharacterClass

```
alphanumeric()  
"0123456789  
ABCDEFGHIJKLMNOPQRSTUVWXYZabcdefghijklmnopqrstuvwxyz" (8)
```

Type: CharacterClass

You can quickly test whether a character belongs to a class.

```
member?(char "a", c11)  
true (9)
```

Type: Boolean

```
member?(char "a", c12)  
false (10)
```

Type: Boolean

Classes have the usual set operations because the `CharacterClass` domain belongs to the category `FiniteSetAggregate(Character)`.

```
intersect(c11, c12)
"y"
(11)
Type: CharacterClass
```

```
union(c11,c12)
"abcdefghijklmnopqrstuvwxyz"
(12)
Type: CharacterClass
```

```
difference(c11,c12)
"aeiou"
(13)
Type: CharacterClass
```

```
intersect(complement(c11),c12)
"bcdfghjklmnpqrstvwxyz"
(14)
Type: CharacterClass
```

You can modify character classes by adding or removing characters.

```
insert!(char "a", c12)
"abcdefghijklmnopqrstvwxyz"
(15)
Type: CharacterClass
```

```
remove!(char "b", c12)
"acdfghjklmnpqrstvwxyz"
(16)
Type: CharacterClass
```

For more information on related topics, see ‘`Character`’ on page 374 and ‘`String`’ on page 577. Issue the system command `)show CharacterClass` to display the full list of operations defined by `CharacterClass`.

## 9.10 CliffordAlgebra

CliffordAlgebra(n,K,Q) defines a vector space of dimension  $2^n$  over the field  $K$  with a given quadratic form  $Q$ . If  $\{e_1, \dots, e_n\}$  is a basis for  $K^n$  then

$$\left\{ \begin{array}{ll} 1 & \\ e_i & \text{for } 1 \leq i \leq n \\ e_{i_1} e_{i_2} & \text{for } 1 \leq i_1 < i_2 \leq n \\ \dots & \\ e_1 e_2 \dots e_n & \end{array} \right\}$$

is a basis for the Clifford algebra. The algebra is defined by the relations

$$\begin{aligned} e_i e_i &= Q(e_i) \\ e_i e_j &= -e_j e_i \text{ for } i \neq j \end{aligned}$$

Examples of Clifford Algebras are gaussians (complex numbers), quaternions, exterior algebras and spin algebras.

### 9.10.1 The Complex Numbers as a Clifford Algebra

This is the field over which we will work, rational functions with integer coefficients.

```
K := Fraction Polynomial Integer
```

```
Fraction Polynomial Integer
```

Type: Domain

We use this matrix for the quadratic form.

```
m := matrix [[-1]]
[ -1 ]
```

(2)

Type: Matrix Integer

We get complex arithmetic by using this domain.

```
C := CliffordAlgebra(1, K, quadraticForm m)
```

```
CliffordAlgebra (1, Fraction Polynomial Integer , MATRIX )
```

Type: Domain

Here is  $i$ , the usual square root of  $-1$ .

```
i: C := e(1)
```

```
e_1
```

(4)

Type: CliffordAlgebra(1, Fraction Polynomial Integer, MATRIX)

Here are some examples of the arithmetic.

```
x := a + b * i
```

```
a + b e_1
```

(5)

Type: CliffordAlgebra(1, Fraction Polynomial Integer, MATRIX)

```
y := c + d * i
```

```
c + d e_1
```

(6)

Type: CliffordAlgebra(1, Fraction Polynomial Integer, MATRIX)



See ‘Complex’ on page 383 for examples of AXIOM’s constructor implementing complex numbers.

$$\begin{aligned} & \mathbf{x} * \mathbf{y} \\ & -b\,d + a\,c + (a\,d + b\,c)\,e_1 \end{aligned} \tag{7}$$

Type: CliffordAlgebra(1, Fraction Polynomial Integer, MATRIX)

### 9.10.2 The Quaternion Numbers as a Clifford Algebra

This is the field over which we will work, rational functions with integer coefficients.

$$\begin{aligned} & \mathbf{K} := \text{Fraction Polynomial Integer} \\ & \text{Fraction Polynomial Integer} \end{aligned} \tag{1}$$

Type: Domain

We use this matrix for the quadratic form.

$$\begin{aligned} & \mathbf{m} := \text{matrix} \, [[-1,0],[0,-1]] \\ & \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \end{aligned} \tag{2}$$

Type: Matrix Integer

The resulting domain is the quaternions.

$$\begin{aligned} & \mathbf{H} := \text{CliffordAlgebra}(2, \mathbf{K}, \text{quadraticForm } \mathbf{m}) \\ & \text{CliffordAlgebra} \, (2, \text{Fraction Polynomial Integer}, \text{MATRIX}) \end{aligned} \tag{3}$$

Type: Domain

We use Hamilton’s notation for i,j,k.

$$\begin{aligned} & \mathbf{i}: \mathbf{H} := \mathbf{e}(1) \\ & e_1 \end{aligned} \tag{4}$$

Type: CliffordAlgebra(2, Fraction Polynomial Integer, MATRIX)

$$\begin{aligned} & \mathbf{j}: \mathbf{H} := \mathbf{e}(2) \\ & e_2 \end{aligned} \tag{5}$$

Type: CliffordAlgebra(2, Fraction Polynomial Integer, MATRIX)

$$\begin{aligned} & \mathbf{k}: \mathbf{H} := \mathbf{i} * \mathbf{j} \\ & e_1\,e_2 \end{aligned} \tag{6}$$

Type: CliffordAlgebra(2, Fraction Polynomial Integer, MATRIX)

$$\begin{aligned} & \mathbf{x} := \mathbf{a} + \mathbf{b} * \mathbf{i} + \mathbf{c} * \mathbf{j} + \mathbf{d} * \mathbf{k} \\ & a + b\,e_1 + c\,e_2 + d\,e_1\,e_2 \end{aligned} \tag{7}$$

Type: CliffordAlgebra(2, Fraction Polynomial Integer, MATRIX)

$$\begin{aligned} & \mathbf{y} := \mathbf{e} + \mathbf{f} * \mathbf{i} + \mathbf{g} * \mathbf{j} + \mathbf{h} * \mathbf{k} \\ & e + f\,e_1 + g\,e_2 + h\,e_1\,e_2 \end{aligned} \tag{8}$$

Type: CliffordAlgebra(2, Fraction Polynomial Integer, MATRIX)

$$\mathbf{x} + \mathbf{y} \\ e + a + (f + b) e_1 + (g + c) e_2 + (h + d) e_1 e_2 \quad (9)$$

Type: CliffordAlgebra(2, Fraction Polynomial Integer, MATRIX)

$$\mathbf{x} * \mathbf{y} \\ -d h - c g - b f + a e + (c h - d g + a f + b e) e_1 + \\ (-b h + a g + d f + c e) e_2 + (a h + b g - c f + d e) e_1 e_2 \quad (10)$$

Type: CliffordAlgebra(2, Fraction Polynomial Integer, MATRIX)

See ‘Quaternion’ on page 535 for examples of AXIOM’s constructor implementing quaternions.

$$\mathbf{y} * \mathbf{x} \\ -d h - c g - b f + a e + (-c h + d g + a f + b e) e_1 + \\ (b h + a g - d f + c e) e_2 + (a h - b g + c f + d e) e_1 e_2 \quad (11)$$

Type: CliffordAlgebra(2, Fraction Polynomial Integer, MATRIX)

### 9.10.3 The Exterior Algebra on a Three Space

This is the field over which we will work, rational functions with integer coefficients.

`K := Fraction Polynomial Integer`

Fraction Polynomial Integer (1)

Type: Domain

If we chose the three by three zero quadratic form, we obtain the exterior algebra on  $e(1), e(2), e(3)$ .

`Ext := CliffordAlgebra(3, K, quadraticForm 0)`

CliffordAlgebra (3, Fraction Polynomial Integer , MATRIX ) (2)

Type: Domain

This is a three dimensional vector algebra. We define  $i, j, k$  as the unit vectors.

`i: Ext := e(1)`

$e_1$  (3)

Type: CliffordAlgebra(3, Fraction Polynomial Integer, MATRIX)

`j: Ext := e(2)`

$e_2$  (4)

Type: CliffordAlgebra(3, Fraction Polynomial Integer, MATRIX)

`k: Ext := e(3)`

$e_3$  (5)

Type: CliffordAlgebra(3, Fraction Polynomial Integer, MATRIX)

Now it is possible to do arithmetic.

`x := x1*i + x2*j + x3*k`

$x1 e_1 + x2 e_2 + x3 e_3$  (6)

Type: CliffordAlgebra(3, Fraction Polynomial Integer, MATRIX)

$$y := y1*i + y2*j + y3*k$$

$$y1 e_1 + y2 e_2 + y3 e_3 \quad (7)$$

Type: CliffordAlgebra(3, Fraction Polynomial Integer, MATRIX)

$$x + y$$

$$(y1 + x1) e_1 + (y2 + x2) e_2 + (y3 + x3) e_3 \quad (8)$$

Type: CliffordAlgebra(3, Fraction Polynomial Integer, MATRIX)

$$x * y + y * x$$

$$0 \quad (9)$$

Type: CliffordAlgebra(3, Fraction Polynomial Integer, MATRIX)

On an  $n$  space, a grade  $p$  form has a dual  $n-p$  form. In particular, in three space the dual of a grade two element identifies  $e1*e2 \rightarrow e3$ ,  $e2*e3 \rightarrow e1$ ,  $e3*e1 \rightarrow e2$ .

$$\text{dual2 } a == \text{coefficient}(a, [2,3]) * i + \text{coefficient}(a, [3,1]) * j + \text{coefficient}(a, [1,2]) * k$$

Type: Void

The vector cross product is then given by this.

$$\text{dual2}(x*y)$$

Compiling function dual2 with type CliffordAlgebra(3, Fraction Polynomial Integer, MATRIX) -> CliffordAlgebra(3, Fraction Polynomial Integer, MATRIX)

$$(x2 y3 - x3 y2) e_1 + (-x1 y3 + x3 y1) e_2 + (x1 y2 - x2 y1) e_3 \quad (11)$$

Type: CliffordAlgebra(3, Fraction Polynomial Integer, MATRIX)

## 9.10.4 The Dirac Spin Algebra

In this section we will work over the field of rational numbers.

$$K := \text{Fraction Integer}$$

$$\text{Fraction Integer} \quad (1)$$

Type: Domain

We define the quadratic form to be the Minkowski space-time metric.

$$g := \text{matrix} \begin{bmatrix} 1, 0, 0, 0 \\ 0, -1, 0, 0 \\ 0, 0, -1, 0 \\ 0, 0, 0, -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad (2)$$

Type: Matrix Integer

We obtain the Dirac spin algebra used in Relativistic Quantum Field Theory.

```
D := CliffordAlgebra(4,K, quadraticForm g)
CliffordAlgebra (4, Fraction Integer , MATRIX )
```

(3)

Type: Domain

The usual notation for the basis is  $\gamma$  with a superscript. For AXIOM input we will use `gam(i)`:

```
gam := [e(i)$D for i in 1..4]
[e1, e2, e3, e4]
```

(4)

Type: List CliffordAlgebra(4, Fraction Integer, MATRIX)

There are various contraction identities of the form

$$g(1,t) \cdot \text{gam}(1) \cdot \text{gam}(m) \cdot \text{gam}(n) \cdot \text{gam}(r) \cdot \text{gam}(s) \cdot \text{gam}(t) = 2 \cdot (\text{gam}(s) \text{gam}(m) \text{gam}(n) \text{gam}(r) + \text{gam}(r) \text{gam}(n) \text{gam}(m) \text{gam}(s))$$

where a sum over 1 and t is implied.

Verify this identity for particular values of `m,n,r,s`.

```
m := 1; n:= 2; r := 3; s := 4;
```

(5)

Type: PositiveInteger

```
lhs := reduce(+, [reduce(+, [
  g(1,t)*gam(1)*gam(m)*gam(n)*gam(r)*gam(s)*gam(t) for t
  in 1..4]) for t in 1..4])
-4 e1 e2 e3 e4
```

(6)

Type: CliffordAlgebra(4, Fraction Integer, MATRIX)

```
rhs := 2*(gam s * gam m*gam n*gam r + gam r*gam n*gam
  m*gam s)
-4 e1 e2 e3 e4
```

(7)

Type: CliffordAlgebra(4, Fraction Integer, MATRIX)

## 9.11 Complex

The Complex constructor implements complex objects over a commutative ring  $R$ . Typically, the ring  $R$  is Integer, Fraction Integer, Float or DoubleFloat.  $R$  can also be a symbolic type, like Polynomial Integer. For more information about the numerical and graphical aspects of complex numbers, see Section 8.1 on page 264.

Complex objects are created by the **complex** operation.

```
a := complex(4/3, 5/2)
```

$$\frac{4}{3} + \frac{5}{2}i \quad (1)$$

Type: Complex Fraction Integer

```
b := complex(4/3, -5/2)
```

$$\frac{4}{3} - \frac{5}{2}i \quad (2)$$

Type: Complex Fraction Integer

The standard arithmetic operations are available.

```
a + b
```

$$\frac{8}{3} \quad (3)$$

Type: Complex Fraction Integer

```
a - b
```

$$5i \quad (4)$$

Type: Complex Fraction Integer

```
a * b
```

$$\frac{289}{36} \quad (5)$$

Type: Complex Fraction Integer

If  $R$  is a field, you can also divide the complex objects.

```
a / b
```

$$-\frac{161}{289} + \frac{240}{289}i \quad (6)$$

Type: Complex Fraction Integer

Use a conversion (Section 2.7 on page 113) to view the last object as a fraction of complex integers.

```
% :: Fraction Complex Integer
```

$$\frac{-15 + 8i}{15 + 8i} \quad (7)$$

Type: Fraction Complex Integer

The predefined macro **%i** is defined to be **complex(0,1)**.

```
3.4 + 6.7 * %i
```

$$3.4 + 6.7i \quad (8)$$

Type: Complex Float

You can also compute the **conjugate** and **norm** of a complex number.

`conjugate a`

$$\frac{4}{3} - \frac{5}{2} i$$

(9)

Type: Complex Fraction Integer

`norm a`

$$\frac{289}{36}$$

(10)

Type: Fraction Integer

The **real** and **imag** operations are provided to extract the real and imaginary parts, respectively.

`real a`

$$\frac{4}{3}$$

(11)

Type: Fraction Integer

`imag a`

$$\frac{5}{2}$$

(12)

Type: Fraction Integer

The domain Complex Integer is also called the Gaussian integers. If **R** is the integers (or, more generally, a EuclideanDomain), you can compute greatest common divisors.

`gcd(13 - 13*i, 31 + 27*i)`

$$5 + i$$

(13)

Type: Complex Integer

You can also compute least common multiples.

`lcm(13 - 13*i, 31 + 27*i)`

$$143 - 39 i$$

(14)

Type: Complex Integer

You can **factor** Gaussian integers.

`factor(13 - 13*i)`

$$-(1 + i) (2 + 3 i) (3 + 2 i)$$

(15)

Type: Factored Complex Integer

`factor complex(2,0)`

$$-i (1 + i)^2$$

(16)

Type: Factored Complex Integer

## 9.12 Continued-Fraction

The **continuedFraction** operation converts its fractional argument to a continued fraction.

Continued fractions have been a fascinating and useful tool in mathematics for well over three hundred years. AXIOM implements continued fractions for fractions of any Euclidean domain. In practice, this usually means rational numbers. In this section we demonstrate some of the operations available for manipulating both finite and infinite continued fractions. It may be helpful if you review ‘Stream’ on page 575 to remind yourself of some of the operations with streams.

The ContinuedFraction domain is a field and therefore you can add, subtract, multiply and divide the fractions.

```
c := continuedFraction(314159/100000)
```

$$3 + \frac{1}{7} + \frac{1}{15} + \frac{1}{1} + \frac{1}{25} + \frac{1}{1} + \frac{1}{7} + \frac{1}{4} \quad (1)$$

Type: ContinuedFraction Integer

This display is a compact form of the bulkier

$$3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{1 + \frac{1}{25 + \frac{1}{1 + \frac{1}{7 + \frac{1}{4}}}}}}}$$

You can write any rational number in a similar form. The fraction will be finite and you can always take the “numerators” to be 1. That is, any rational number can be written as a simple, finite continued fraction of the form

$$a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{\ddots a_{n-1} + \frac{1}{a_n}}}}$$

The  $a_i$  are called partial quotients and the operation **partialQuotients** creates a stream of them.

```
partialQuotients c
```

$$[3, 7, 15, 1, 25, 1, 7, \dots] \quad (2)$$

Type: Stream Integer

By considering more and more of the fraction, you get the **convergents**. For example, the first convergent is  $a_1$ , the second is  $a_1 + 1/a_2$  and so on.

```
convergents c
```

$$\left[ 3, \frac{22}{7}, \frac{333}{106}, \frac{355}{113}, \frac{9208}{2931}, \frac{9563}{3044}, \frac{76149}{24239}, \dots \right] \quad (3)$$

Type: Stream Fraction Integer

Since this is a finite continued fraction, the last convergent is the original rational number, in reduced form. The result of **approximants** is always an infinite stream, though it may just repeat the “last” value.

$$\text{approximants } c \quad \left[ 3, \frac{22}{7}, \frac{333}{106}, \frac{355}{113}, \frac{9208}{2931}, \frac{9563}{3044}, \frac{76149}{24239}, \dots \right] \quad (4)$$

Type: Stream Fraction Integer

Inverting  $c$  only changes the partial quotients of its fraction by inserting a 0 at the beginning of the list.

$$pq := \text{partialQuotients}(1/c) \quad [0, 3, 7, 15, 1, 25, 1, \dots] \quad (5)$$

Type: Stream Integer

Do this to recover the original continued fraction from this list of partial quotients. The three-argument form of the **continuedFraction** operation takes an element which is the whole part of the fraction, a stream of elements which are the numerators of the fraction, and a stream of elements which are the denominators of the fraction.

$$\text{continuedFraction}(\text{first } pq, \text{repeating } [1], \text{rest } pq) \quad \frac{1}{3} + \frac{1}{7} + \frac{1}{15} + \frac{1}{1} + \frac{1}{25} + \frac{1}{1} + \frac{1}{7} + \dots \quad (6)$$

Type: ContinuedFraction Integer

The streams need not be finite for **continuedFraction**. Can you guess which irrational number has the following continued fraction? See the end of this section for the answer.

$$z := \text{continuedFraction}(3, \text{repeating } [1], \text{repeating } [3, 6]) \quad 3 + \frac{1}{3} + \frac{1}{6} + \frac{1}{3} + \frac{1}{6} + \frac{1}{3} + \frac{1}{6} + \frac{1}{3} + \dots \quad (7)$$

Type: ContinuedFraction Integer

In 1737 Euler discovered the infinite continued fraction expansion

$$\frac{e-1}{2} = \frac{1}{1 + \frac{1}{6 + \frac{1}{10 + \frac{1}{14 + \dots}}}}$$

We use this expansion to compute rational and floating point approximations of  $e$ .<sup>2</sup>

By looking at the above expansion, we see that the whole part is 0 and the numerators are all equal to 1. This constructs the stream of denominators.

$$\text{dens:Stream Integer} := \text{cons}(1, \text{generate}((x \mapsto x+4), 6)) \quad [1, 6, 10, 14, 18, 22, 26, \dots] \quad (8)$$

Type: Stream Integer

<sup>2</sup>For this and other interesting expansions, see C. D. Olds, *Continued Fractions*, New Mathematical Library, (New York: Random House, 1963), pp. 134–139.



Therefore this is the continued fraction expansion for  $(e-1)/2$ .

$$\text{cf} := \text{continuedFraction}(0, \text{repeating } [1], \text{dens})$$

$$\frac{1}{1} + \frac{1}{6} + \frac{1}{10} + \frac{1}{14} + \frac{1}{18} + \frac{1}{22} + \frac{1}{26} + \dots \quad (9)$$

Type: ContinuedFraction Integer

These are the rational number convergents.

$$\text{ccf} := \text{convergents } \text{cf}$$

$$\left[0, 1, \frac{6}{7}, \frac{61}{71}, \frac{860}{1001}, \frac{15541}{18089}, \frac{342762}{398959}, \dots\right] \quad (10)$$

Type: Stream Fraction Integer

You can get rational convergents for  $e$  by multiplying by 2 and adding 1.

$$\text{eConvergents} := [2 * e + 1 \text{ for } e \text{ in } \text{ccf}]$$

$$\left[1, 3, \frac{19}{7}, \frac{193}{71}, \frac{2721}{1001}, \frac{49171}{18089}, \frac{1084483}{398959}, \dots\right] \quad (11)$$

Type: Stream Fraction Integer

You can also compute the floating point approximations to these convergents.

$$\text{eConvergents} :: \text{Stream Float}$$

$$[1.0, 3.0, 2.7142857142857142857, 2.7183098591549295775, 2.7182817182817182817, 2.7182818287356957267, 2.7182818284585634113, \dots] \quad (12)$$

Type: Stream Float

Compare this to the value of  $e$  computed by the **exp** operation in Float.

$$\text{exp } 1.0$$

$$2.7182818284590452354 \quad (13)$$

Type: Float

In about 1658, Lord Brouncker established the following expansion for  $4/\pi$ .

$$1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \dots}}}}}$$

Let's use this expansion to compute rational and floating point approximations for  $\pi$ .

$$\text{cf} := \text{continuedFraction}(1, [(2*i+1)**2 \text{ for } i \text{ in } 0..], \text{repeating } [2])$$

$$1 + \frac{1}{2} + \frac{9}{2} + \frac{25}{2} + \frac{49}{2} + \frac{81}{2} + \frac{121}{2} + \frac{169}{2} + \dots \quad (14)$$

Type: ContinuedFraction Integer

$$\text{ccf} := \text{convergents } \text{cf}$$

$$\left[1, \frac{3}{2}, \frac{15}{13}, \frac{105}{76}, \frac{315}{263}, \frac{3465}{2578}, \frac{45045}{36979}, \dots\right] \quad (15)$$

Type: Stream Fraction Integer

$$\text{piConvergents} := [4/p \text{ for } p \text{ in ccf}]$$

$$\left[4, \frac{8}{3}, \frac{52}{15}, \frac{304}{105}, \frac{1052}{315}, \frac{10312}{3465}, \frac{147916}{45045}, \dots\right] \quad (16)$$

Type: Stream Fraction Integer

As you can see, the values are converging to  $\pi = 3.14159265358979323846\dots$ , but not very quickly.

$$\text{piConvergents} :: \text{Stream Float}$$

$$[4.0, 2.6666666666666667, 3.4666666666666667, 2.8952380952380952381, 3.3396825396825396825, 2.9760461760461760462, 3.2837384837384837385, \dots] \quad (17)$$

Type: Stream Float

You need not restrict yourself to continued fractions of integers. Here is an expansion for a quotient of Gaussian integers.

$$\text{continuedFraction}((-122 + 597\%i)/(4 - 4\%i))$$

$$-90 + 59\ i + \frac{1|}{|1 - 2\ i|} + \frac{1|}{|-1 + 2\ i|} \quad (18)$$

Type: ContinuedFraction Complex Integer

This is an expansion for a quotient of polynomials in one variable with rational number coefficients.

$$r : \text{Fraction UnivariatePolynomial}(x, \text{Fraction Integer})$$

Type: Void

$$r := ((x - 1) * (x - 2)) / ((x - 3) * (x - 4))$$

$$\frac{x^2 - 3x + 2}{x^2 - 7x + 12} \quad (20)$$

Type: Fraction UnivariatePolynomial(x, Fraction Integer)

$$\text{continuedFraction } r$$

$$1 + \frac{1|}{|\frac{1}{4}x - \frac{9}{8}|} + \frac{1|}{|\frac{16}{3}x - \frac{40}{3}|} \quad (21)$$

Type: ContinuedFraction UnivariatePolynomial(x, Fraction Integer)

To conclude this section, we give you evidence that

$$z = 3 + \frac{1|}{|3|} + \frac{1|}{|6|} + \frac{1|}{|3|} + \frac{1|}{|6|} + \frac{1|}{|3|} + \frac{1|}{|6|} + \frac{1|}{|3|} + \frac{1|}{|6|} + \frac{1|}{|3|} + \frac{1|}{|6|} + \dots$$

is the expansion of  $\sqrt{11}$ .

$$[i*i \text{ for } i \text{ in convergents}(z) :: \text{Stream Float}]$$

$$[9.0, 11.111111111111111, 10.99445983379501385, 11.000277777777778, 10.999986076398799786, 11.000000697929731039, 10.999999965015834446, \dots] \quad (22)$$

Type: Stream Float

## 9.13 CycleIndicators

This section is based upon the paper J. H. Redfield, “The Theory of Group-Reduced Distributions”, American J. Math., 49 (1927) 433-455, and is an application of group theory to enumeration problems. It is a development of the work by P. A. MacMahon on the application of symmetric functions and Hammond operators to combinatorial theory.

The theory is based upon the power sum symmetric functions  $s_i$  which are the sum of the  $i^{\text{th}}$  powers of the variables. The cycle index of a permutation is an expression that specifies the sizes of the cycles of a permutation, and may be represented as a partition. A partition of a non-negative integer  $n$  is a collection of positive integers called its parts whose sum is  $n$ . For example, the partition  $(3^2 \ 2 \ 1^2)$  will be used to represent  $s_3^2 s_2 s_1^2$  and will indicate that the permutation has two cycles of length 3, one of length 2 and two of length 1. The cycle index of a permutation group is the sum of the cycle indices of its permutations divided by the number of permutations. The cycle indices of certain groups are provided.

We first expose something from the library.

)expose EVALCYC

EvaluateCycleIndicators is now explicitly exposed in

The operation **complete** returns the cycle index of the symmetric group of order  $n$  for argument  $n$ . Alternatively, it is the  $n^{\text{th}}$  complete homogeneous symmetric function expressed in terms of power sum symmetric functions.

complete 1

$$(1) \tag{1}$$

Type: SymmetricPolynomial Fraction Integer

complete 2

$$\frac{1}{2} (2) + \frac{1}{2} (1^2) \tag{2}$$

Type: SymmetricPolynomial Fraction Integer

complete 3

$$\frac{1}{3} (3) + \frac{1}{2} (2 \ 1) + \frac{1}{6} (1^3) \tag{3}$$

Type: SymmetricPolynomial Fraction Integer

complete 7

$$\begin{aligned} & \frac{1}{7} (7) + \frac{1}{6} (6 \ 1) + \frac{1}{10} (5 \ 2) + \frac{1}{10} (5 \ 1^2) + \frac{1}{12} (4 \ 3) + \\ & \frac{1}{8} (4 \ 2 \ 1) + \frac{1}{24} (4 \ 1^3) + \frac{1}{18} (3^2 \ 1) + \frac{1}{24} (3 \ 2^2) + \frac{1}{12} (3 \ 2 \ 1^2) \quad (4) \\ & + \frac{1}{72} (3 \ 1^4) + \frac{1}{48} (2^3 \ 1) + \frac{1}{48} (2^2 \ 1^3) + \frac{1}{240} (2 \ 1^5) + \frac{1}{5040} (1^7) \end{aligned}$$

Type: SymmetricPolynomial Fraction Integer

The operation **elementary** computes the  $n^{\text{th}}$  elementary symmetric function for argument  $n$ .

elementary 7

$$\begin{aligned} & \frac{1}{7} (7) - \frac{1}{6} (6 \ 1) - \frac{1}{10} (5 \ 2) + \frac{1}{10} (5 \ 1^2) - \frac{1}{12} (4 \ 3) + \frac{1}{8} (4 \ 2 \ 1) \\ & - \frac{1}{24} (4 \ 1^3) + \frac{1}{18} (3^2 \ 1) + \frac{1}{24} (3 \ 2^2) - \frac{1}{12} (3 \ 2 \ 1^2) \quad (5) \\ & + \frac{1}{72} (3 \ 1^4) - \frac{1}{48} (2^3 \ 1) + \frac{1}{48} (2^2 \ 1^3) - \frac{1}{240} (2 \ 1^5) + \frac{1}{5040} (1^7) \end{aligned}$$

Type: SymmetricPolynomial Fraction Integer

The operation **alternating** returns the cycle index of the alternating group having an even number of even parts in each cycle partition.

alternating 7

$$\begin{aligned} & \frac{2}{7} (7) + \frac{1}{5} (5 \ 1^2) + \frac{1}{4} (4 \ 2 \ 1) + \frac{1}{9} (3^2 \ 1) + \frac{1}{12} (3 \ 2^2) + \\ & \frac{1}{36} (3 \ 1^4) + \frac{1}{24} (2^2 \ 1^3) + \frac{1}{2520} (1^7) \quad (6) \end{aligned}$$

Type: SymmetricPolynomial Fraction Integer

The operation **cyclic** returns the cycle index of the cyclic group.

cyclic 7

$$\frac{6}{7} (7) + \frac{1}{7} (1^7) \quad (7)$$

Type: SymmetricPolynomial Fraction Integer

The operation **dihedral** is the cycle index of the dihedral group.

dihedral 7

$$\frac{3}{7} (7) + \frac{1}{2} (2^3 \ 1) + \frac{1}{14} (1^7) \quad (8)$$

Type: SymmetricPolynomial Fraction Integer

The operation **graphs** for argument  $n$  returns the cycle index of the group of permutations on the edges of the complete graph with  $n$  nodes induced by applying the symmetric group to the nodes.

graphs 5

$$\begin{aligned} & \frac{1}{6} (6 \ 3 \ 1) + \frac{1}{5} (5^2) + \frac{1}{4} (4^2 \ 2) + \frac{1}{6} (3^3 \ 1) + \frac{1}{8} (2^4 \ 1^2) + \\ & \frac{1}{12} (2^3 \ 1^4) + \frac{1}{120} (1^{10}) \quad (9) \end{aligned}$$

Type: SymmetricPolynomial Fraction Integer

The cycle index of a direct product of two groups is the product of the cycle indices of the groups. Redfield provided two operations on two cycle indices which will be called “cup” and “cap” here. The **cup** of two cycle indices is a kind of scalar product that combines monomials for permutations with the same cycles. The **cap** operation provides the sum of the coefficients of the result of the **cup** operation which will be an integer that enumerates what Redfield called group-reduced distributions.

We can, for example, represent `complete 2 * complete 2` as the set of objects `a a b b` and `complete 2 * complete 1 * complete 1` as `c c d e`.

This integer is the number of different sets of four pairs.

```
cap(complete 2**2, complete 2*complete 1**2)
```

4 (10)

Type: Fraction Integer

For example,

a a b b	a a b b	a a b b	a a b b
c c d e	c d c e	c e c d	d e c c

This integer is the number of different sets of four pairs no two pairs being equal.

```
cap(elementary 2**2, complete 2*complete 1**2)
```

2 (11)

Type: Fraction Integer

For example,

a a b b	a a b b
c d c e	c e c d

In this case the configurations enumerated are easily constructed, however the theory merely enumerates them providing little help in actually constructing them.

Here are the number of 6-pairs, first from `a a a b b c`, second from `d d e e f g`.

```
cap(complete 3*complete 2*complete 1,complete
2**2*complete 1**2)
```

24 (12)

Type: Fraction Integer

Here it is again, but with no equal pairs.

```
cap(elementary 3*elementary 2*elementary 1,complete
2**2*complete 1**2)
```

8 (13)

Type: Fraction Integer

```
cap(complete 3*complete 2*complete 1,elementary
2**2*elementary 1**2)
```

8 (14)

Type: Fraction Integer

The number of 6-triples, first from $a a a b b c$ , second from $d d e e f g$ , third from $h h i i j j$ .	<pre>eval(cup(complete 3*complete 2*complete 1, cup(complete 2**2*complete 1**2,complete 2**3)))</pre> 1500 <div style="text-align: right;">(15)</div> <div style="text-align: right;">Type: Fraction Integer</div>
The cycle index of vertices of a square is dihedral 4.	<pre>square:=dihedral 4</pre> $\frac{1}{4} (4) + \frac{3}{8} (2^2) + \frac{1}{4} (2 \ 1^2) + \frac{1}{8} (1^4)$ <div style="text-align: right;">(16)</div> <div style="text-align: right;">Type: SymmetricPolynomial Fraction Integer</div>
The number of different squares with 2 red vertices and 2 blue vertices.	<pre>cap(complete 2**2,square)</pre> 2 <div style="text-align: right;">(17)</div> <div style="text-align: right;">Type: Fraction Integer</div>
The number of necklaces with 3 red beads, 2 blue beads and 2 green beads.	<pre>cap(complete 3*complete 2**2,dihedral 7)</pre> 18 <div style="text-align: right;">(18)</div> <div style="text-align: right;">Type: Fraction Integer</div>
The number of graphs with 5 nodes and 7 edges.	<pre>cap(graphs 5,complete 7*complete 3)</pre> 4 <div style="text-align: right;">(19)</div> <div style="text-align: right;">Type: Fraction Integer</div>
The cycle index of rotations of vertices of a cube.	<pre>s(x) == powerSum(x)</pre> <div style="text-align: right;">Type: Void</div> <pre>cube:=(1/24)*(s 1**8+9*s 2**4 + 8*s 3**2*s 1**2+6*s 4**2)</pre> Compiling function <i>s</i> with type PositiveInteger -> SymmetricPolynomial Fraction Integer $\frac{1}{4} (4^2) + \frac{1}{3} (3^2 \ 1^2) + \frac{3}{8} (2^4) + \frac{1}{24} (1^8)$ <div style="text-align: right;">(21)</div> <div style="text-align: right;">Type: SymmetricPolynomial Fraction Integer</div>
The number of cubes with 4 red vertices and 4 blue vertices.	<pre>cap(complete 4**2,cube)</pre> 7 <div style="text-align: right;">(22)</div> <div style="text-align: right;">Type: Fraction Integer</div>
The number of labeled graphs with degree sequence 2 2 2 1 1 with no loops or multiple edges.	<pre>cap(complete 2**3*complete 1**2,wreath(elementary 4,elementary 2))</pre> 7 <div style="text-align: right;">(23)</div> <div style="text-align: right;">Type: Fraction Integer</div>

Again, but with loops allowed  
but not multiple edges.

```
cap(complete 2**3*complete 1**2,wreath(elementary
4,complete 2))
```

17

(24)

Type: Fraction Integer

Again, but with multiple edges  
allowed, but not loops

```
cap(complete 2**3*complete 1**2,wreath(complete
4,elementary 2))
```

10

(25)

Type: Fraction Integer

Again, but with both multiple  
edges and loops allowed

```
cap(complete 2**3*complete 1**2,wreath(complete 4,complete
2))
```

23

(26)

Type: Fraction Integer

Having constructed a cycle index for a configuration we are at liberty to evaluate the  $s_i$  components any way we please. For example we can produce enumerating generating functions. This is done by providing a function `f` on an integer `i` to the value required of  $s_i$ , and then evaluating `eval(f, cycleindex)`.

```
x: ULS(FRAC INT,'x,0) := 'x
```

$x$

(27)

Type: UnivariateLaurentSeries(Fraction Integer, x, 0)

```
ZeroOrOne: INT -> ULS(FRAC INT, 'x, 0)
```

Type: Void

```
Integers: INT -> ULS(FRAC INT, 'x, 0)
```

Type: Void

For the integers 0 and 1, or two  
colors.

```
ZeroOrOne n == 1+x**n
```

Type: Void

```
ZeroOrOne 5
```

```
Compiling function ZeroOrOne with type Integer ->
UnivariateLaurentSeries(Fraction Integer,x,0)
```

$1 + x^5$

(31)

Type: UnivariateLaurentSeries(Fraction Integer, x, 0)

For the integers 0, 1, 2, ...  
we have this.

Integers n == 1/(1-x\*\*n)

Type: Void

Integers 5

Compiling function Integers with type Integer ->  
UnivariateLaurentSeries(Fraction Integer,x,0)

$$1 + x^5 + O(x^8) \quad (33)$$

Type: UnivariateLaurentSeries(Fraction Integer, x, 0)

The coefficient of  $x^n$  is the  
number of graphs with 5 nodes  
and n edges.

eval(ZeroOrOne, graphs 5)

$$1 + x + 2x^2 + 4x^3 + 6x^4 + 6x^5 + 6x^6 + 4x^7 + O(x^8) \quad (34)$$

Type: UnivariateLaurentSeries(Fraction Integer, x, 0)

The coefficient of  $x^n$  is the  
number of necklaces with n red  
beads and n-8 green beads.

eval(ZeroOrOne, dihedral 8)

$$1 + x + 4x^2 + 5x^3 + 8x^4 + 5x^5 + 4x^6 + x^7 + O(x^8) \quad (35)$$

Type: UnivariateLaurentSeries(Fraction Integer, x, 0)

The coefficient of  $x^n$  is the  
number of partitions of n into 4  
or fewer parts.

eval(Integers, complete 4)

$$1 + x + 2x^2 + 3x^3 + 5x^4 + 6x^5 + 9x^6 + 11x^7 + O(x^8) \quad (36)$$

Type: UnivariateLaurentSeries(Fraction Integer, x, 0)

The coefficient of  $x^n$  is the  
number of partitions of n into 4  
boxes containing ordered  
distinct parts.

eval(Integers, elementary 4)

$$x^6 + x^7 + 2x^8 + 3x^9 + 5x^{10} + 6x^{11} + 9x^{12} + 11x^{13} + O(x^{14}) \quad (37)$$

Type: UnivariateLaurentSeries(Fraction Integer, x, 0)

The coefficient of  $x^n$  is the  
number of different cubes with n  
red vertices and 8-n green ones.

eval(ZeroOrOne, cube)

$$1 + x + 3x^2 + 3x^3 + 7x^4 + 3x^5 + 3x^6 + x^7 + O(x^8) \quad (38)$$

Type: UnivariateLaurentSeries(Fraction Integer, x, 0)

The coefficient of  $x^n$  is the  
number of different cubes with  
integers on the vertices whose  
sum is n.

eval(Integers, cube)

$$1 + x + 4x^2 + 7x^3 + 21x^4 + 37x^5 + 85x^6 + 151x^7 + O(x^8) \quad (39)$$

Type: UnivariateLaurentSeries(Fraction Integer, x, 0)

The coefficient of  $x^n$  is the  
number of graphs with 5 nodes  
and with integers on the edges  
whose sum is n. In other words,  
the enumeration is of  
multigraphs with 5 nodes and n  
edges.

eval(Integers, graphs 5)

$$1 + x + 3x^2 + 7x^3 + 17x^4 + 35x^5 + 76x^6 + 149x^7 + O(x^8) \quad (40)$$

Type: UnivariateLaurentSeries(Fraction Integer, x, 0)



Graphs with 15 nodes  
enumerated with respect to  
number of edges.

$$\text{eval}(\text{ZeroOrOne}, \text{graphs } 15)$$

$$1 + x + 2x^2 + 5x^3 + 11x^4 + 26x^5 + 68x^6 + 177x^7 + O(x^8) \quad (41)$$

Type: UnivariateLaurentSeries(Fraction Integer, x, 0)

Necklaces with 7 green beads, 8  
white beads, 5 yellow beads and  
10 red beads.

$$\text{cap}(\text{dihedral } 30, \text{complete } 7 * \text{complete } 8 * \text{complete } 5 * \text{complete } 10)$$

$$49958972383320 \quad (42)$$

Type: Fraction Integer

The operation **SFunction** is the S-function or Schur function of a partition written as a descending list of integers expressed in terms of power sum symmetric functions.

In this case the argument partition represents a tableau shape. For example **3,2,2,1** represents a tableau with three boxes in the first row, two boxes in the second and third rows, and one box in the fourth row. **SFunction [3,2,2,1]** counts the number of different tableaux of shape **3, 2, 2, 1** filled with objects with an ascending order in the columns and a non-descending order in the rows.

$$\text{sf3221} := \text{SFunction } [3, 2, 2, 1]$$

$$\frac{1}{12} (6 \ 2) - \frac{1}{12} (6 \ 1^2) - \frac{1}{16} (4^2) + \frac{1}{12} (4 \ 3 \ 1) + \frac{1}{24} (4 \ 1^4) -$$

$$\frac{1}{36} (3^2 \ 2) + \frac{1}{36} (3^2 \ 1^2) - \frac{1}{24} (3 \ 2^2 \ 1) - \frac{1}{36} (3 \ 2 \ 1^3) -$$

$$\frac{1}{72} (3 \ 1^5) - \frac{1}{192} (2^4) + \frac{1}{48} (2^3 \ 1^2) + \frac{1}{96} (2^2 \ 1^4) -$$

$$\frac{1}{144} (2 \ 1^6) + \frac{1}{576} (1^8) \quad (43)$$

Type: SymmetricPolynomial Fraction Integer

This is the number filled with a  
a b b c c d d.

$$\text{cap}(\text{sf3221}, \text{complete } 2 ** 4)$$

$$3 \quad (44)$$

Type: Fraction Integer

The configurations enumerated above are:

a a b	a a c	a a d
b c	b b	b b
c d	c d	c c
d	d	d

This is the number of tableaux  
filled with 1..8.

$$\text{cap}(\text{sf3221}, \text{powerSum } 1 ** 8)$$

$$70 \quad (45)$$

Type: Fraction Integer

The coefficient of  $x^n$  is the  
number of column strict reverse  
plane partitions of **n** of shape **3  
2 2 1**.

$$\text{eval}(\text{Integers}, \text{sf3221})$$

$$x^9 + 3x^{10} + 7x^{11} + 14x^{12} + 27x^{13} + 47x^{14} + O(x^{15}) \quad (46)$$

Type: UnivariateLaurentSeries(Fraction Integer, x, 0)

The smallest is

0 0 0  
1 1  
2 2  
3

## 9.14 DeRhamComplex

The domain constructor `DeRhamComplex` creates the class of differential forms of arbitrary degree over a coefficient ring. The De Rham complex constructor takes two arguments: a ring, `coefRing`, and a list of coordinate variables.

This is the ring of coefficients.

```
coefRing := Integer
```

Integer

(1)

Type: Domain

These are the coordinate variables.

```
lv : List Symbol := [x,y,z]
```

$[x, y, z]$

(2)

Type: List Symbol

This is the De Rham complex of Euclidean three-space using coordinates  $x$ ,  $y$  and  $z$ .

```
der := DERHAM(coefRing,lv)
```

DeRhamComplex (Integer ,  $[x, y, z]$ )

(3)

Type: Domain

This complex allows us to describe differential forms having expressions of integers as coefficients. These coefficients can involve any number of variables, for example,  $f(x, t, r, y, u, z)$ . As we've chosen to work with ordinary Euclidean three-space, expressions involving these forms are treated as functions of  $x$ ,  $y$  and  $z$  with the additional arguments  $t$ ,  $r$  and  $u$  regarded as symbolic constants.

Here are some examples of coefficients.

```
R := Expression coefRing
```

Expression Integer

(4)

Type: Domain

```
f : R := x**2*y*z-5*x**3*y**2*z**5
```

$-5x^3y^2z^5 + x^2yz$

(5)

Type: Expression Integer

```
g : R := z**2*y*cos(z)-7*sin(x**3*y**2)*z**2
```

$-7z^2 \sin(x^3y^2) + yz^2 \cos(z)$

(6)

Type: Expression Integer

```
h : R :=x*y*z-2*x**3*y*z**2
```

$-2x^3yz^2 + xyz$

(7)

Type: Expression Integer

We now define the multiplicative basis elements for the exterior algebra over  $R$ .

```
dx : der := generator(1)
```

$dx$

(8)

Type: DeRhamComplex(Integer,  $[x, y, z]$ )

```
dy : der := generator(2)
dy
```

Type: DeRhamComplex(Integer, [x, y, z])

```
dz : der := generator(3)
dz
```

Type: DeRhamComplex(Integer, [x, y, z])

This is an alternative way to give the above assignments.

```
[dx,dy,dz] := [generator(i)$der for i in 1..3]
[dx, dy, dz]
```

Type: List DeRhamComplex(Integer, [x, y, z])

Now we define some one-forms.

```
alpha : der := f*dx + g*dy + h*dz
(-2 x^3 y z^2 + x y z) dz + (-7 z^2 sin(x^3 y^2) + y z^2 cos(z)) dy
+ (-5 x^3 y^2 z^5 + x^2 y z) dx
```

Type: DeRhamComplex(Integer, [x, y, z])

```
beta : der := cos(tan(x*y*z)+x*y*z)*dx + x*dy
x dy + cos(tan(x y z) + x y z) dx
```

Type: DeRhamComplex(Integer, [x, y, z])

A well-known theorem states that the composition of **exteriorDifferential** with itself is the zero map for continuous forms. Let's verify this theorem for **alpha**.

```
exteriorDifferential alpha;
```

Type: DeRhamComplex(Integer, [x, y, z])

We suppressed the lengthy output of the last expression, but nevertheless, the composition is zero.

```
exteriorDifferential %
0
```

Type: DeRhamComplex(Integer, [x, y, z])

Now we check that **exteriorDifferential** is a “graded derivation” **D**, that is, **D** satisfies:

```
gamma := alpha * beta
(2 x^4 y z^2 - x^2 y z) dy dz + (2 x^3 y z^2 - x y z) *
cos(tan(x y z) + x y z) dx dz +
```

$D(ab) = D(a)b + (-1)^{\text{degree}(a)} aD(b)$

```
((7 z^2 sin(x^3 y^2) - y z^2 cos(z)) cos(tan(x y z) + x y z)
- 5 x^4 y^2 z^5 + x^3 y z) dx dy
```

Type: DeRhamComplex(Integer, [x, y, z])

We try this for the one-forms  $\alpha$  and  $\beta$ .

$$\begin{aligned} & \text{exteriorDifferential}(\gamma) \\ & - (\text{exteriorDifferential}(\alpha) * \beta - \alpha * \\ & \quad \text{exteriorDifferential}(\beta)) \\ & 0 \end{aligned} \tag{17}$$

Type: DeRhamComplex(Integer, [x, y, z])

Now we define some “basic operators” (see ‘Operator’ on page 516).

$$\begin{aligned} a : \text{BOP} &:= \text{operator}('a) \\ a \end{aligned} \tag{18}$$

Type: BasicOperator

$$\begin{aligned} b : \text{BOP} &:= \text{operator}('b) \\ b \end{aligned} \tag{19}$$

Type: BasicOperator

$$\begin{aligned} c : \text{BOP} &:= \text{operator}('c) \\ c \end{aligned} \tag{20}$$

Type: BasicOperator

We also define some indeterminate one- and two-forms using these operators.

$$\begin{aligned} \sigma &:= a(x, y, z) * dx + b(x, y, z) * dy + c(x, y, z) * dz \\ c(x, y, z) dz + b(x, y, z) dy + a(x, y, z) dx \end{aligned} \tag{21}$$

Type: DeRhamComplex(Integer, [x, y, z])

$$\begin{aligned} \theta &:= a(x, y, z) * dx * dy + b(x, y, z) * dx * dz + \\ & \quad c(x, y, z) * dy * dz \\ c(x, y, z) dy dz + b(x, y, z) dx dz + a(x, y, z) dx dy \end{aligned} \tag{22}$$

Type: DeRhamComplex(Integer, [x, y, z])

This allows us to get formal definitions for the “gradient” ...

$$\begin{aligned} & \text{totalDifferential}(a(x, y, z))\$der \\ & a_{,3}(x, y, z) dz + a_{,2}(x, y, z) dy + a_{,1}(x, y, z) dx \end{aligned} \tag{23}$$

Type: DeRhamComplex(Integer, [x, y, z])

the “curl” ...

$$\begin{aligned} & \text{exteriorDifferential } \sigma \\ & (c_{,2}(x, y, z) - b_{,3}(x, y, z)) dy dz + \\ & (c_{,1}(x, y, z) - a_{,3}(x, y, z)) dx dz + \\ & (b_{,1}(x, y, z) - a_{,2}(x, y, z)) dx dy \end{aligned} \tag{24}$$

Type: DeRhamComplex(Integer, [x, y, z])

and the “divergence.”

$$\begin{aligned} & \text{exteriorDifferential } \theta \\ & (c_{,1}(x, y, z) - b_{,2}(x, y, z) + a_{,3}(x, y, z)) dx dy dz \end{aligned} \tag{25}$$

Type: DeRhamComplex(Integer, [x, y, z])

Note that the De Rham complex is an algebra with unity. This element 1 is the basis for elements for zero-forms, that is, functions in our space.

$$\begin{aligned} \text{one} &: \text{der} := 1 \\ 1 & \\ \text{Type: DeRhamComplex(Integer, [x, y, z])} \end{aligned} \quad (26)$$

To convert a function to a function lying in the De Rham complex, multiply the function by “one.”

$$\begin{aligned} \text{g1} &: \text{der} := a([x, t, y, u, v, z, e]) * \text{one} \\ a(x, t, y, u, v, z, e) & \\ \text{Type: DeRhamComplex(Integer, [x, y, z])} \end{aligned} \quad (27)$$

A current limitation of AXIOM forces you to write functions with more than four arguments using square brackets in this way.

$$\begin{aligned} \text{h1} &: \text{der} := a([x, y, x, t, x, z, y, r, u, x]) * \text{one} \\ a(x, y, x, t, x, z, y, r, u, x) & \\ \text{Type: DeRhamComplex(Integer, [x, y, z])} \end{aligned} \quad (28)$$

Now note how the system keeps track of where your coordinate functions are located in expressions.

$$\begin{aligned} \text{exteriorDifferential g1} & \\ a_{,6}(x, t, y, u, v, z, e) dz + a_{,3}(x, t, y, u, v, z, e) dy + & \\ a_{,1}(x, t, y, u, v, z, e) dx & \\ \text{Type: DeRhamComplex(Integer, [x, y, z])} \end{aligned} \quad (29)$$

$$\begin{aligned} \text{exteriorDifferential h1} & \\ a_{,6}(x, y, x, t, x, z, y, r, u, x) dz + & \\ (a_{,7}(x, y, x, t, x, z, y, r, u, x) + a_{,2}(x, y, x, t, x, z, y, r, u, x)) \cdot & \\ dy + \left( \begin{array}{l} a_{,10}(x, y, x, t, x, z, y, r, u, x) + \\ a_{,5}(x, y, x, t, x, z, y, r, u, x) + \\ a_{,3}(x, y, x, t, x, z, y, r, u, x) + \\ a_{,1}(x, y, x, t, x, z, y, r, u, x) \end{array} \right) dx & \\ \text{Type: DeRhamComplex(Integer, [x, y, z])} \end{aligned} \quad (30)$$

In this example of Euclidean three-space, the basis for the De Rham complex consists of the eight forms: 1, dx, dy, dz, dx\*dy, dx\*dz, dy\*dz, and dx\*dy\*dz.

$$\begin{aligned} \text{coefficient}(\text{gamma}, dx*dy) & \\ \left( 7 z^2 \sin(x^3 y^2) - y z^2 \cos(z) \right) \cos(\tan(x y z) + x y z) - & \\ 5 x^4 y^2 z^5 + x^3 y z & \\ \text{Type: Expression Integer} \end{aligned} \quad (31)$$

$$\begin{aligned} \text{coefficient}(\text{gamma}, \text{one}) & \\ 0 & \\ \text{Type: Expression Integer} \end{aligned} \quad (32)$$

$$\begin{aligned} \text{coefficient}(\text{g1}, \text{one}) & \\ a(x, t, y, u, v, z, e) & \\ \text{Type: Expression Integer} \end{aligned} \quad (33)$$

## 9.15 Decimal- Expansion

---

All rationals have repeating decimal expansions. Operations to access the individual digits of a decimal expansion can be obtained by converting the value to `RadixExpansion(10)`. More examples of expansions are available in ‘`BinaryExpansion`’ on page 359, ‘`HexadecimalExpansion`’ on page 444, and ‘`RadixExpansion`’ on page 537. Issue the system command `)show DecimalExpansion` to display the full list of operations defined by `DecimalExpansion`.

The operation **decimal** is used to create this expansion of type `DecimalExpansion`.

```
r := decimal(22/7)
3.142857
(1)
```

Type: `DecimalExpansion`

Arithmetic is exact.

```
r + decimal(6/7)
4
(2)
```

Type: `DecimalExpansion`

The period of the expansion can be short or long ...

```
[decimal(1/i) for i in 350..354]
[0.00285714, 0.002849, 0.0028409,
0.00283286118980169971671388101983,
0.00282485875706214689265536723163841807909604519774011299435]
(3)
```

Type: `List DecimalExpansion`

or very long.

```
decimal(1/2049)
0.0004880429477794045876037091264031234748657881893606
637384089799902391410444119082479258174719375305026842362
127867252318204001952171791117618350414836505612493899463
152757442654953635919960956564177647632991703269887750122
010736944851146900927281600780868716447047340165934602244
997559785261102977061981454367984382625671059053196681307
9551
(4)
```

Type: `DecimalExpansion`

These numbers are bona fide algebraic objects.

```
p := decimal(1/4)*x**2 + decimal(2/3)*x + decimal(4/9)
0.25 x^2 + 0.6 x + 0.4
(5)
```

Type: `Polynomial DecimalExpansion`

```
q := differentiate(p, x)
0.5 x + 0.6
(6)
```

Type: `Polynomial DecimalExpansion`

```
g := gcd(p, q)
x + 1.3
(7)
```

Type: `Polynomial DecimalExpansion`

## 9.16 Distributed- Multivariate- Polynomial

---

The constructor DMP orders its monomials lexicographically while HDMP orders them by total order refined by reverse lexicographic order.

These constructors are mostly used in Gröbner basis calculations.

DistributedMultivariatePolynomial and HomogeneousDistributedMultivariatePolynomial, abbreviated DMP and HDMP, respectively, are very similar to MultivariatePolynomial except that they are represented and displayed in a non-recursive manner.

(d1,d2,d3) : DMP([z,y,x],FRAC INT)

Type: Void

d1 := -4\*z + 4\*y\*\*2\*x + 16\*x\*\*2 + 1  
 $-4z + 4y^2x + 16x^2 + 1$   
 Type: DistributedMultivariatePolynomial([z, y, x], Fraction Integer)

d2 := 2\*z\*y\*\*2 + 4\*x + 1  
 $2zy^2 + 4x + 1$   
 Type: DistributedMultivariatePolynomial([z, y, x], Fraction Integer)

d3 := 2\*z\*x\*\*2 - 2\*y\*\*2 - x  
 $2zx^2 - 2y^2 - x$   
 Type: DistributedMultivariatePolynomial([z, y, x], Fraction Integer)

groebner [d1,d2,d3]  

$$\left[ z - \frac{1568}{2745}x^6 - \frac{1264}{305}x^5 + \frac{6}{305}x^4 + \frac{182}{549}x^3 - \frac{2047}{610}x^2 - \frac{103}{2745}x - \frac{2857}{10980}, \right.$$

$$y^2 + \frac{112}{2745}x^6 - \frac{84}{305}x^5 - \frac{1264}{305}x^4 - \frac{13}{549}x^3 + \frac{84}{305}x^2 + \frac{1772}{2745}x + \frac{2}{2745},$$

$$\left. x^7 + \frac{29}{4}x^6 - \frac{17}{16}x^4 - \frac{11}{8}x^3 + \frac{1}{32}x^2 + \frac{15}{16}x + \frac{1}{4} \right]$$
  
 Type: List DistributedMultivariatePolynomial([z, y, x], Fraction Integer)

(n1,n2,n3) : HDMP([z,y,x],FRAC INT)

Type: Void

(n1,n2,n3) := (d1,d2,d3)  
 $2zx^2 - 2y^2 - x$   
 Type: HomogeneousDistributedMultivariatePolynomial([z, y, x], Fraction Integer)



Note that we get a different Gröbner basis when we use the HDMP polynomials, as expected.

```
groebner [n1,n2,n3]
```

$$\left[ y^4 + 2x^3 - \frac{3}{2}x^2 + \frac{1}{2}z - \frac{1}{8}, \right. \\ \left. x^4 + \frac{29}{4}x^3 - \frac{1}{8}y^2 - \frac{7}{4}zx - \frac{9}{16}x - \frac{1}{4}, z y^2 + 2x + \frac{1}{2}, \right. \\ \left. y^2x + 4x^2 - z + \frac{1}{4}, z x^2 - y^2 - \frac{1}{2}x, z^2 - 4y^2 + 2x^2 - \frac{1}{4}z - \frac{3}{2}x \right] \quad (8)$$

Type: List HomogeneousDistributedMultivariatePolynomial([z, y, x], Fraction Integer)

GeneralDistributedMultivariatePolynomial is somewhat more flexible in the sense that as well as accepting a list of variables to specify the variable ordering, it also takes a predicate on exponent vectors to specify the term ordering. With this polynomial type the user can experiment with the effect of using completely arbitrary term orderings. This flexibility is mostly important for algorithms such as Gröbner basis calculations which can be very sensitive to term ordering.

For more information on related topics, see Section 1.9 on page 73, Section 2.7 on page 113, ‘Polynomial’ on page 529, ‘UnivariatePolynomial’ on page 594, and ‘MultivariatePolynomial’ on page 508. Issue the system command `)show DistributedMultivariatePolynomial` to display the full list of operations defined by DistributedMultivariatePolynomial.

## 9.17 DoubleFloat

---

AXIOM provides two kinds of floating point numbers. The domain Float (abbreviation FLOAT) implements a model of arbitrary precision floating point numbers. The domain DoubleFloat (abbreviation DFLOAT) is intended to make available hardware floating point arithmetic in AXIOM. The actual model of floating point DoubleFloat that provides is system-dependent. For example, on the IBM system 370 AXIOM uses IBM double precision which has fourteen hexadecimal digits of precision or roughly sixteen decimal digits. Arbitrary precision floats allow the user to specify the precision at which arithmetic operations are computed. Although this is an attractive facility, it comes at a cost. Arbitrary-precision floating-point arithmetic typically takes twenty to two hundred times more time than hardware floating point.

The usual arithmetic and elementary functions are available for DoubleFloat. Use `)show DoubleFloat` to get a list of operations or the HyperDoc Browse facility to get more extensive documentation about DoubleFloat.

By default, floating point numbers that you enter into AXIOM are of type Float.

```
2.71828
2.71828
```

(1)  
Type: Float

You must therefore tell AXIOM that you want to use DoubleFloat values and operations. The following are some conservative guidelines for getting AXIOM to use DoubleFloat.

To get a value of type DoubleFloat, use a target with “@”, ...

```
2.71828@DoubleFloat
2.71828
```

(2)  
Type: DoubleFloat

a conversion, ...

```
2.71828 :: DoubleFloat
2.71828
```

(3)  
Type: DoubleFloat

or an assignment to a declared variable. It is more efficient if you use a target rather than an explicit or implicit conversion.

```
eApprox : DoubleFloat := 2.71828
2.71828
```

(4)  
Type: DoubleFloat

You also need to declare functions that work with DoubleFloat.

```
avg : List DoubleFloat -> DoubleFloat
```

Type: Void

```

avg 1 ==
  empty? 1 => 0 :: DoubleFloat
  reduce(_+,1) / #1

```

Type: Void

```
avg []
```

```

Compiling function avg with type List DoubleFloat ->
  DoubleFloat

```

```
0.0
```

(7)

Type: DoubleFloat

```
avg [3.4,9.7,-6.8]
```

```
2.1
```

(8)

Type: DoubleFloat

Use package-calling for  
operations from DoubleFloat  
unless the arguments themselves  
are already of type DoubleFloat.

```
cos(3.1415926)$DoubleFloat
```

```
-0.9999999999999999
```

(9)

Type: DoubleFloat

```
cos(3.1415926 :: DoubleFloat)
```

```
-0.9999999999999999
```

(10)

Type: DoubleFloat

By far, the most common usage of DoubleFloat is for functions to be graphed. For more information about AXIOM's numerical and graphical facilities, see Section 7 on page 235, Section 8.1 on page 264, and 'Float' on page 427.

## 9.18 EqTable

---

The EqTable domain provides tables where the keys are compared using **eq?**. Keys are considered equal only if they are the same instance of a structure. This is useful if the keys are themselves updatable structures. Otherwise, all operations are the same as for type Table. See ‘Table’ on page 585 for general information about tables. Issue the system command **)show EqTable** to display the full list of operations defined by EqTable.

The operation **table** is here used to create a table where the keys are lists of integers.

```
e: EqTable(List Integer, Integer) := table()
table()
(1)
```

Type: EqTable(List Integer, Integer)

These two lists are equal according to “=”, but not according to **eq?**.

```
l1 := [1,2,3]
[1, 2, 3]
(2)
```

Type: List PositiveInteger

```
l2 := [1,2,3]
[1, 2, 3]
(3)
```

Type: List PositiveInteger

Because the two lists are not **eq?**, separate values can be stored under each.

```
e.l1 := 111
111
(4)
```

Type: PositiveInteger

```
e.l2 := 222
222
(5)
```

Type: PositiveInteger

```
e.l1
111
(6)
```

Type: PositiveInteger

## 9.19 Equation

Equations are created using the equals symbol, “=”.

The Equation domain provides equations as mathematical objects. These are used, for example, as the input to various **solve** operations.

```
eq1 := 3*x + 4*y = 5
4 y + 3 x = 5
```

(1)

Type: Equation Polynomial Integer

```
eq2 := 2*x + 2*y = 3
2 y + 2 x = 3
```

(2)

Type: Equation Polynomial Integer

The left- and right-hand sides of an equation are accessible using the operations **lhs** and **rhs**.

```
lhs eq1
4 y + 3 x
```

(3)

Type: Polynomial Integer

```
rhs eq1
5
```

(4)

Type: Polynomial Integer

Arithmetic operations are supported and operate on both sides of the equation.

```
eq1 + eq2
6 y + 5 x = 8
```

(5)

Type: Equation Polynomial Integer

```
eq1 * eq2
8 y^2 + 14 x y + 6 x^2 = 15
```

(6)

Type: Equation Polynomial Integer

```
2*eq2 - eq1
x = 1
```

(7)

Type: Equation Polynomial Integer

Equations may be created for any type so the arithmetic operations will be defined only when they make sense. For example, exponentiation is not defined for equations involving non-square matrices.

```
eq1**2
16 y^2 + 24 x y + 9 x^2 = 25
```

(8)

Type: Equation Polynomial Integer

Note that an equals symbol is also used to *test* for equality of values in certain contexts. For example, **x+1** and **y** are unequal as polynomials.

```
if x+1 = y then "equal" else "unequal"
"unequal"
```

(9)

Type: String

`eqpol := x+1 = y`

$x + 1 = y$  (10)

Type: Equation Polynomial Integer

If an equation is used where a Boolean value is required, then it is evaluated using the equality test from the operand type.

`if eqpol then "equal" else "unequal"`  
`"unequal"`

(11)

Type: String

If one wants a Boolean value rather than an equation, all one has to do is ask!

`eqpol::Boolean`  
`false`

(12)

Type: Boolean

## 9.20 Exit

A function that does not return directly to its caller has `Exit` as its return type. The operation **error** is an example of one which does not return to its caller. Instead, it causes a return to top-level.

```
n := 0
0
```

(1)

Type: NonNegativeInteger

The function **gasp** is given return type `Exit` since it is guaranteed never to return a value to its caller.

```
gasp(): Exit ==
  free n
  n := n + 1
  error "Oh no!"
```

Function declaration `gasp : () -> Exit` has been added to workspace.

Type: Void

The return type of **half** is determined by resolving the types of the two branches of the `if`.

```
half(k) ==
  if odd? k then gasp()
  else k quo 2
```

Type: Void

Because **gasp** has the return type `Exit`, the type of `if` in **half** is resolved to be `Integer`.

```
half 4
Compiling function gasp with type () -> Exit
Compiling function half with type PositiveInteger ->
Integer
```

```
2
```

(4)

Type: PositiveInteger

```
half 3
Error signalled from user code in function gasp:
Oh no!
```

```
n
1
```

(5)

Type: NonNegativeInteger

For functions which return no value at all, use `Void`. See Section 6 on page 177 and ‘`Void`’ on page 603 for more information. Issue the system command `)show Exit` to display the full list of operations defined by `Exit`.

## 9.21 Expression

This is an object of type  
Expression Integer.

Expression is a constructor that creates domains whose objects can have very general symbolic forms. Here are some examples:

$$\sin(\mathbf{x}) + 3 \cos(\mathbf{x})^2$$

$$\sin(x) + 3 \cos(x)^2 \quad (1)$$

Type: Expression Integer

This is an object of type  
Expression Float.

$$\tan(\mathbf{x}) - 3.45 \cdot \mathbf{x}$$

$$\tan(x) - 3.45 x \quad (2)$$

Type: Expression Float

This object contains symbolic  
function applications, sums,  
products, square roots, and a  
quotient.

$$(\tan \text{ sqrt } 7 - \sin \text{ sqrt } 11)^2 / (4 - \cos(\mathbf{x} - \mathbf{y}))$$

$$\frac{-\tan(\sqrt{7})^2 + 2 \sin(\sqrt{11}) \tan(\sqrt{7}) - \sin(\sqrt{11})^2}{\cos(y - x) - 4} \quad (3)$$

Type: Expression Integer

As you can see, Expression actually takes an argument domain. The *coefficients* of the terms within the expression belong to the argument domain. Integer and Float, along with Complex Integer and Complex Float are the most common coefficient domains.

The choice of whether to use a  
Complex coefficient domain or  
not is important since AXIOM  
can perform some simplifications  
on real-valued objects

$$\log(\exp \mathbf{x}) @ \text{Expression(Integer)}$$

$$x \quad (4)$$

Type: Expression Integer

... which are not valid on  
complex ones.

$$\log(\exp \mathbf{x}) @ \text{Expression(Complex Integer)}$$

$$\log(e^x) \quad (5)$$

Type: Expression Complex Integer

Many potential coefficient  
domains, such as  
AlgebraicNumber, are not  
usually used because Expression  
can subsume them.

$$\text{sqrt } 3 + \text{sqrt}(2 + \text{sqrt}(-5))$$

$$\sqrt{\sqrt{-5} + 2} + \sqrt{3} \quad (6)$$

Type: AlgebraicNumber

$$\% :: \text{Expression Integer}$$

$$\sqrt{\sqrt{-5} + 2} + \sqrt{3} \quad (7)$$

Type: Expression Integer

Note that we sometimes talk about “an object of type Expression.” This is not really correct because we should say, for example, “an object of type Expression Integer” or “an object of type Expression Float.” By a similar abuse of language, when we refer to an “expression” in this section we will mean an object of type Expression R for some domain **R**.



The AXIOM documentation contains many examples of the use of Expression. For the rest of this section, we'll give you some pointers to those examples plus give you some idea of how to manipulate expressions.

It is important for you to know that Expression creates domains that have category Field. Thus you can invert any non-zero expression and you shouldn't expect an operation like **factor** to give you much information. You can imagine expressions as being represented as quotients of "multivariate" polynomials where the "variables" are kernels (see 'Kernel' on page 457). A kernel can either be a symbol such as **x** or a symbolic function application like **sin(x + 4)**. The second example is actually a nested kernel since the argument to **sin** contains the kernel **x**.

```
height mainKernel sin(x + 4)
2
```

(8)  
Type: PositiveInteger

Actually, the argument to **sin** is an expression, and so the structure of Expression is recursive. 'Kernel' on page 457 demonstrates how to extract the kernels in an expression.

Use the HyperDoc Browse facility to see what operations are applicable to expression. At the time of this writing, there were 262 operations with 147 distinct name in Expression Integer. For example, **numer** and **denom** extract the numerator and denominator of an expression.

```
e := (sin(x) - 4)**2 / ( 1 - 2*y*sqrt(- y) )
```

$$\frac{-\sin(x)^2 + 8 \sin(x) - 16}{2 y \sqrt{-y} - 1}$$

(9)  
Type: Expression Integer

```
numer e
```

$$-\sin(x)^2 + 8 \sin(x) - 16$$

(10)  
Type: SparseMultivariatePolynomial(Integer, Kernel Expression Integer)

```
denom e
```

$$2 y \sqrt{-y} - 1$$

(11)  
Type: SparseMultivariatePolynomial(Integer, Kernel Expression Integer)

Use **D** to compute partial derivatives.

```
D(e, x)
```

$$\frac{(4 y \cos(x) \sin(x) - 16 y \cos(x)) \sqrt{-y} - 2 \cos(x) \sin(x) + 8 \cos(x)}{4 y \sqrt{-y} + 4 y^3 - 1}$$

(12)  
Type: Expression Integer

See Section 1.12 on page 78 for more examples of expressions and derivatives.

$$D(e, [x, y], [1, 2])$$

$$\frac{\left( \begin{aligned} &((-2304 y^7 + 960 y^4) \cos(x) \sin(x) \\ &+ (9216 y^7 - 3840 y^4) \cos(x)) \sqrt{-y} \\ &+ (-960 y^9 + 2160 y^6 - 180 y^3 - 3) \cos(x) \sin(x) \\ &+ (3840 y^9 - 8640 y^6 + 720 y^3 + 12) \cos(x) \end{aligned} \right)}{\left( \begin{aligned} &(256 y^{12} - 1792 y^9 + 1120 y^6 - 112 y^3 + 1) \sqrt{-y} \\ &- 1024 y^{11} + 1792 y^8 - 448 y^5 + 16 y^2 \end{aligned} \right)} \quad (13)$$

Type: Expression Integer

See Section 1.10 on page 75 and Section 1.11 on page 76 for more examples of expressions and calculus. Differential equations involving expressions are discussed in Section 8.10 on page 308. Chapter 8 has many advanced examples: see Section 8.8 on page 292 for a discussion of AXIOM's integration facilities.

When an expression involves no “symbol kernels” (for example,  $x$ ), it may be possible to numerically evaluate the expression.

If you suspect the evaluation will create a complex number, use **complexNumeric**.

```
complexNumeric(cos(2 - 3*i))
```

$$-4.1896256909688072301 + 9.109227893755336598 i \quad (14)$$

Type: Complex Float

If you know it will be real, use **numeric**.

```
numeric(tan 3.8)
```

$$0.77355609050312607286 \quad (15)$$

Type: Float

The **numeric** operation will display an error message if the evaluation yields a value with a non-zero imaginary part. Both of these operations have an optional second argument  $n$  which specifies that the accuracy of the approximation be up to  $n$  decimal places.

When an expression involves no “symbolic application” kernels, it may be possible to convert it a polynomial or rational function in the variables that are present.

```
e2 := cos(x**2 - y + 3)
```

$$\cos(y - x^2 - 3) \quad (16)$$

Type: Expression Integer

```
e3 := asin(e2) - %pi/2
```

$$-y + x^2 + 3 \quad (17)$$

Type: Expression Integer

```
e3 :: Polynomial Integer
```

$$-y + x^2 + 3 \quad (18)$$

Type: Polynomial Integer

This also works for the polynomial types where specific variables and their ordering are given.

```
e3 :: DMP([x, y], Integer)
```

$$x^2 - y + 3 \quad (19)$$

Type: DistributedMultivariatePolynomial([x, y], Integer)

Finally, a certain amount of simplification takes place as expressions are constructed.

```
sin %pi
```

$$0 \quad (20)$$

Type: Expression Integer

```
cos(%pi / 4)
```

$$\frac{\sqrt{2}}{2} \quad (21)$$

Type: Expression Integer

For simplifications that involve multiple terms of the expression, use **simplify**.

```
tan(x)**6 + 3*tan(x)**4 + 3*tan(x)**2 + 1
```

$$\tan(x)^6 + 3 \tan(x)^4 + 3 \tan(x)^2 + 1 \quad (22)$$

Type: Expression Integer

```
simplify %
```

$$\frac{1}{\cos(x)^6} \quad (23)$$

Type: Expression Integer

See Section 6.21 on page 228 for examples of how to write your own rewrite rules for expressions.

## 9.22 Factored

---

Factored creates a domain whose objects are kept in factored form as long as possible. Thus certain operations like “\*” (multiplication) and **gcd** are relatively easy to do. Others, such as addition, require somewhat more work, and the result may not be completely factored unless the argument domain **R** provides a **factor** operation. Each object consists of a unit and a list of factors, where each factor consists of a member of **R** (the *base*), an exponent, and a flag indicating what is known about the base. A flag may be one of “nil”, “sqfr”, “irred” or “prime”, which mean that nothing is known about the base, it is square-free, it is irreducible, or it is prime, respectively. The current restriction to factored objects of integral domains allows simplification to be performed without worrying about multiplication order.

### 9.22.1 Decomposing Factored Objects

---

In this section we will work with a factored integer.

```
g := factor(4312)
```

$$2^3 7^2 11$$

(1)

Type: Factored Integer

Let’s begin by decomposing **g** into pieces. The only possible units for integers are 1 and -1.

```
unit(g)
```

1

(2)

Type: PositiveInteger

There are three factors.

```
numberOfFactors(g)
```

3

(3)

Type: PositiveInteger

We can make a list of the bases, ...

```
[nthFactor(g,i) for i in 1..numberOfFactors(g)]
```

[2, 7, 11]

(4)

Type: List Integer

and the exponents, ...

```
[nthExponent(g,i) for i in 1..numberOfFactors(g)]
```

[3, 2, 1]

(5)

Type: List Integer

and the flags. You can see that all the bases (factors) are prime.

```
[nthFlag(g,i) for i in 1..numberOfFactors(g)]
```

["prime", "prime", "prime"]

(6)

Type: List Union("nil", "sqfr", "irred", "prime")

A useful operation for pulling apart a factored object into a list of records of the components is **factorList**.

```
factorList(g)
[[flg = "prime", fctr = 2, xpnt = 3],
 [flg = "prime", fctr = 7, xpnt = 2],
 [flg = "prime", fctr = 11, xpnt = 1]]
Type: List Record(flg: Union("nil", "sqfr", "irred", "prime"), fctr: Integer, xpnt: Integer)
```

(7)

If you don't care about the flags, use **factors**.

```
factors(g)
[[factor = 2, exponent = 3],
 [factor = 7, exponent = 2],
 [factor = 11, exponent = 1]]
Type: List Record(factor: Integer, exponent: Integer)
```

(8)

Neither of these operations returns the unit.

```
first(%).factor
2
Type: PositiveInteger
```

(9)

### 9.22.2 Expanding Factored Objects

Recall that we are working with this factored integer.

```
g := factor(4312)
2^3 7^2 11
Type: Factored Integer
```

(1)

To multiply out the factors with their multiplicities, use **expand**.

```
expand(g)
4312
Type: PositiveInteger
```

(2)

If you would like, say, the distinct factors multiplied together but with multiplicity one, you could do it this way.

```
reduce(*,[t.factor for t in factors(g)])
154
Type: PositiveInteger
```

(3)

### 9.22.3 Arithmetic with Factored Objects

We're still working with this factored integer.

```
g := factor(4312)
2^3 7^2 11
Type: Factored Integer
```

(1)

We'll also define this factored integer.

```
f := factor(246960)
```

$2^4 3^2 5 7^3$  (2)

Type: Factored Integer

Operations involving multiplication and division are particularly easy with factored objects.

```
f * g
```

$2^7 3^2 5 7^5 11$  (3)

Type: Factored Integer

```
f**500
```

$2^{2000} 3^{1000} 5^{500} 7^{1500}$  (4)

Type: Factored Integer

```
gcd(f, g)
```

$2^3 7^2$  (5)

Type: Factored Integer

```
lcm(f, g)
```

$2^4 3^2 5 7^3 11$  (6)

Type: Factored Integer

If we use addition and subtraction things can slow down because we may need to compute greatest common divisors.

```
f + g
```

$2^3 7^2 641$  (7)

Type: Factored Integer

```
f - g
```

$2^3 7^2 619$  (8)

Type: Factored Integer

Test for equality with 0 and 1 by using **zero?** and **one?**, respectively.

```
zero?(factor(0))
```

true (9)

Type: Boolean

```
zero?(g)
```

false (10)

Type: Boolean

```
one?(factor(1))
```

true (11)

Type: Boolean

```
one? (f)
false
```

(12)

Type: Boolean

Another way to get the zero and one factored objects is to use package calling (see Section 2.9 on page 119).

```
0$Factored(Integer)
0
```

(13)

Type: Factored Integer

```
1$Factored(Integer)
1
```

(14)

Type: Factored Integer

### 9.22.4 Creating New Factored Objects

The **map** operation is used to iterate across the unit and bases of a factored object. See ‘FactoredFunctions2’ on page 419 for a discussion of **map**.

The following four operations take a base and an exponent and create a factored object. They differ in handling the flag component.

```
nilFactor(24,2)
242
```

(1)

Type: Factored Integer

This factor has no associated information.

```
nthFlag(% ,1)
"nil"
```

(2)

Type: Union("nil", ...)

This factor is asserted to be square-free.

```
sqfrFactor(30,2)
302
```

(3)

Type: Factored Integer

This factor is asserted to be irreducible.

```
irreducibleFactor(13,10)
1310
```

(4)

Type: Factored Integer

This factor is asserted to be prime.

```
primeFactor(11,5)
115
```

(5)

Type: Factored Integer

A partial inverse to **factorList** is **makeFR**.

```
h := factor(-720)
-24 32 5
```

(6)

Type: Factored Integer

The first argument is the unit and the second is a list of records as returned by **factorList**.

```
h - makeFR(unit(h),factorList(h))
0
(7)
Type: Factored Integer
```

## 9.22.5 Factored Objects with Variables

Some of the operations available for polynomials are also available for factored polynomials.

```
p := (4*x*x-12*x+9)*y*y + (4*x*x-12*x+9)*y + 28*x*x - 84*x + 63
(4 x^2 - 12 x + 9) y^2 + (4 x^2 - 12 x + 9) y + 28 x^2 - 84 x + 63
(1)
Type: Polynomial Integer
```

```
fp := factor(p)
(2 x - 3)^2 (y^2 + y + 7)
(2)
Type: Factored Polynomial Integer
```

You can differentiate with respect to a variable.

```
D(p,x)
(8 x - 12) y^2 + (8 x - 12) y + 56 x - 84
(3)
Type: Polynomial Integer
```

```
D(fp,x)
4 (2 x - 3) (y^2 + y + 7)
(4)
Type: Factored Polynomial Integer
```

```
numberOfFactors(%)
3
(5)
Type: PositiveInteger
```



## 9.23 Factored- Functions2

---

The FactoredFunctions2 package implements one operation, **map**, for applying an operation to every base in a factored object and to the unit.

```
double(x) == x + x
```

Type: Void

```
f := factor(720)
```

```
24 32 5
```

(2)

Type: Factored Integer

Actually, the **map** operation used in this example comes from Factored itself, since **double** takes an integer argument and returns an integer result.

```
map(double,f)
```

```
Compiling function double with type Integer ->
Integer
```

```
2 44 62 10
```

(3)

Type: Factored Integer

If we want to use an operation that returns an object that has a type different from the operation's argument, the **map** in Factored cannot be used and we use the one in FactoredFunctions2.

```
makePoly(b) == x + b
```

Type: Void

In fact, the “2” in the name of the package means that we might be using factored objects of two different types.

```
g := map(makePoly,f)
```

```
Compiling function makePoly with type Integer ->
Polynomial Integer
```

```
(x + 1) (x + 2)4 (x + 3)2 (x + 5)
```

(5)

Type: Factored Polynomial Integer

It is important to note that both versions of **map** destroy any information known about the bases (the fact that they are prime, for instance).

The flags for each base are set to “nil” in the object returned by **map**.

```
nthFlag(g,1)
```

```
"nil"
```

(6)

Type: Union("nil", ...)

For more information about factored objects and their use, see ‘Factored’ on page 414 and Section 8.13 on page 338.

## 9.24 File

Before working with a file, it must be made accessible to AXIOM with the **open** operation.

The File(S) domain provides a basic interface to read and write values of type S in files.

```
ifile:File List Integer:=open("/tmp/jazz1","output")
"/tmp/jazz1"
(1)
Type: File List Integer
```

The **open** function arguments are a FileName and a String specifying the mode. If a full pathname is not specified, the current default directory is assumed. The mode must be one of "input" or "output". If it is not specified, "input" is assumed. Once the file has been opened, you can read or write data.

The operations **read!** and **write!** are provided.

```
write!(ifile, [-1,2,3])
[-1, 2, 3]
(2)
Type: List Integer
```

```
write!(ifile, [10,-10,0,111])
[10, -10, 0, 111]
(3)
Type: List Integer
```

```
write!(ifile, [7])
[7]
(4)
Type: List Integer
```

You can change from writing to reading (or vice versa) by reopening a file.

```
reopen!(ifile, "input")
"/tmp/jazz1"
(5)
Type: File List Integer
```

```
read! ifile
[-1, 2, 3]
(6)
Type: List Integer
```

```
read! ifile
[10, -10, 0, 111]
(7)
Type: List Integer
```

The **read!** operation can cause an error if one tries to read more data than is in the file. To guard against this possibility the **readIfCan!** operation should be used.

```
readIfCan! ifile
[7]
(8)
Type: Union(List Integer, ...)
```

	<code>readIfCan! ifile</code>	
	<code>"failed"</code>	(9)
		Type: Union("failed", ...)
You can find the current mode of the file, and the file's name.	<code>iomode ifile</code>	
	<code>"input"</code>	(10)
		Type: String
	<code>name ifile</code>	
	<code>"/tmp/jazz1"</code>	(11)
		Type: FileName
When you are finished with a file, you should close it.	<code>close! ifile</code>	
	<code>"/tmp/jazz1"</code>	(12)
		Type: File List Integer
	<code>)system rm /tmp/jazz1</code>	

A limitation of the underlying LISP system is that not all values can be represented in a file. In particular, delayed values containing compiled functions cannot be saved.

For more information on related topics, see 'TextFile' on page 588, 'KeyedAccessFile' on page 460, 'Library' on page 474, and 'FileName' on page 422. Issue the system command `)show File` to display the full list of operations defined by File.

## 9.25 FileName

---

The FileName domain provides an interface to the computer's file system. Functions are provided to manipulate file names and to test properties of files.

The simplest way to use file names in the AXIOM interpreter is to rely on conversion to and from strings. The syntax of these strings depends on the operating system.

fn: FileName

Type: Void

On AIX, this is a proper file syntax:

```
fn := "/spad/src/input/fname.input"
"/spad/src/input/fname.input"
```

(2)

Type: FileName

Although it is very convenient to be able to use string notation for file names in the interpreter, it is desirable to have a portable way of creating and manipulating file names from within programs.

A measure of portability is obtained by considering a file name to consist of three parts: the *directory*, the *name*, and the *extension*.

```
directory fn
"/spad/src/input"
```

(3)

Type: String

```
name fn
"fname"
```

(4)

Type: String

```
extension fn
"input"
```

(5)

Type: String

The meaning of these three parts depends on the operating system. For example, on CMS the file "SPADPROF INPUT M" would have directory "M", name "SPADPROF" and extension "INPUT".

It is possible to create a filename from its parts.

```
fn := filename("/u/smwatt/work", "fname", "input")
"/u/smwatt/work/fname.input"
```

(6)

Type: FileName

When writing programs, it is helpful to refer to directories via variables.

```
objdir := "/tmp"
"/tmp"
```

(7)

Type: String

	<pre>fn := filename(objdir, "table", "spad") "/tmp/table.spad"</pre>	(8)	Type: FileName
If the directory or the extension is given as an empty string, then a default is used. On AIX, the defaults are the current directory and no extension.	<pre>fn := filename("", "letter", "") "letter"</pre>	(9)	Type: FileName
Three tests provide information about names in the file system.			
The <b>exists?</b> operation tests whether the named file exists.	<pre>exists? "/etc/passwd" true</pre>	(10)	Type: Boolean
The operation <b>readable?</b> tells whether the named file can be read. If the file does not exist, then it cannot be read.	<pre>readable? "/etc/passwd" true</pre>	(11)	Type: Boolean
	<pre>readable? "/etc/security/passwd" false</pre>	(12)	Type: Boolean
	<pre>readable? "/ect/passwd" false</pre>	(13)	Type: Boolean
Likewise, the operation <b>writable?</b> tells whether the named file can be written. If the file does not exist, the test is determined by the properties of the directory.	<pre>writable? "/etc/passwd" false</pre>	(14)	Type: Boolean
	<pre>writable? "/dev/null" true</pre>	(15)	Type: Boolean
	<pre>writable? "/etc/DoesNotExist" false</pre>	(16)	Type: Boolean
	<pre>writable? "/tmp/DoesNotExist" true</pre>	(17)	Type: Boolean

The **new** operation constructs the name of a new writable file. The argument sequence is the same as for **filename**, except that the name part is actually a prefix for a constructed unique name.

The resulting file is in the specified directory with the given extension, and the same defaults are used.

```
fn := new(objdir, "xxx", "yy")  
"/tmp/xxx82222.yy"
```

(18)

Type: FileName

## 9.26 FlexibleArray

The FlexibleArray domain constructor creates one-dimensional arrays of elements of the same type. Flexible arrays are an attempt to provide a data type that has the best features of both one-dimensional arrays (fast, random access to elements) and lists (flexibility). They are implemented by a fixed block of storage. When necessary for expansion, a new, larger block of storage is allocated and the elements from the old storage area are copied into the new block.

Flexible arrays have available most of the operations provided by OneDimensionalArray (see ‘OneDimensionalArray’ on page 514 and ‘Vector’ on page 601). Since flexible arrays are also of category ExtensibleLinearAggregate, they have operations **concat!**, **delete!**, **insert!**, **merge!**, **remove!**, **removeDuplicates!**, and **select!**. In addition, the operations **physicalLength** and **physicalLength!** provide user-control over expansion and contraction.

A convenient way to create a flexible array is to apply the operation **flexibleArray** to a list of values.

```
flexibleArray [i for i in 1..6]
[1, 2, 3, 4, 5, 6]
(1)
```

Type: FlexibleArray PositivelInteger

Create a flexible array of six zeroes.

```
f : FARRAY INT := new(6,0)
[0, 0, 0, 0, 0, 0]
(2)
```

Type: FlexibleArray Integer

For  $i = 1 \dots 6$ , set the  $i^{\text{th}}$  element to  $i$ . Display **f**.

```
for i in 1..6 repeat f.i := i; f
[1, 2, 3, 4, 5, 6]
(3)
```

Type: FlexibleArray Integer

Initially, the physical length is the same as the number of elements.

```
physicalLength f
6
(4)
```

Type: PositivelInteger

Add an element to the end of **f**.

```
concat!(f,11)
[1, 2, 3, 4, 5, 6, 11]
(5)
```

Type: FlexibleArray Integer

See that its physical length has grown.

```
physicalLength f
10
(6)
```

Type: PositivelInteger

Make **f** grow to have room for 15 elements.

```
physicalLength!(f,15)
[1, 2, 3, 4, 5, 6, 11]
(7)
```

Type: FlexibleArray Integer

Concatenate the elements of <b>f</b> to itself. The physical length allows room for three more values at the end.	<code>concat!(f,f)</code> [1, 2, 3, 4, 5, 6, 11, 1, 2, 3, 4, 5, 6, 11]	(8) Type: FlexibleArray Integer
Use <b>insert!</b> to add an element to the front of a flexible array.	<code>insert!(22,f,1)</code> [22, 1, 2, 3, 4, 5, 6, 11, 1, 2, 3, 4, 5, 6, 11]	(9) Type: FlexibleArray Integer
Create a second flexible array from <b>f</b> consisting of the elements from index 10 forward.	<code>g := f(10..)</code> [2, 3, 4, 5, 6, 11]	(10) Type: FlexibleArray Integer
Insert this array at the front of <b>f</b> .	<code>insert!(g,f,1)</code> [2, 3, 4, 5, 6, 11, 22, 1, 2, 3, 4, 5, 6, 11, 1, 2, 3, 4, 5, 6, 11]	(11) Type: FlexibleArray Integer
Merge the flexible array <b>f</b> into <b>g</b> after sorting each in place.	<code>merge!(sort! f, sort! g)</code> [1, 1, 2, 2, 2, 2, 3, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 6, 6, 6, 6, 11, 11, 11, 11, 22]	(12) Type: FlexibleArray Integer
Remove duplicates in place.	<code>removeDuplicates! f</code> [1, 2, 3, 4, 5, 6, 11, 22]	(13) Type: FlexibleArray Integer
Remove all odd integers.	<code>select!(i +-&gt; even? i,f)</code> [2, 4, 6, 22]	(14) Type: FlexibleArray Integer
All these operations have shrunk the physical length of <b>f</b> .	<code>physicalLength f</code> 8	(15) Type: PositiveInteger
To force AXIOM not to shrink flexible arrays call the <b>shrinkable</b> operation with the argument <b>false</b> . You must package call this operation. The previous value is returned.	<code>shrinkable(false)\$FlexibleArray(Integer)</code> <code>true</code>	(16) Type: Boolean



## 9.27 Float

---

AXIOM provides two kinds of floating point numbers. The domain Float (abbreviation FLOAT) implements a model of arbitrary precision floating point numbers. The domain DoubleFloat (abbreviation DFLOAT) is intended to make available hardware floating point arithmetic in AXIOM. The actual model of floating point that DoubleFloat provides is system-dependent. For example, on the IBM system 370 AXIOM uses IBM double precision which has fourteen hexadecimal digits of precision or roughly sixteen decimal digits. Arbitrary precision floats allow the user to specify the precision at which arithmetic operations are computed. Although this is an attractive facility, it comes at a cost. Arbitrary-precision floating-point arithmetic typically takes twenty to two hundred times more time than hardware floating point.

For more information about AXIOM's numeric and graphic facilities, see Section 7 on page 235, Section 8.1 on page 264, and 'DoubleFloat' on page 404.

### 9.27.1 Introduction to Float

Scientific notation is supported for input and output of floating point numbers. A floating point number is written as a string of digits containing a decimal point optionally followed by the letter "E", and then the exponent.

We begin by doing some calculations using arbitrary precision floats. The default precision is twenty decimal digits.

`1.234`

`1.234`

(1)

Type: Float

A decimal base for the exponent is assumed, so the number `1.234E2` denotes  $1.234 \cdot 10^2$ .

`1.234E2`

`123.4`

(2)

Type: Float

The normal arithmetic operations are available for floating point numbers.

`sqrt(1.2 + 2.3 / 3.4 ** 4.5)`

`1.0996972790671286226`

(3)

Type: Float

### 9.27.2 Conversion Functions

---

You can use conversion (Section 2.7 on page 113) to go back and forth between Integer, Fraction Integer and Float, as appropriate.

`i := 3 :: Float`

`3.0`

(1)

Type: Float

	<pre>i :: Integer</pre> $3$	(2)	Type: Integer
	<pre>i :: Fraction Integer</pre> $3$	(3)	Type: Fraction Integer
Since you are explicitly asking for a conversion, you must take responsibility for any loss of exactness.	<pre>r := 3/7 :: Float</pre> $0.42857142857142857143$	(4)	Type: Float
	<pre>r :: Fraction Integer</pre> $\frac{3}{7}$	(5)	Type: Fraction Integer
This conversion cannot be performed: use <b>truncate</b> or <b>round</b> if that is what you intend.	<pre>r :: Integer</pre> <p>Cannot convert from type Float to Integer for value 0.4285714285 7142857143</p>		
The operations <b>truncate</b> and <b>round</b> truncate ...	<pre>truncate 3.6</pre> $3.0$	(6)	Type: Float
and round to the nearest integral Float respectively.	<pre>round 3.6</pre> $4.0$	(7)	Type: Float
	<pre>truncate(-3.6)</pre> $-3.0$	(8)	Type: Float
	<pre>round(-3.6)</pre> $-4.0$	(9)	Type: Float
The operation <b>fractionPart</b> computes the fractional part of <b>x</b> , that is, <b>x - truncate x</b> .	<pre>fractionPart 3.6</pre> $0.6$	(10)	Type: Float

The operation **digits** allows the user to set the precision. It returns the previous value it was using.

```
digits 40
20
```

(11)  
Type: PositiveInteger

```
sqrt 0.2
0.4472135954999579392818347337462552470881
```

(12)  
Type: Float

```
pi()$Float
3.141592653589793238462643383279502884197
```

(13)  
Type: Float

The precision is only limited by the computer memory available. Calculations at 500 or more digits of precision are not difficult.

```
digits 500
40
```

(14)  
Type: PositiveInteger

```
pi()$Float
3.1415926535897932384626433832795028841971693993751058
209749445923078164062862089986280348253421170679821480
865132823066470938446095505822317253594081284811174502
841027019385211055596446229489549303819644288109756659
334461284756482337867831652712019091456485669234603486
104543266482133936072602491412737245870066063155881748
815209209628292540917153643678925903600113305305488204
665213841469519415116094330572703657595919530921861173
819326117931051185480744623799627495673518857527248912
279381830119491
```

(15)  
Type: Float

Reset **digits** to its default value.

```
digits 20
500
```

(16)  
Type: PositiveInteger

Numbers of type Float are represented as a record of two integers, namely, the mantissa and the exponent where the base of the exponent is binary. That is, the floating point number  $(m, e)$  represents the number  $m \cdot 2^e$ . A consequence of using a binary base is that decimal numbers can not, in general, be represented exactly.

### 9.27.3 Output Functions

Output spacing can be modified with the **outputSpacing** operation. This inserts no spaces and then displays the value of **x**.

Issue this to have the spaces inserted every 5 digits.

By default, the system displays floats in either fixed format or scientific format, depending on the magnitude of the number.

A particular format may be requested with the operations **outputFloating** and **outputFixed**.

Additionally, you can ask for **n** digits to be displayed after the decimal point.

This resets the output printing to the default behavior.

A number of operations exist for specifying how numbers of type **Float** are to be displayed. By default, spaces are inserted every ten digits in the output for readability.<sup>3</sup>

```
outputSpacing 0; x := sqrt 0.2
```

```
(1) 0.44721359549995793928 (0)
Type: Float
```

```
outputSpacing 5; x
```

```
(2) 0.44721 35954 99957 93928 (0)
Type: Float
```

```
y := x/10**10
```

```
(3) 0.44721 35954 99957 93928 E -10 (0)
Type: Float
```

```
outputFloating(); x
```

```
(4) 0.44721 35954 99957 93928 E 0 (0)
Type: Float
```

```
outputFixed(); y
```

```
(5) 0.00000 00000 44721 35954 99957 93928 (0)
Type: Float
```

```
outputFloating 2; y
```

```
(6) 0.45 E -10 (0)
Type: Float
```

```
outputFixed 2; x
```

```
(7) 0.45 (0)
Type: Float
```

```
outputGeneral()
```

Type: Void

---

<sup>3</sup>Note that you cannot include spaces in the input form of a floating point number, though you can use underscores.

### 9.27.4 An Example: Determinant of a Hilbert Matrix

First do the computation using rational numbers to obtain the exact result.

Consider the problem of computing the determinant of a 10 by 10 Hilbert matrix. The  $(i,j)^{\text{th}}$  entry of a Hilbert matrix is given by  $1/(i+j+1)$ .

```
a: Matrix Fraction Integer := matrix [[1/(i+j+1) for j in
0..9] for i in 0..9]
```

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10} & \frac{1}{11} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10} & \frac{1}{11} & \frac{1}{12} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10} & \frac{1}{11} & \frac{1}{12} & \frac{1}{13} \\ \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10} & \frac{1}{11} & \frac{1}{12} & \frac{1}{13} & \frac{1}{14} \\ \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10} & \frac{1}{11} & \frac{1}{12} & \frac{1}{13} & \frac{1}{14} & \frac{1}{15} \\ \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10} & \frac{1}{11} & \frac{1}{12} & \frac{1}{13} & \frac{1}{14} & \frac{1}{15} & \frac{1}{16} \\ \frac{1}{8} & \frac{1}{9} & \frac{1}{10} & \frac{1}{11} & \frac{1}{12} & \frac{1}{13} & \frac{1}{14} & \frac{1}{15} & \frac{1}{16} & \frac{1}{17} \\ \frac{1}{9} & \frac{1}{10} & \frac{1}{11} & \frac{1}{12} & \frac{1}{13} & \frac{1}{14} & \frac{1}{15} & \frac{1}{16} & \frac{1}{17} & \frac{1}{18} \\ \frac{1}{10} & \frac{1}{11} & \frac{1}{12} & \frac{1}{13} & \frac{1}{14} & \frac{1}{15} & \frac{1}{16} & \frac{1}{17} & \frac{1}{18} & \frac{1}{19} \end{bmatrix}$$

(1)

Type: Matrix Fraction Integer

This version of **determinant** uses Gaussian elimination.

```
d:= determinant a
```

$$\frac{1}{46206893947914691316295628839036278726983680000000000}$$

(2)

Type: Fraction Integer

```
d :: Float
```

$$0.21641792264314918691E - 52$$

(3)

Type: Float

Now use hardware floats. Note that a semicolon (;) is used to prevent the display of the matrix.

```
b: Matrix DoubleFloat := matrix [[1/(i+j+1$DoubleFloat)
for j in 0..9] for i in 0..9];
```

(4)

Type: Matrix DoubleFloat

The result given by hardware floats is correct only to four significant digits of precision. In the jargon of numerical analysis, the Hilbert matrix is said to be “ill-conditioned.”

```
determinant b
```

$$2.1643677945721411e - 53$$

(5)

Type: DoubleFloat

Now repeat the computation at a higher precision using Float.

```
digits 40
```

$$20$$

(6)

Type: PositiveInteger

```
c: Matrix Float := matrix [[1/(i+j+1$Float) for j in 0..9]
  for i in 0..9];
```

(7)

Type: Matrix Float

```
determinant c
```

$0.2164179226431491869060594983622617436159E - 52$  (8)

Type: Float

Reset **digits** to its default value. `digits 20`

40 (9)

Type: PositiveInteger

## 9.28 Fraction

---

The Fraction domain implements quotients. The elements must belong to a domain of category `IntegralDomain`: multiplication must be commutative and the product of two non-zero elements must not be zero. This allows you to make fractions of most things you would think of, but don't expect to create a fraction of two matrices! The abbreviation for Fraction is `FRAC`.

Use “/” to create a fraction.

```
a := 11/12
```

$$\frac{11}{12}$$

(1)

Type: Fraction Integer

```
b := 23/24
```

$$\frac{23}{24}$$

(2)

Type: Fraction Integer

The standard arithmetic operations are available.

```
3 - a*b**2 + a + b/a
```

$$\frac{313271}{76032}$$

(3)

Type: Fraction Integer

Extract the numerator and denominator by using **numer** and **denom**, respectively.

```
numer(a)
```

$$11$$

(4)

Type: PositiveInteger

```
denom(b)
```

$$24$$

(5)

Type: PositiveInteger

Operations like **max**, **min**, **negative?**, **positive?** and **zero?** are all available if they are provided for the numerators and denominators. See ‘Integer’ on page 445 for examples.

Don't expect a useful answer from **factor**, **gcd** or **lcm** if you apply them to fractions.

```
r := (x**2 + 2*x + 1)/(x**2 - 2*x + 1)
```

$$\frac{x^2 + 2x + 1}{x^2 - 2x + 1}$$

(6)

Type: Fraction Polynomial Integer

Since all non-zero fractions are invertible, these operations have trivial definitions.

```
factor(r)
```

$$\frac{x^2 + 2x + 1}{x^2 - 2x + 1}$$

(7)

Type: Factored Fraction Polynomial Integer

Use <b>map</b> to apply <b>factor</b> to the numerator and denominator, which is probably what you mean.	<pre>map(factor,r)</pre> $\frac{(x+1)^2}{(x-1)^2}$ <div>Type: Fraction Factored Polynomial Integer</div>	(8)
Other forms of fractions are available. Use <b>continuedFraction</b> to create a continued fraction.	<pre>continuedFraction(7/12)</pre> $\frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \frac{1}{2}$ <div>Type: ContinuedFraction Integer</div>	(9)
Use <b>partialFraction</b> to create a partial fraction. See ‘ContinuedFraction’ on page 385 and ‘PartialFraction’ on page 525 for additional information and examples.	<pre>partialFraction(7,12)</pre> $1 - \frac{3}{2^2} + \frac{1}{3}$ <div>Type: PartialFraction Integer</div>	(10)
Use conversion to create alternative views of fractions with objects moved in and out of the numerator and denominator.	<pre>g := 2/3 + 4/5*%i</pre> $\frac{2}{3} + \frac{4}{5}i$ <div>Type: Complex Fraction Integer</div>	(11)
Conversion is discussed in detail in Section 2.7 on page 113.	<pre>g :: FRAC COMPLEX INT</pre> $\frac{10 + 12i}{15}$ <div>Type: Fraction Complex Integer</div>	(12)



## 9.29 FullPartial- Fraction- Expansion

Our examples will all involve quotients of univariate polynomials with rational number coefficients.

Here is a simple-looking rational function.

We use **fullPartialFraction** to convert it to an object of type **FullPartialFractionExpansion**.

Use a coercion to change it back into a quotient.

Full partial fractions differentiate faster than rational functions.

The domain **FullPartialFractionExpansion** implements factor-free conversion of quotients to full partial fractions.

**Fx** := **FRAC UP**(**x**, **FRAC INT**)

Fraction UnivariatePolynomial (**x**, Fraction Integer) (1)

Type: Domain

**f** : **Fx** := 36 / (**x**\*\*5-2\***x**\*\*4-2\***x**\*\*3+4\***x**\*\*2+**x**-2)

$$\frac{36}{x^5 - 2x^4 - 2x^3 + 4x^2 + x - 2} \quad (2)$$

Type: Fraction UnivariatePolynomial(x, Fraction Integer)

**g** := **fullPartialFraction f**

$$\frac{4}{x-2} - \frac{4}{x+1} + \sum_{\%A^2-1=0} \frac{-3\%A-6}{(x-\%A)^2} \quad (3)$$

Type: FullPartialFractionExpansion(Fraction Integer, UnivariatePolynomial(x, Fraction Integer))

**g** :: **Fx**

$$\frac{36}{x^5 - 2x^4 - 2x^3 + 4x^2 + x - 2} \quad (4)$$

Type: Fraction UnivariatePolynomial(x, Fraction Integer)

**g5** := **D**(**g**, 5)

$$-\frac{480}{(x-2)^6} + \frac{480}{(x+1)^6} + \sum_{\%A^2-1=0} \frac{2160\%A+4320}{(x-\%A)^7} \quad (5)$$

Type: FullPartialFractionExpansion(Fraction Integer, UnivariatePolynomial(x, Fraction Integer))

**f5** := **D**(**f**, 5)

$$\frac{\begin{pmatrix} -544320 x^{10} + 4354560 x^9 - 14696640 x^8 + 28615680 x^7 \\ -40085280 x^6 + 46656000 x^5 - 39411360 x^4 + 18247680 x^3 \\ -5870880 x^2 + 3317760 x + 246240 \end{pmatrix}}{\begin{pmatrix} x^{20} - 12 x^{19} + 53 x^{18} - 76 x^{17} - 159 x^{16} + 676 x^{15} - 391 x^{14} \\ -1596 x^{13} + 2527 x^{12} + 1148 x^{11} - 4977 x^{10} + 1372 x^9 + 4907 x^8 \\ -3444 x^7 - 2381 x^6 + 2924 x^5 + 276 x^4 - 1184 x^3 + 208 x^2 \\ + 192 x - 64 \end{pmatrix}} \quad (6)$$

Type: Fraction UnivariatePolynomial(x, Fraction Integer)

We can check that the two forms  
represent the same function.

$$\begin{aligned} & \text{g5}::\text{Fx} - \text{f5} \\ & 0 \end{aligned} \tag{7}$$

Type: Fraction UnivariatePolynomial(x, Fraction Integer)

Here are some examples that are  
more complicated.

$$\begin{aligned} \text{f} : \text{Fx} &:= (\mathbf{x}^{**5} * (\mathbf{x}-1)) / ((\mathbf{x}^{**2} + \mathbf{x} + 1)^{**2} * (\mathbf{x}-2)^{**3}) \\ & \frac{x^6 - x^5}{x^7 - 4x^6 + 3x^5 + 9x^3 - 6x^2 - 4x - 8} \end{aligned} \tag{8}$$

Type: Fraction UnivariatePolynomial(x, Fraction Integer)

$$\begin{aligned} \text{g} &:= \text{fullPartialFraction f} \\ & \frac{\frac{1952}{2401}}{x-2} + \frac{\frac{464}{343}}{(x-2)^2} + \frac{\frac{32}{49}}{(x-2)^3} + \\ & \sum \frac{-\frac{179}{2401} \%A + \frac{135}{2401}}{x - \%A} + \\ \%A^2 + \%A + 1 &= 0 \end{aligned} \tag{9}$$

$$\begin{aligned} & \sum \frac{\frac{37}{1029} \%A + \frac{20}{1029}}{(x - \%A)^2} \\ \%A^2 + \%A + 1 &= 0 \end{aligned}$$

Type: FullPartialFractionExpansion(Fraction Integer, UnivariatePolynomial(x, Fraction Integer))

$$\begin{aligned} \text{g} &:: \text{Fx} - \text{f} \\ & 0 \end{aligned} \tag{10}$$

Type: Fraction UnivariatePolynomial(x, Fraction Integer)

$$\begin{aligned} \text{f} : \text{Fx} &:= (2*\mathbf{x}^{**7}-7*\mathbf{x}^{**5}+26*\mathbf{x}^{**3}+8*\mathbf{x}) / (\mathbf{x}^{**8}- \\ & 5*\mathbf{x}^{**6}+6*\mathbf{x}^{**4}+4*\mathbf{x}^{**2}-8) \\ & \frac{2x^7 - 7x^5 + 26x^3 + 8x}{x^8 - 5x^6 + 6x^4 + 4x^2 - 8} \end{aligned} \tag{11}$$

Type: Fraction UnivariatePolynomial(x, Fraction Integer)

$$\begin{aligned} \text{g} &:= \text{fullPartialFraction f} \\ & \sum \frac{\frac{1}{2}}{x - \%A} + \sum \frac{1}{(\%A^2 - 2)^3} + \\ \%A^2 - 2 &= 0 \end{aligned} \tag{12}$$

$$\begin{aligned} & \sum \frac{\frac{1}{2}}{x - \%A} \\ \%A^2 + 1 &= 0 \end{aligned}$$

Type: FullPartialFractionExpansion(Fraction Integer, UnivariatePolynomial(x, Fraction Integer))

g :: Fx - f

0

(13)

Type: Fraction UnivariatePolynomial(x, Fraction Integer)

f:Fx := x\*\*3 / (x\*\*21 + 2\*x\*\*20 + 4\*x\*\*19 + 7\*x\*\*18 + 10\*x\*\*17 + 17\*x\*\*16 + 22\*x\*\*15 + 30\*x\*\*14 + 36\*x\*\*13 + 40\*x\*\*12 + 47\*x\*\*11 + 46\*x\*\*10 + 49\*x\*\*9 + 43\*x\*\*8 + 38\*x\*\*7 + 32\*x\*\*6 + 23\*x\*\*5 + 19\*x\*\*4 + 10\*x\*\*3 + 7\*x\*\*2 + 2\*x + 1)

$$\frac{x^3}{\left( \begin{array}{l} x^{21} + 2 x^{20} + 4 x^{19} + 7 x^{18} + 10 x^{17} + 17 x^{16} + 22 x^{15} + \\ 30 x^{14} + 36 x^{13} + 40 x^{12} + 47 x^{11} + 46 x^{10} + 49 x^9 + 43 x^8 + \\ 38 x^7 + 32 x^6 + 23 x^5 + 19 x^4 + 10 x^3 + 7 x^2 + 2 x + 1 \end{array} \right)} \quad (14)$$

Type: Fraction UnivariatePolynomial(x, Fraction Integer)

g := fullPartialFraction f

$$\begin{aligned} & \sum_{\%A^2 + 1 = 0} \frac{\frac{1}{2} \%A}{x - \%A} + \sum_{\%A^2 + \%A + 1 = 0} \frac{\frac{1}{9} \%A - \frac{19}{27}}{x - \%A} + \\ & \sum_{\%A^2 + \%A + 1 = 0} \frac{\frac{1}{27} \%A - \frac{1}{27}}{(x - \%A)^2} + \\ & \sum_{\%A^5 + \%A^2 + 1 = 0} \frac{\left( -\frac{96556567040}{912390759099} \%A^4 + \frac{420961732891}{912390759099} \%A^3 \right.}{x - \%A} + \\ & \quad \left. -\frac{59101056149}{912390759099} \%A^2 - \frac{373545875923}{912390759099} \%A \right) \\ & \quad + \frac{529673492498}{912390759099} \\ & \sum_{\%A^5 + \%A^2 + 1 = 0} \frac{\left( -\frac{5580868}{94070601} \%A^4 - \frac{2024443}{94070601} \%A^3 \right.}{(x - \%A)^2} + \\ & \quad \left. + \frac{4321919}{94070601} \%A^2 - \frac{84614}{1542141} \%A \right) \\ & \sum_{\%A^5 + \%A^2 + 1 = 0} \frac{\left( -\frac{5070620}{94070601} \right)}{(x - \%A)^2} + \\ & \sum_{\%A^5 + \%A^2 + 1 = 0} \frac{\left( \frac{1610957}{94070601} \%A^4 + \frac{2763014}{94070601} \%A^3 - \right.}{(x - \%A)^3} \\ & \quad \left. \frac{2016775}{94070601} \%A^2 + \frac{266953}{94070601} \%A + \frac{4529359}{94070601} \right) \end{aligned} \quad (15)$$

Type: FullPartialFractionExpansion(Fraction Integer, UnivariatePolynomial(x, Fraction Integer))

This verification takes much longer than the conversion to partial fractions.

$$g :: \frac{f}{0} \quad \text{Type: Fraction UnivariatePolynomial(x, Fraction Integer)} \quad (16)$$

For more information, see the paper: Bronstein, M and Salvy, B. "Full Partial Fraction Decomposition of Rational Functions," *Proceedings of ISSAC'93, Kiev*, ACM Press. All see 'PartialFraction' on page 525 for standard partial fraction decompositions.

## 9.30 GeneralSparse- Table

---

Sometimes when working with tables there is a natural value to use as the entry in all but a few cases. The `GeneralSparseTable` constructor can be used to provide any table type with a default value for entries. See ‘Table’ on page 585 for general information about tables. Issue the system command `)show GeneralSparseTable` to display the full list of operations defined by `GeneralSparseTable`.

Suppose we launched a fund-raising campaign to raise fifty thousand dollars. To record the contributions, we want a table with strings as keys (for the names) and integer entries (for the amount). In a data base of cash contributions, unless someone has been explicitly entered, it is reasonable to assume they have made a zero dollar contribution.

This creates a keyed access file with default entry 0.

```
patrons: GeneralSparseTable(String, Integer,  
    KeyedAccessFile(Integer), 0) := table() ;
```

(1)

Type: GeneralSparseTable(String, Integer, KeyedAccessFile Integer, 0)

Now `patrons` can be used just as any other table. Here we record two gifts.

```
patrons."Smith" := 10500  
10500
```

(2)

Type: PositiveInteger

```
patrons."Jones" := 22000  
22000
```

(3)

Type: PositiveInteger

Now let us look up the size of the contributions from Jones and Stingy.

```
patrons."Jones"  
22000
```

(4)

Type: PositiveInteger

```
patrons."Stingy"  
0
```

(5)

Type: NonNegativeInteger

Have we met our seventy thousand dollar goal?

```
reduce(+, entries patrons)  
32500
```

(6)

Type: PositiveInteger

So the project is cancelled and we can delete the data base:

```
)system rm -r kaf*.sdata
```

### 9.31 Groebner- Factorization- Package

---

Solving systems of polynomial equations with the Gröbner basis algorithm can often be very time consuming because, in general, the algorithm has exponential run-time. These systems, which often come from concrete applications, frequently have symmetries which are not taken advantage of by the algorithm. However, it often happens in this case that the polynomials which occur during the Gröbner calculations are reducible. Since AXIOM has an excellent polynomial factorization algorithm, it is very natural to combine the Gröbner and factorization algorithms.

GroebnerFactorizationPackage exports the **groebnerFactorize** operation which implements a modified Gröbner basis algorithm. In this algorithm, each polynomial that is to be put into the partial list of the basis is first factored. The remaining calculation is split into as many parts as there are irreducible factors. Call these factors  $p_1, \dots, p_n$ . In the branches corresponding to  $p_2, \dots, p_n$ , the factor  $p_1$  can be divided out, and so on. This package also contains operations that allow you to specify the polynomials that are not zero on the common roots of the final Gröbner basis.

Here is an example from chemistry. In a theoretical model of the cyclohexan  $C_6H_{12}$ , the six carbon atoms each sit in the center of gravity of a tetrahedron that has two hydrogen atoms and two carbon atoms at its corners. We first normalize and set the length of each edge to 1. Hence, the distances of one fixed carbon atom to each of its immediate neighbours is 1. We will denote the distances to the other three carbon atoms by  $x$ ,  $y$  and  $z$ .

A. Dress developed a theory to decide whether a set of points and distances between them can be realized in an  $n$ -dimensional space. Here, of course, we have  $n = 3$ .

```
mfzn : SQMATRIX(6,DMP([x,y,z],Fraction INT)) :=
  [[0,1,1,1,1,1], [1,0,1,8/3,x,8/3], [1,1,0,1,8/3,y],
   [1,8/3,1,0,1,8/3], [1,x,8/3,1,0,1], [1,8/3,y,8/3,1,0]]
```

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & \frac{8}{3} & x & \frac{8}{3} \\ 1 & 1 & 0 & 1 & \frac{8}{3} & y \\ 1 & \frac{8}{3} & 1 & 0 & 1 & \frac{8}{3} \\ 1 & x & \frac{8}{3} & 1 & 0 & 1 \\ 1 & \frac{8}{3} & y & \frac{8}{3} & 1 & 0 \end{bmatrix} \quad (1)$$

```
Type: SquareMatrix(6, DistributedMultivariatePolynomial([x, y, z], Fraction
Integer))
```

For the cyclohexan, the distances have to satisfy this equation.

$$\begin{aligned} \text{eq} &:= \text{determinant mfzn} \\ &-x^2 y^2 + \frac{22}{3} x^2 y - \frac{25}{9} x^2 + \frac{22}{3} x y^2 - \frac{388}{9} x y - \frac{250}{27} x \\ &-\frac{25}{9} y^2 - \frac{250}{27} y + \frac{14575}{81} \end{aligned} \quad (2)$$

Type: DistributedMultivariatePolynomial([x, y, z], Fraction Integer)

They also must satisfy the equations given by cyclic shifts of the indeterminates.

$$\begin{aligned} &\text{groebnerFactorize} [\text{eq}, \text{eval}(\text{eq}, [\mathbf{x}, \mathbf{y}, \mathbf{z}], [\mathbf{y}, \mathbf{z}, \mathbf{x}]), \\ &\quad \text{eval}(\text{eq}, [\mathbf{x}, \mathbf{y}, \mathbf{z}], [\mathbf{z}, \mathbf{x}, \mathbf{y}])] \\ &\left[ \left[ x y + x z - \frac{22}{3} x + y z - \frac{22}{3} y - \frac{22}{3} z + \frac{121}{3}, \right. \right. \\ &\quad x z^2 - \frac{22}{3} x z + \frac{25}{9} x + y z^2 - \frac{22}{3} y z + \frac{25}{9} y - \frac{22}{3} z^2 + \\ &\quad \frac{388}{9} z + \frac{250}{27}, \\ &\quad y^2 z^2 - \frac{22}{3} y^2 z + \frac{25}{9} y^2 - \frac{22}{3} y z^2 + \frac{388}{9} y z + \frac{250}{27} y + \\ &\quad \left. \frac{25}{9} z^2 + \frac{250}{27} z - \frac{14575}{81} \right], \quad (3) \\ &\left[ x + y - \frac{21994}{5625}, y^2 - \frac{21994}{5625} y + \frac{4427}{675}, z - \frac{463}{87} \right], \\ &\left[ x^2 - \frac{1}{2} x z - \frac{11}{2} x - \frac{5}{6} z + \frac{265}{18}, y - z, z^2 - \frac{38}{3} z + \frac{265}{9} \right], \\ &\left[ x - \frac{25}{9}, y - \frac{11}{3}, z - \frac{11}{3} \right], \left[ x - \frac{11}{3}, y - \frac{11}{3}, z - \frac{11}{3} \right], \\ &\left[ x + \frac{5}{3}, y + \frac{5}{3}, z + \frac{5}{3} \right], \left[ x - \frac{19}{3}, y + \frac{5}{3}, z + \frac{5}{3} \right] \end{aligned}$$

Type: List List DistributedMultivariatePolynomial([x, y, z], Fraction Integer)

The union of the solutions of this list is the solution of our original problem. If we impose positivity conditions, we get two relevant ideals. One ideal is zero-dimensional, namely  $x = y = z = 11/3$ , and this determines the “boat” form of the cyclohexan. The other ideal is one-dimensional, which means that we have a solution space given by one parameter. This gives the “chair” form of the cyclohexan. The parameter describes the angle of the “back of the chair.”

**groebnerFactorize** has an optional Boolean-valued second argument.

When it is `true` partial results are displayed, since it may happen that the calculation does not terminate in a reasonable time. See the source code for `GroebnerFactorizationPackage` in **groebf.spad** for more details about the algorithms used.



## 9.32 Heap

---

The domain `Heap(S)` implements a priority queue of objects of type `S` such that the operation **extract!** removes and returns the maximum element. The implementation represents heaps as flexible arrays (see ‘FlexibleArray’ on page 425). The representation and algorithms give complexity of  $O(\log(n))$  for insertion and extractions, and  $O(n)$  for construction.

Create a heap of six elements.	<pre>h := heap [-4,9,11,2,7,-7]</pre> <pre>[11, 7, 9, -4, 2, -7]</pre> <div style="text-align: right;">(1)</div> <div style="text-align: right;">Type: Heap Integer</div>
Use <b>insert!</b> to add an element.	<pre>insert!(3,h)</pre> <pre>[11, 7, 9, -4, 2, -7, 3]</pre> <div style="text-align: right;">(2)</div> <div style="text-align: right;">Type: Heap Integer</div>
The operation <b>extract!</b> removes and returns the maximum element.	<pre>extract! h</pre> <pre>11</pre> <div style="text-align: right;">(3)</div> <div style="text-align: right;">Type: PositiveInteger</div>
The internal structure of <code>h</code> has been appropriately adjusted.	<pre>h</pre> <pre>[9, 7, 3, -4, 2, -7]</pre> <div style="text-align: right;">(4)</div> <div style="text-align: right;">Type: Heap Integer</div>
Now <b>extract!</b> elements repeatedly until none are left, collecting the elements in a list.	<pre>[extract!(h) while not empty?(h)]</pre> <pre>[9, 7, 3, 2, -4, -7]</pre> <div style="text-align: right;">(5)</div> <div style="text-align: right;">Type: List Integer</div>
Another way to produce the same result is by defining a <b>heapsort</b> function.	<pre>heapsort(x) == (empty? x =&gt; []; cons(extract!(x),heapsort x))</pre> <div style="text-align: right;">Type: Void</div>
Create another sample heap.	<pre>h1 := heap [17,-4,9,-11,2,7,-7]</pre> <pre>[17, 2, 9, -11, -4, 7, -7]</pre> <div style="text-align: right;">(7)</div> <div style="text-align: right;">Type: Heap Integer</div>
Apply <b>heapsort</b> to present elements in order.	<pre>heapsort h1</pre> <pre>Compiling function heapsort with type Heap Integer</pre> <pre>-&gt; List Integer</pre> <pre>[17, 9, 7, 2, -4, -7, -11]</pre> <div style="text-align: right;">(8)</div> <div style="text-align: right;">Type: List Integer</div>

### 9.33 Hexadecimal- Expansion

---

All rationals have repeating hexadecimal expansions. The operation **hex** returns these expansions of type `HexadecimalExpansion`. Operations to access the individual numerals of a hexadecimal expansion can be obtained by converting the value to `RadixExpansion(16)`. More examples of expansions are available in the ‘`DecimalExpansion`’ on page 401, ‘`BinaryExpansion`’ on page 359, and ‘`RadixExpansion`’ on page 537.

Issue the system command `)show HexadecimalExpansion` to display the full list of operations defined by `HexadecimalExpansion`.

This is a hexadecimal expansion of a rational number.

```
r := hex(22/7)
3.249
(1)
```

Type: HexadecimalExpansion

Arithmetic is exact.

```
r + hex(6/7)
4
(2)
```

Type: HexadecimalExpansion

The period of the expansion can be short or long ...

```
[hex(1/i) for i in 350..354]
[0.00BB3EE721A54D88, 0.00BAB 6561, 0.00BA2E8,
0.00B9A7862A0FF465879D5F,
0.00B92143FA36F5E02E4850FE8DBD78]
(3)
```

Type: List HexadecimalExpansion

or very long!

```
hex(1/1007)
0.0041149783F0BF2C7D13933192AF6980619EE345E91
EC2BB9D5CCA5C071E40926E54E8DDAE24196C0B2F8A0
AAD60DBA57F5D4C8536262210C74F1
(4)
```

Type: HexadecimalExpansion

These numbers are bona fide algebraic objects.

```
p := hex(1/4)*x**2 + hex(2/3)*x + hex(4/9)
0.4 x^2 + 0.A x + 0.71C
(5)
```

Type: Polynomial HexadecimalExpansion

```
q := D(p, x)
0.8 x + 0.A
(6)
```

Type: Polynomial HexadecimalExpansion

```
g := gcd(p, q)
x + 1.5
(7)
```

Type: Polynomial HexadecimalExpansion

## 9.34 Integer

---

AXIOM provides many operations for manipulating arbitrary precision integers. In this section we will show some of those that come from Integer itself plus some that are implemented in other packages. More examples of using integers are in the following sections: ‘Some Numbers’ in Section 1.5 on page 56, ‘IntegerNumberTheoryFunctions’ on page 453, ‘DecimalExpansion’ on page 401, ‘BinaryExpansion’ on page 359, ‘HexadecimalExpansion’ on page 444, and ‘RadixExpansion’ on page 537.

### 9.34.1 Basic Functions

---

The size of an integer in AXIOM is only limited by the amount of computer storage you have available. The usual arithmetic operations are available.

```
2**(5678 - 4856 + 2 * 17)
4804810770435008147181540925125924391239526139871682263
4738556100880842000763082930863425270914120837430745722
7821149607627692202643343568752733498024953930242542523
0458177649495442143929053063884787051467457680738771416
9885981549563293528878334250628775936
```

(1)

Type: PositiveInteger

There are a number of ways of working with the sign of an integer. Let’s use this `x` as an example.

```
x := -101
-101
```

(2)

Type: Integer

First of all, there is the absolute value function.

```
abs(x)
101
```

(3)

Type: PositiveInteger

The **sign** operation returns -1 if its argument is negative, 0 if zero and 1 if positive.

```
sign(x)
-1
```

(4)

Type: Integer

You can determine if an integer is negative in several other ways.

```
x < 0
true
```

(5)

Type: Boolean

```
x <= -1
true
```

(6)

Type: Boolean

```
negative?(x)
true
```

(7)

Type: Boolean

Similarly, you can find out if it is positive.

```
x > 0
false
```

(8)  
Type: Boolean

```
x >= 1
false
```

(9)  
Type: Boolean

```
positive?(x)
false
```

(10)  
Type: Boolean

This is the recommended way of determining whether an integer is zero.

```
zero?(x)
false
```

(11)  
Type: Boolean

Use the **zero?** operation whenever you are testing any mathematical object for equality with zero. This is usually more efficient than using “=” (think of matrices: it is easier to tell if a matrix is zero by just checking term by term than constructing another “zero” matrix and comparing the two matrices term by term) and also avoids the problem that “=” is usually used for creating equations.

This is the recommended way of determining whether an integer is equal to one.

```
one?(x)
false
```

(12)  
Type: Boolean

This syntax is used to test equality using “=”. It says that you want a Boolean (**true** or **false**) answer rather than an equation.

```
(x = -101)@Boolean
true
```

(13)  
Type: Boolean

The operations **odd?** and **even?** determine whether an integer is odd or even, respectively. They each return a Boolean object.

```
odd?(x)
true
```

(14)  
Type: Boolean

```
even?(x)
false
```

(15)  
Type: Boolean

The operation <b>gcd</b> computes the greatest common divisor of two integers.	<code>gcd(56788,43688)</code> 4	(16) Type: PositiveInteger
The operation <b>lcm</b> computes their least common multiple.	<code>lcm(56788,43688)</code> 620238536	(17) Type: PositiveInteger
To determine the maximum of two integers, use <b>max</b> .	<code>max(678,567)</code> 678	(18) Type: PositiveInteger
To determine the minimum, use <b>min</b> .	<code>min(678,567)</code> 567	(19) Type: PositiveInteger
The <b>reduce</b> operation is used to extend binary operations to more than two arguments. For example, you can use <b>reduce</b> to find the maximum integer in a list or compute the least common multiple of all integers in the list.	<code>reduce(max,[2,45,-89,78,100,-45])</code> 100	(20) Type: PositiveInteger
	<code>reduce(min,[2,45,-89,78,100,-45])</code> -89	(21) Type: Integer
	<code>reduce(gcd,[2,45,-89,78,100,-45])</code> 1	(22) Type: PositiveInteger
	<code>reduce(lcm,[2,45,-89,78,100,-45])</code> 1041300	(23) Type: PositiveInteger
The infix operator “/” is <i>not</i> used to compute the quotient of integers. Rather, it is used to create rational numbers as described in ‘Fraction’ on page 433.	<code>13 / 4</code> $\frac{13}{4}$	(24) Type: Fraction Integer
The infix operation <b>quo</b> computes the integer quotient.	<code>13 quo 4</code> 3	(25) Type: PositiveInteger

The infix operation <b>rem</b> computes the integer remainder.	<code>13 rem 4</code> <code>1</code>	(26) Type: PositiveInteger
One integer is evenly divisible by another if the remainder is zero. The operation <b>exquo</b> can also be used. See Section 2.5 on page 108 for an example.	<code>zero?(167604736446952 rem 2003644)</code> <code>true</code>	(27) Type: Boolean
The operation <b>divide</b> returns a record of the quotient and remainder and thus is more efficient when both are needed.	<code>d := divide(13,4)</code> <code>[quotient = 3, remainder = 1]</code>	(28) Type: Record(quotient: Integer, remainder: Integer)
	<code>d.quotient</code> <code>3</code>	(29) Type: PositiveInteger
Records are discussed in detail in Section 2.4 on page 105.	<code>d.remainder</code> <code>1</code>	(30) Type: PositiveInteger

### 9.34.2 Primes and Factorization

Use the operation <b>factor</b> to factor integers. It returns an object of type Factored Integer. See ‘Factored’ on page 414 for a discussion of the manipulation of factored objects.	<code>factor 102400</code> <code>2<sup>12</sup> 5<sup>2</sup></code>	(1) Type: Factored Integer
The operation <b>prime?</b> returns <b>true</b> or <b>false</b> depending on whether its argument is a prime.	<code>prime? 7</code> <code>true</code>	(2) Type: Boolean
	<code>prime? 8</code> <code>false</code>	(3) Type: Boolean
The operation <b>nextPrime</b> returns the least prime number greater than its argument.	<code>nextPrime 100</code> <code>101</code>	(4) Type: PositiveInteger

The operation **prevPrime** returns the greatest prime number less than its argument.

```
prevPrime 100
97
```

(5)

Type: PositiveInteger

To compute all primes between two integers (inclusively), use the operation **primes**.

```
primes(100,175)
[173, 167, 163, 157, 151, 149, 139, 137, 131, 127, 113, 109,
107, 103, 101]
```

(6)

Type: List Integer

You might sometimes want to see the factorization of an integer when it is considered a *Gaussian integer*. See ‘Complex’ on page 383 for more details.

```
factor(2 :: Complex Integer)
−i (1 + i)2
```

(7)

Type: Factored Complex Integer

### 9.34.3 Some Number Theoretic Functions

AXIOM provides several number theoretic operations for integers. More examples are in ‘IntegerNumberTheoryFunctions’ on page 453.

The operation **fibonacci** computes the Fibonacci numbers. The algorithm has running time  $O(\log^3(n))$  for argument **n**.

```
[fibonacci(k) for k in 0..]
[0, 1, 1, 2, 3, 5, 8, ...]
```

(1)

Type: Stream Integer

The operation **legendre** computes the Legendre symbol for its two integer arguments where the second one is prime. If you know the second argument to be prime, use **jacobi** instead where no check is made.

```
[legendre(i,11) for i in 0..10]
[0, 1, −1, 1, 1, 1, −1, −1, −1, 1, −1]
```

(2)

Type: List Integer

The operation **jacobi** computes the Jacobi symbol for its two integer arguments. By convention, 0 is returned if the greatest common divisor of the numerator and denominator is not 1.

```
[jacobi(i,15) for i in 0..9]
[0, 1, 1, 0, 1, 0, 0, −1, 1, 0]
```

(3)

Type: List Integer

The operation **eulerPhi** computes the values of Euler’s  $\phi$ -function where  $\phi(n)$  equals the number of positive integers less than or equal to **n** that are relatively prime to the positive integer **n**.

```
[eulerPhi i for i in 1..]
[1, 1, 2, 2, 4, 2, 6, ...]
```

(4)

Type: Stream Integer

The operation <b>moebiusMu</b> computes the Möbius $\mu$ function.	<pre>[moebiusMu i for i in 1..]</pre> <pre>[1, -1, -1, 0, -1, 1, -1, ...]</pre>	(5)
	Type: Stream Integer	
Although they have somewhat limited utility, AXIOM provides Roman numerals.	<pre>a := roman(78)</pre> <pre>LXXVIII</pre>	(6)
	Type: RomanNumeral	
	<pre>b := roman(87)</pre> <pre>LXXXVII</pre>	(7)
	Type: RomanNumeral	
	<pre>a + b</pre>	
	<pre>CLXV</pre>	(8)
	Type: RomanNumeral	
	<pre>a * b</pre>	
	<pre>MMMMMMDCCLXXXVI</pre>	(9)
	Type: RomanNumeral	
	<pre>b rem a</pre>	
	<pre>IX</pre>	(10)
	Type: RomanNumeral	



## 9.35 IntegerLinear- Dependence

The elements  $v_1, \dots, v_n$  of a module  $M$  over a ring  $R$  are said to be *linearly dependent over  $R$*  if there exist  $c_1, \dots, c_n$  in  $R$ , not all 0, such that  $c_1 v_1 + \dots + c_n v_n = 0$ . If such  $c_i$ 's exist, they form what is called a *linear dependence relation over  $R$*  for the  $v_i$ 's.

The package IntegerLinearDependence provides functions for testing whether some elements of a module over the integers are linearly dependent over the integers, and to find the linear dependence relations, if any.

Consider the domain of two by two square matrices with integer entries.

```
M := SQMATRIX(2,INT)
```

```
SquareMatrix (2, Integer )
```

Type: Domain

Now create three such matrices.

```
m1: M := squareMatrix matrix [[1, 2], [0, -1]]
```

$$\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$$

Type: SquareMatrix(2, Integer)

```
m2: M := squareMatrix matrix [[2, 3], [1, -2]]
```

$$\begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix}$$

Type: SquareMatrix(2, Integer)

```
m3: M := squareMatrix matrix [[3, 4], [2, -3]]
```

$$\begin{bmatrix} 3 & 4 \\ 2 & -3 \end{bmatrix}$$

Type: SquareMatrix(2, Integer)

This tells you whether  $m1$ ,  $m2$  and  $m3$  are linearly dependent over the integers.

```
linearlyDependentOverZ? vector [m1, m2, m3]
```

```
true
```

Type: Boolean

Since they are linearly dependent, you can ask for the dependence relation.

```
c := linearDependenceOverZ vector [m1, m2, m3]
```

```
[1, -2, 1]
```

Type: Union(Vector Integer, ...)

This means that the following linear combination should be 0.

```
c.1 * m1 + c.2 * m2 + c.3 * m3
```

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Type: SquareMatrix(2, Integer)

When a given set of elements are linearly dependent over  $R$ , this also means that at least one of them can be rewritten as a linear combination of the others with coefficients in the quotient field of  $R$ .

To express a given element in terms of other elements, use the operation **solveLinearlyOverQ**.

`solveLinearlyOverQ(vector [m1, m3], m2)`

$$\left[ \frac{1}{2}, \frac{1}{2} \right]$$

(8)

Type: Union(Vector Fraction Integer, ...)

## 9.36 IntegerNumber- TheoryFunctions

The operation **divisors** returns a list of the divisors of an integer.

You can now compute the number of divisors of 144 and the sum of the divisors of 144 by counting and summing the elements of the list we just created.

In AXIOM, you can simply call the operations **numberOfDivisors** and **sumOfDivisors**.

The IntegerNumberTheoryFunctions package contains a variety of operations of interest to number theorists. Many of these operations deal with divisibility properties of integers. (Recall that an integer  $a$  divides an integer  $b$  if there is an integer  $c$  such that  $b = a * c$ .)

```
div144 := divisors(144)
```

[1, 2, 3, 4, 6, 8, 9, 12, 16, 18, 24, 36, 48, 72, 144] (1)

Type: List Integer

```
#(div144)
```

15 (2)

Type: PositiveInteger

```
reduce(+,div144)
```

403 (3)

Type: PositiveInteger

Of course, you can compute the number of divisors of an integer  $n$ , usually denoted  $d(n)$ , and the sum of the divisors of an integer  $n$ , usually denoted  $\sigma(n)$ , without ever listing the divisors of  $n$ .

```
numberOfDivisors(144)
```

15 (4)

Type: PositiveInteger

```
sumOfDivisors(144)
```

403 (5)

Type: PositiveInteger

The key is that  $d(n)$  and  $\sigma(n)$  are “multiplicative functions.” This means that when  $n$  and  $m$  are relatively prime, that is, when  $n$  and  $m$  have no prime factor in common, then  $d(nm) = d(n) d(m)$  and  $\sigma(nm) = \sigma(n) \sigma(m)$ . Note that these functions are trivial to compute when  $n$  is a prime power and are computed for general  $n$  from the prime factorization of  $n$ . Other examples of multiplicative functions are  $\sigma_k(n)$ , the sum of the  $k^{\text{th}}$  powers of the divisors of  $n$  and  $\varphi(n)$ , the number of integers between 1 and  $n$  which are prime to  $n$ . The corresponding AXIOM operations are called **sumOfKthPowerDivisors** and **eulerPhi**.

An interesting function is  $\mu(n)$ , the Möbius  $\mu$  function, defined as follows:  $\mu(1) = 1$ ,  $\mu(n) = 0$ , when  $n$  is divisible by a square, and  $\mu = (-1)^k$ , when  $n$  is the product of  $k$  distinct primes. The corresponding AXIOM operation is **moebiusMu**. This function occurs in the following theorem:

**Theorem** (Möbius Inversion Formula):

Let  $f(n)$  be a function on the positive integers and let  $F(n)$  be defined by

$$F(n) = \sum_{d|n} f(d)$$

where the sum is taken over the positive divisors of  $n$ . Then the values of  $f(n)$  can be recovered from the values of  $F(n)$ :

$$f(n) = \sum_{d|n} \mu(d) F\left(\frac{n}{d}\right)$$

where again the sum is taken over the positive divisors of  $n$ .

When  $f(n) = 1$ , then  $F(n) = d(n)$ . Thus, if you sum  $\mu(d) \cdot d(n/d)$  over the positive divisors  $d$  of  $n$ , you should always get 1.

```
f1(n) == reduce(+,[moebiusMu(d) * numberOfDivisors(quo(n,d))
for d in divisors(n)])
```

Type: Void

```
f1(200)
```

```
Compiling function f1 with type PositiveInteger ->
Integer
```

```
1 (7)
```

Type: PositiveInteger

```
f1(846)
```

```
1 (8)
```

Type: PositiveInteger

Similarly, when  $f(n) = n$ , then  $F(n) = \sigma(n)$ . Thus, if you sum  $\mu(d) \cdot \sigma(n/d)$  over the positive divisors  $d$  of  $n$ , you should always get  $n$ .

```
f2(n) == reduce(+,[moebiusMu(d) * sumOfDivisors(quo(n,d))
for d in divisors(n)])
```

Type: Void

```
f2(200)
```

```
Compiling function f2 with type PositiveInteger ->
Integer
```

```
200 (10)
```

Type: PositiveInteger

```
f2(846)
```

```
846 (11)
```

Type: PositiveInteger

The Möbius inversion formula is derived from the multiplication of formal Dirichlet series. A Dirichlet series is an infinite series of the form

$$\sum_{n=1}^{\infty} a(n)n^{-s}$$

When

$$\sum_{n=1}^{\infty} a(n)n^{-s} \cdot \sum_{n=1}^{\infty} b(n)n^{-s} = \sum_{n=1}^{\infty} c(n)n^{-s}$$

then  $c(n) = \sum_{d|n} a(d)b(n/d)$ . Recall that the Riemann  $\zeta$  function is defined by

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1} = \sigma_{n=1}^{\infty} n^{-s}$$

where the product is taken over the set of (positive) primes. Thus,

$$\zeta(s)^{-1} = \prod_p (1 - p^{-s}) = \sigma_{n=1}^{\infty} \mu(n)n^{-s}$$

Now if  $F(n) = \sum_{d|n} f(d)$ , then

$$\sum f(n)n^{-s} \cdot \zeta(s) = \sum F(n)n^{-s}$$

Thus,

$$\zeta(s)^{-1} \cdot \sum F(n)n^{-s} = \sum f(n)n^{-s}$$

and  $f(n) = \sum_{d|n} \mu(d)F(n/d)$ .

The Fibonacci numbers are defined by  $F(1) = F(2) = 1$  and  $F(n) = F(n-1) + F(n-2)$  for  $n = 3, 4, \dots$

The operation **fibonacci**  
computes the  $n^{\text{th}}$  Fibonacci  
number.

```
fibonacci(25)
75025
```

(12)

Type: PositiveInteger

```
[fibonacci(n) for n in 1..15]
[1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610]
```

(13)

Type: List Integer

Fibonacci numbers can also be  
expressed as sums of binomial  
coefficients.

```
fib(n) == reduce(+,[binomial(n-1-k,k) for k in 0..quo(n-
1,2)])
```

Type: Void

```
fib(25)
Compiling function fib with type PositiveInteger ->
Integer
75025
```

(15)

Type: PositiveInteger

```
[fib(n) for n in 1..15]
```

```
[1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610]
```

Type: List Integer

Quadratic symbols can be computed with the operations **legendre** and **jacobi**. The Legendre symbol  $\left(\frac{a}{p}\right)$  is defined for integers **a** and **p** with **p** an odd prime number. By definition,  $\left(\frac{a}{p}\right)$ , when **a** is a square (mod **p**),  $\left(\frac{a}{p}\right)$ , when **a** is not a square (mod **p**), and  $\left(\frac{a}{p}\right)$ , when **a** is divisible by **p**.

You compute  $\left(\frac{a}{p}\right)$  via the command **legendre(a,p)**.

```
legendre(3,5)
```

```
-1
```

Type: Integer

```
legendre(23,691)
```

```
-1
```

Type: Integer

The Jacobi symbol  $\left(\frac{a}{n}\right)$  is the usual extension of the Legendre symbol, where **n** is an arbitrary integer. The most important property of the Jacobi symbol is the following: if **K** is a quadratic field with discriminant **d** and quadratic character  $\chi$ , then  $\chi(\mathbf{n}) = (\mathbf{d}/\mathbf{n})$ . Thus, you can use the Jacobi symbol to compute, say, the class numbers of imaginary quadratic fields from a standard class number formula.

This function computes the class number of the imaginary quadratic field with discriminant **d**.

```
h(d) == quo(reduce(+, [jacobi(d,k) for k in 1..quo(-d, 2)]), 2 - jacobi(d,2))
```

Type: Void

```
h(-163)
```

```
Compiling function h with type Integer -> Integer
```

```
1
```

Type: PositiveInteger

```
h(-499)
```

```
3
```

Type: PositiveInteger

```
h(-1832)
```

```
26
```

Type: PositiveInteger

## 9.37 Kernel

A *kernel* is a symbolic function application (such as `sin(x + y)`) or a symbol (such as `x`). More precisely, a non-symbol kernel over a set  $S$  is an operator applied to a given list of arguments from  $S$ . The operator has type `BasicOperator` (see ‘BasicOperator’ on page 356) and the kernel object is usually part of an expression object (see ‘Expression’ on page 410).

Kernels are created implicitly for you when you create expressions.

	<code>x :: Expression Integer</code>	
	$x$	(1)
		Type: Expression Integer
You can directly create a “symbol” kernel by using the <b>kernel</b> operation.	<code>kernel x</code>	
	$x$	(2)
		Type: Kernel Expression Integer
This expression has two different kernels.	<code>sin(x) + cos(x)</code>	
	$\sin(x) + \cos(x)$	(3)
		Type: Expression Integer
The operator <b>kernels</b> returns a list of the kernels in an object of type <code>Expression</code> .	<code>kernels %</code>	
	<code>[sin(x), cos(x)]</code>	(4)
		Type: List Kernel Expression Integer
This expression also has two different kernels.	<code>sin(x)**2 + sin(x) + cos(x)</code>	
	$\sin(x)^2 + \sin(x) + \cos(x)$	(5)
		Type: Expression Integer
The <code>sin(x)</code> kernel is used twice.	<code>kernels %</code>	
	<code>[sin(x), cos(x)]</code>	(6)
		Type: List Kernel Expression Integer
An expression need not contain any kernels.	<code>kernels(1 :: Expression Integer)</code>	
	<code>[]</code>	(7)
		Type: List Kernel Expression Integer
If one or more kernels are present, one of them is designated the <i>main</i> kernel.	<code>mainKernel(cos(x) + tan(x))</code>	
	$\tan(x)$	(8)
		Type: Union(Kernel Expression Integer, ...)
Kernels can be nested. Use <b>height</b> to determine the nesting depth.	<code>height kernel x</code>	
	<code>1</code>	(9)
		Type: PositiveInteger

This has height 2 because the  $x$  has height 1 and then we apply an operator to that.

```
height mainKernel(sin x)
2
```

(10)

Type: PositiveInteger

```
height mainKernel(sin cos x)
3
```

(11)

Type: PositiveInteger

```
height mainKernel(sin cos (tan x + sin x))
4
```

(12)

Type: PositiveInteger

Use the **operator** operation to extract the operator component of the kernel. The operator has type BasicOperator.

```
operator mainKernel(sin cos (tan x + sin x))
sin
```

(13)

Type: BasicOperator

Use the **name** operation to extract the name of the operator component of the kernel. The name has type Symbol. This is really just a shortcut for a two-step process of extracting the operator and then calling **name** on the operator.

```
name mainKernel(sin cos (tan x + sin x))
sin
```

(14)

Type: Symbol

AXIOM knows about functions such as **sin**, **cos** and so on and can make kernels and then expressions using them. To create a kernel and expression using an arbitrary operator, use **operator**.

Now  $f$  can be used to create symbolic function applications.

```
f := operator 'f
f
```

(15)

Type: BasicOperator

```
e := f(x, y, 10)
f(x, y, 10)
```

(16)

Type: Expression Integer

Use the **is?** operation to learn if the operator component of a kernel is equal to a given operator.

```
is?(e, f)
true
```

(17)

Type: Boolean

You can also use a symbol or a string as the second argument to **is?**.

```
is?(e, 'f)
true
```

(18)

Type: Boolean



Use the **argument** operation to get a list containing the argument component of a kernel.

```
argument mainKernel e
```

```
[x, y, 10]
```

(19)

Type: List Expression Integer

Conceptually, an object of type Expression can be thought of a quotient of multivariate polynomials, where the “variables” are kernels. The arguments of the kernels are again expressions and so the structure recurses. See ‘Expression’ on page 410 for examples of using kernels to take apart expression objects.

## 9.38 KeyedAccessFile

The domain KeyedAccessFile(S) provides files which can be used as associative tables. Data values are stored in these files and can be retrieved according to their keys. The keys must be strings so this type behaves very much like the StringTable(S) domain. The difference is that keyed access files reside in secondary storage while string tables are kept in memory. For more information on table-oriented operations, see the description of Table.

Before a keyed access file can be used, it must first be opened. A new file can be created by opening it for output.

```
ey: KeyedAccessFile(Integer) := open("/tmp/editor.year",
  "output")
"/tmp/editor.year" (1)
```

Type: KeyedAccessFile Integer

Just as for vectors, tables or lists, values are saved in a keyed access file by setting elements.

```
ey."Char" := 1986
1986 (2)
```

Type: PositiveInteger

```
ey."Caviness" := 1985
1985 (3)
```

Type: PositiveInteger

```
ey."Fitch" := 1984
1984 (4)
```

Type: PositiveInteger

Values are retrieved using application, in any of its syntactic forms.

```
ey."Char"
1986 (5)
```

Type: PositiveInteger

```
ey("Char")
1986 (6)
```

Type: PositiveInteger

```
ey "Char"
1986 (7)
```

Type: PositiveInteger

Attempting to retrieve a non-existent element in this way causes an error. If it is not known whether a key exists, you should use the **search** operation.

```
search("Char", ey)
1986 (8)
```

Type: Union(Integer, ...)

	<code>search("Smith", ey)</code>	
	<code>"failed"</code>	(9)
		Type: Union("failed", ...)
When an entry is no longer needed, it can be removed from the file.	<code>remove!("Char", ey)</code>	
	<code>1986</code>	(10)
		Type: Union(Integer, ...)
The <b>keys</b> operation returns a list of all the keys for a given file.	<code>keys ey</code>	
	<code>["Fitch", "Caviness"]</code>	(11)
		Type: List String
The <b>#</b> operation gives the number of entries.	<code>#ey</code>	
	<code>2</code>	(12)
		Type: PositiveInteger
	The table view of keyed access files provides safe operations. That is, if the AXIOM program is terminated between file operations, the file is left in a consistent, current state. This means, however, that the operations are somewhat costly. For example, after each update the file is closed.	
Here we add several more items to the file, then check its contents.	<code>KE := Record(key: String, entry: Integer)</code>	
	<code>Record (key : String , entry : Integer )</code>	(13)
		Type: Domain
	<code>reopen!(ey, "output")</code>	
	<code>"/tmp/editor.year"</code>	(14)
		Type: KeyedAccessFile Integer
If many items are to be added to a file at the same time, then it is more efficient to use the <b>write!</b> operation.	<code>write!(ey, ["van Hulzen", 1983]\$KE)</code>	
	<code>[key = "van Hulzen", entry = 1983]</code>	(15)
		Type: Record(key: String, entry: Integer)
	<code>write!(ey, ["Calmet", 1982]\$KE)</code>	
	<code>[key = "Calmet", entry = 1982]</code>	(16)
		Type: Record(key: String, entry: Integer)
	<code>write!(ey, ["Wang", 1981]\$KE)</code>	
	<code>[key = "Wang", entry = 1981]</code>	(17)
		Type: Record(key: String, entry: Integer)

```
close! ey
"/tmp/editor.year" (18)
```

Type: KeyedAccessFile Integer

The **read!** operation is also available from the file view, but it returns elements in a random order. It is generally clearer and more efficient to use the **keys** operation and to extract elements by key.

```
keys ey
["Wang", "Calmet", "van Hulzen", "Fitch", "Caviness"] (19)
```

Type: List String

```
members ey
[1981, 1982, 1983, 1984, 1985] (20)
```

Type: List Integer

```
)system rm -r /tmp/editor.year
```

For more information on related topics, see ‘File’ on page 420, ‘TextFile’ on page 588, and ‘Library’ on page 474. Issue the system command `)show KeyedAccessFile` to display the full list of operations defined by KeyedAccessFile.

### 9.39

## LazardSetSolvingPackage

---

The LazardSetSolvingPackage package constructor solves polynomial systems by means of Lazard triangular sets. However one condition is relaxed: Regular triangular sets whose saturated ideals have positive dimension are not necessarily normalized.

The decompositions are computed in two steps. First the algorithm of Moreno Maza (implemented in the RegularTriangularSet domain constructor) is called. Then the resulting decompositions are converted into lists of square-free regular triangular sets and the redundant components are removed. Moreover, zero-dimensional regular triangular sets are normalized.

This constructor takes six arguments. The first one, **R**, is the coefficient ring of the polynomials; it must belong to the category GcdDomain. The second one, **E**, is the exponent monoid of the polynomials; it must belong to the category OrderedAbelianMonoidSup. the third one, **V**, is the ordered set of variables; it must belong to the category OrderedSet. The fourth one is the polynomial ring; it must belong to the category RecursivePolynomialCategory(R,E,V). The fifth one is a domain of the category RegularTriangularSetCategory(R,E,V,P) and the last one is a domain of the category SquareFreeRegularTriangularSetCategory(R,E,V,P). The abbreviation for LazardSetSolvingPackage is LAZM3PK.

**N.B.** For the purpose of solving zero-dimensional algebraic systems, the package ZDSOLVE is easier to call and provides operations to compute either the complex roots or the real roots.

We illustrate now the use of the LazardSetSolvingPackage package constructor with two examples (Butcher and Vermeer).

Define the coefficient ring.

```
R := Integer
Integer
Type: Domain
```

Define the list of variables,

```
ls : List Symbol := [b1,x,y,z,t,v,u,w]
[b1, x, y, z, t, v, u, w]
Type: List Symbol
```

and make it an ordered set;

```
V := OVAR(ls)
OrderedVariableList [b1, x, y, z, t, v, u, w]
Type: Domain
```

then define the exponent monoid.

```
E := IndexedExponents V
IndexedExponents OrderedVariableList [b1 ,x ,y ,z ,t ,v ,u ,w]
Type: Domain
```

Define the polynomial ring.  $P := \text{NSMP}(R, V)$   
 $\text{NewSparseMultivariatePolynomial}(\text{Integer}, \text{OrderedVariableList} [b1, x, y, z, t, v, u, w])$  (5)  
Type: Domain

Let the variables be polynomial.  $b1: P := 'b1$   
 $b1$  (6)  
Type:  $\text{NewSparseMultivariatePolynomial}(\text{Integer}, \text{OrderedVariableList} [b1, x, y, z, t, v, u, w])$

$x: P := 'x$   
 $x$  (7)  
Type:  $\text{NewSparseMultivariatePolynomial}(\text{Integer}, \text{OrderedVariableList} [b1, x, y, z, t, v, u, w])$

$y: P := 'y$   
 $y$  (8)  
Type:  $\text{NewSparseMultivariatePolynomial}(\text{Integer}, \text{OrderedVariableList} [b1, x, y, z, t, v, u, w])$

$z: P := 'z$   
 $z$  (9)  
Type:  $\text{NewSparseMultivariatePolynomial}(\text{Integer}, \text{OrderedVariableList} [b1, x, y, z, t, v, u, w])$

$t: P := 't$   
 $t$  (10)  
Type:  $\text{NewSparseMultivariatePolynomial}(\text{Integer}, \text{OrderedVariableList} [b1, x, y, z, t, v, u, w])$

$u: P := 'u$   
 $u$  (11)  
Type:  $\text{NewSparseMultivariatePolynomial}(\text{Integer}, \text{OrderedVariableList} [b1, x, y, z, t, v, u, w])$

$v: P := 'v$   
 $v$  (12)  
Type:  $\text{NewSparseMultivariatePolynomial}(\text{Integer}, \text{OrderedVariableList} [b1, x, y, z, t, v, u, w])$

$w: P := 'w$   
 $w$  (13)  
Type:  $\text{NewSparseMultivariatePolynomial}(\text{Integer}, \text{OrderedVariableList} [b1, x, y, z, t, v, u, w])$

Now call the RegularTriangularSet domain constructor.

```
T := REGSET(R,E,V,P)
RegularTriangularSet(Integer, IndexedExponents
OrderedVariableList [b1, x, y, z, t, v, u, w], OrderedVariableList
[b1, x, y, z, t, v, u, w], NewSparseMultivariatePolynomial(Integer,
OrderedVariableList [b1, x, y, z, t, v, u, w]))
```

Type: Domain

Define a polynomial system (the Butcher example).

```
p0 := b1 + y + z - t - w
b1 + y + z - t - w
```

Type: NewSparseMultivariatePolynomial(Integer, OrderedVariableList [b1, x, y, z, t, v, u, w])

```
p1 := 2*z*u + 2*y*v + 2*t*w - 2*w**2 - w - 1
2 v y + 2 u z + 2 w t - 2 w^2 - w - 1
```

Type: NewSparseMultivariatePolynomial(Integer, OrderedVariableList [b1, x, y, z, t, v, u, w])

```
p2 := 3*z*u**2 + 3*y*v**2 - 3*t*w**2 + 3*w**3 + 3*w**2 - t
+ 4*w
3 v^2 y + 3 u^2 z + (-3 w^2 - 1) t + 3 w^3 + 3 w^2 + 4 w
```

Type: NewSparseMultivariatePolynomial(Integer, OrderedVariableList [b1, x, y, z, t, v, u, w])

```
p3 := 6*x*z*v - 6*t*w**2 + 6*w**3 - 3*t*w + 6*w**2 - t +
4*w
6 v z x + (-6 w^2 - 3 w - 1) t + 6 w^3 + 6 w^2 + 4 w
```

Type: NewSparseMultivariatePolynomial(Integer, OrderedVariableList [b1, x, y, z, t, v, u, w])

```
p4 := 4*z*u**3 + 4*y*v**3 + 4*t*w**3 - 4*w**4 - 6*w**3 +
4*t*w - 10*w**2 - w - 1
4 v^3 y + 4 u^3 z + (4 w^3 + 4 w) t - 4 w^4 - 6 w^3 - 10 w^2 - w - 1
```

Type: NewSparseMultivariatePolynomial(Integer, OrderedVariableList [b1, x, y, z, t, v, u, w])

```
p5 := 8*x*z*u*v + 8*t*w**3 - 8*w**4 + 4*t*w**2 - 12*w**3
+ 4*t*w - 14*w**2 - 3*w - 1
8 u v z x + (8 w^3 + 4 w^2 + 4 w) t - 8 w^4 - 12 w^3 - 14 w^2 - 3 w - 1
```

Type: NewSparseMultivariatePolynomial(Integer, OrderedVariableList [b1, x, y, z, t, v, u, w])

$$\begin{aligned}
p6 &:= 12*x*z*v**2+12*t*w**3 -12*w**4 +12*t*w**2 -18*w**3 \\
&\quad +8*t*w -14*w**2 -w -1 \\
12 v^2 z x + (12 w^3 + 12 w^2 + 8 w) t - 12 w^4 - 18 w^3 - 14 w^2 \\
&\quad -w -1
\end{aligned} \tag{21}$$

Type: NewSparseMultivariatePolynomial(Integer, OrderedVariableList [b1, x, y, z, t, v, u, w])

$$\begin{aligned}
p7 &:= -24*t*w**3 + 24*w**4 - 24*t*w**2 + 36*w**3 - 8*t*w + \\
&\quad 26*w**2 + 7*w + 1 \\
(-24 w^3 - 24 w^2 - 8 w) t + 24 w^4 + 36 w^3 + 26 w^2 + 7 w + 1
\end{aligned} \tag{22}$$

Type: NewSparseMultivariatePolynomial(Integer, OrderedVariableList [b1, x, y, z, t, v, u, w])

$$\begin{aligned}
lp &:= [p0, p1, p2, p3, p4, p5, p6, p7] \\
&[b1 + y + z - t - w, 2 v y + 2 u z + 2 w t - 2 w^2 - w - 1, \\
&3 v^2 y + 3 u^2 z + (-3 w^2 - 1) t + 3 w^3 + 3 w^2 + 4 w, \\
&6 v z x + (-6 w^2 - 3 w - 1) t + 6 w^3 + 6 w^2 + 4 w, \\
&4 v^3 y + 4 u^3 z + (4 w^3 + 4 w) t - 4 w^4 - 6 w^3 - 10 w^2 - w - 1, \\
&8 u v z x + (8 w^3 + 4 w^2 + 4 w) t - 8 w^4 - 12 w^3 - 14 w^2 \\
&\quad -3 w - 1, \\
&12 v^2 z x + (12 w^3 + 12 w^2 + 8 w) t - 12 w^4 - 18 w^3 - 14 w^2 \\
&\quad -w - 1, \\
&(-24 w^3 - 24 w^2 - 8 w) t + 24 w^4 + 36 w^3 + 26 w^2 + 7 w + 1]
\end{aligned} \tag{23}$$

Type: List NewSparseMultivariatePolynomial(Integer, OrderedVariableList [b1, x, y, z, t, v, u, w])



First of all, let us solve this system in the sense of Lazard by means of the REGSET constructor:

```
lts := zeroSetSplit(lp,false)$T
[ {w + 1, u, v, t + 1, b1 + y + z + 2},
  {w + 1, v, t + 1, z, b1 + y + 2},
  {w + 1, t + 1, z, y, b1 + 2},
  {w + 1, v - u, t + 1, y + z, x, b1 + 2},
  {w + 1, u, t + 1, y, x, b1 + z + 2},
  {144 w5 + 216 w4 + 96 w3 + 6 w2 - 11 w - 1,
   (12 w2 + 9 w + 1) u - 72 w5 - 108 w4 - 42 w3 - 9 w2 - 3 w,
   (12 w2 + 9 w + 1) v + 36 w4 + 54 w3 + 18 w2,
   (24 w3 + 24 w2 + 8 w) t - 24 w4 - 36 w3 - 26 w2 - 7 w - 1,
   (12 u v - 12 u2) z + (12 w v + 12 w2 + 4) t + (3 w - 5) v +
   36 w4 + 42 w3 + 6 w2 - 16 w,
   2 v y + 2 u z + 2 w t - 2 w2 - w - 1,
   6 v z x + (-6 w2 - 3 w - 1) t + 6 w3 + 6 w2 + 4 w,
   b1 + y + z - t - w } ]
```

Type: List RegularTriangularSet(Integer, IndexedExponents OrderedVariableList [b1, x, y, z, t, v, u, w], OrderedVariableList [b1, x, y, z, t, v, u, w], NewSparseMultivariatePolynomial(Integer, OrderedVariableList [b1, x, y, z, t, v, u, w]))

We can get the dimensions of each component of a decomposition as follows.

```
[coHeight(ts) for ts in lts]
[3, 3, 3, 2, 2, 0]
```

Type: List NonNegativeInteger

The first five sets have a simple shape. However, the last one, which has dimension zero, can be simplified by using Lazard triangular sets.

Thus we call the SquareFreeRegularTriangularSet domain constructor,

```
ST := SREGSET(R,E,V,P)
SquareFreeRegularTriangularSet(Integer, IndexedExponents
OrderedVariableList [b1, x, y, z, t, v, u, w], OrderedVariableList
[b1, x, y, z, t, v, u, w], NewSparseMultivariatePolynomial(Integer,
OrderedVariableList[b1, x, y, z, t, v, u, w]))
```

Type: Domain

and set the LAZM3PK package constructor to our situation.

```
pack := LAZM3PK(R,E,V,P,T,ST)
LazardSetSolvingPackage(Integer, IndexedExponents
OrderedVariableList [b1, x, y, z, t, v, u, w], OrderedVariableList
[b1, x, y, z, t, v, u, w], NewSparseMultivariatePolynomial(Integer,
OrderedVariableList[b1, x, y, z, t, v, u, w]), RegularTriangularSet(
Integer, IndexedExponents OrderedVariableList[b1, x, y, z, t, v, u,
w], OrderedVariableList[b1, x, y, z, t, v, u, w],
NewSparseMultivariatePolynomial(Integer, OrderedVariableList
[b1, x, y, z, t, v, u, w])), SquareFreeRegularTriangularSet(Integer,
IndexedExponents OrderedVariableList[b1, x, y, z, t, v, u, w],
OrderedVariableList[b1, x, y, z, t, v, u, w],
NewSparseMultivariatePolynomial(Integer, OrderedVariableList
[b1, x, y, z, t, v, u, w])))
```

(27)

Type: Domain

We are ready to solve the system by means of Lazard triangular sets:

```
zeroSetSplit(lp,false)$pack
[{w + 1, t + 1, z, y, b1 + 2},
{w + 1, v, t + 1, z, b1 + y + 2},
{w + 1, u, v, t + 1, b1 + y + z + 2},
{w + 1, v - u, t + 1, y + z, x, b1 + 2},
{w + 1, u, t + 1, y, x, b1 + z + 2},
{144 w5 + 216 w4 + 96 w3 + 6 w2 - 11 w - 1,
u - 24 w4 - 36 w3 - 14 w2 + w + 1,
3 v - 48 w4 - 60 w3 - 10 w2 + 8 w + 2,
t - 24 w4 - 36 w3 - 14 w2 - w + 1,
486 z - 2772 w4 - 4662 w3 - 2055 w2 + 30 w + 127,
2916 y - 22752 w4 - 30312 w3 - 8220 w2 + 2064 w + 1561,
356 x - 3696 w4 - 4536 w3 - 968 w2 + 822 w + 371,
2916 b1 - 30600 w4 - 46692 w3 - 20274 w2 - 8076 w + 593} ]
```

(28)

Type: List SquareFreeRegularTriangularSet(Integer, IndexedExponents  
OrderedVariableList [b1, x, y, z, t, v, u, w], OrderedVariableList [b1, x, y, z, t,  
v, u, w], NewSparseMultivariatePolynomial(Integer, OrderedVariableList  
[b1, x, y, z, t, v, u, w]))

We see the sixth triangular set is *nicer* now: each one of its polynomials has a constant initial.

We follow with the Vermeer example. The ordering is the usual one for this system.

Define the polynomial system.

```
f0 := (w - v) ** 2 + (u - t) ** 2 - 1
t2 - 2 u t + v2 - 2 w v + u2 + w2 - 1
```

(29)

Type: NewSparseMultivariatePolynomial(Integer, OrderedVariableList [b1, x, y, z, t,  
v, u, w])

$$f1 := t^{**2} - v^{**3} \quad (30)$$

Type: NewSparseMultivariatePolynomial(Integer, OrderedVariableList [b1, x, y, z, t, v, u, w])

$$f2 := 2 * t * (w - v) + 3 * v^{**2} * (u - t) \quad (31)$$

Type: NewSparseMultivariatePolynomial(Integer, OrderedVariableList [b1, x, y, z, t, v, u, w])

$$f3 := (3 * z * v^{**2} - 1) * (2 * z * t - 1) \quad (32)$$

Type: NewSparseMultivariatePolynomial(Integer, OrderedVariableList [b1, x, y, z, t, v, u, w])

$$lf := [f0, f1, f2, f3] \quad (33)$$

Type: List NewSparseMultivariatePolynomial(Integer, OrderedVariableList [b1, x, y, z, t, v, u, w])

First of all, let us solve this system in the sense of Kalkbrener by means of the REGSET constructor:

$$\text{zeroSetSplit}(lf, \text{true})\$T \quad (34)$$

Type: List RegularTriangularSet(Integer, IndexedExponents OrderedVariableList [b1, x, y, z, t, v, u, w], OrderedVariableList [b1, x, y, z, t, v, u, w], NewSparseMultivariatePolynomial(Integer, OrderedVariableList [b1, x, y, z, t, v, u, w]))

We have obtained one regular chain (i.e. regular triangular set) with dimension 1. This set is in fact a characterist set of the (radical of) the ideal generated by the input system **lf**. Thus we have only the *generic points* of the variety associated with **lf** (for the elimination ordering given

by `ls`).

So let us get now a full description of this variety.

Hence, we solve this system in the sense of Lazard by means of the REGSET constructor:

`zeroSetSplit(lf,false)$T`

$$\left[ \begin{array}{l} \left\{ \begin{array}{l} 729 u^6 + (-1458 w^3 + 729 w^2 - 4158 w - 1685) u^4 + \\ (729 w^6 - 1458 w^5 - 2619 w^4 - 4892 w^3 - 297 w^2 + \\ 5814 w + 427) u^2 + 729 w^8 + 216 w^7 - 2900 w^6 - 2376 \cdot \\ w^5 + 3870 w^4 + 4072 w^3 - 1188 w^2 - 1656 w + 529, \\ \\ (2187 u^4 + (-4374 w^3 - 972 w^2 - 12474 w - 2868) u^2 + \\ 2187 w^6 - 1944 w^5 - 10125 w^4 - 4800 w^3 + 2501 w^2 + \\ 4968 w - 1587) v + (1944 w^3 - 108 w^2) u^2 + 972 w^6 + \\ 3024 w^5 - 1080 w^4 + 496 w^3 + 1116 w^2, \\ \\ (3 v^2 + 2 v - 2 w) t - 3 u v^2, \\ ((4 v - 4 w) t - 6 u v^2) z^2 + (2 t + 3 v^2) z - 1 \\ 27 w^4 + 4 w^3 - 54 w^2 - 36 w + 23, u, \\ (12 w + 2) v - 9 w^2 - 2 w + 9, \\ 6 t^2 - 2 v - 3 w^2 + 2 w + 3, 2 t z - 1 \end{array} \right\}, \\ \\ \left\{ \begin{array}{l} 59049 w^6 + 91854 w^5 - 45198 w^4 + 145152 w^3 + \\ 63549 w^2 + 60922 w + 21420, \\ \\ (31484448266904 w^5 - 18316865522574 w^4 + \\ 23676995746098 w^3 + 6657857188965 w^2 + \\ 8904703998546 w + 3890631403260) u^2 + \\ 94262810316408 w^5 - 82887296576616 w^4 + \\ 89801831438784 w^3 + 28141734167208 w^2 + \\ 38070359425432 w + 16003865949120, \\ \\ (243 w^2 + 36 w + 85) v^2 + (-81 u^2 - 162 w^3 + \\ 36 w^2 + 154 w + 72) v - 72 w^3 + 4 w^2, \\ \\ (3 v^2 + 2 v - 2 w) t - 3 u v^2, \\ ((4 v - 4 w) t - 6 u v^2) z^2 + (2 t + 3 v^2) z - 1 \\ 27 w^4 + 4 w^3 - 54 w^2 - 36 w + 23, u, \\ (12 w + 2) v - 9 w^2 - 2 w + 9, \\ 6 t^2 - 2 v - 3 w^2 + 2 w + 3, 3 v^2 z - 1 \end{array} \right\} \end{array} \right], \quad (35)$$

Type: List RegularTriangularSet(Integer, IndexedExponents OrderedVariableList  
[b1, x, y, z, t, v, u, w], OrderedVariableList [b1, x, y, z, t, v, u, w],  
NewSparseMultivariatePolynomial(Integer, OrderedVariableList [b1, x, y, z,  
t, v, u, w]))

We retrieve our regular chain of dimension 1 and we get three regular

chains of dimension 0 corresponding to the *degenerated cases*. We want now to simplify these zero-dimensional regular chains by using Lazard triangular sets. Moreover, this will allow us to prove that the above decomposition has no redundant component. **N.B.** Generally, decompositions computed by the REGSET constructor do not have redundant components. However, to be sure that no redundant component occurs one needs to use the SREGSET or LAZM3PK constructors.

So let us solve the input system  
in the sense of Lazard by means  
of the LAZM3PK constructor:

`zeroSetSplit(lf,false)$pack`

$$\left[ \begin{array}{l} \left\{ \begin{array}{l} 729 u^6 + (-1458 w^3 + 729 w^2 - 4158 w - 1685) u^4 + \\ (729 w^6 - 1458 w^5 - 2619 w^4 - 4892 w^3 - 297 w^2 + \\ 5814 w + 427) u^2 + 729 w^8 + 216 w^7 - 2900 w^6 - \\ 2376 w^5 + 3870 w^4 + 4072 w^3 - 1188 w^2 - 1656 w + 529, \\ (2187 u^4 + (-4374 w^3 - 972 w^2 - 12474 w - 2868) u^2 + \\ 2187 w^6 - 1944 w^5 - 10125 w^4 - 4800 w^3 + 2501 w^2 + \\ 4968 w - 1587) v + (1944 w^3 - 108 w^2) u^2 + 972 w^6 + \\ 3024 w^5 - 1080 w^4 + 496 w^3 + 1116 w^2, \\ (3 v^2 + 2 v - 2 w) t - 3 u v^2, \\ ((4 v - 4 w) t - 6 u v^2) z^2 + (2 t + 3 v^2) z - 1 \end{array} \right\}, \\ \left\{ \begin{array}{l} 81 w^2 + 18 w + 28, 729 u^2 - 1890 w - 533, \\ 81 v^2 + (-162 w + 27) v - 72 w - 112, \\ 11881 t + (972 w + 2997) u v + (-11448 w - 11536) u, \\ 641237934604288 z^2 + (((78614584763904 w + \\ 26785578742272) u + 236143618655616 w + \\ 70221988585728) v + (358520253138432 w + \\ 101922133759488) u + 142598803536000 w + \\ 54166419595008) z + \\ (32655103844499 w - 44224572465882) u v + \\ (43213900115457 w - 32432039102070) u \end{array} \right\}, \\ \left\{ \begin{array}{l} 27 w^4 + 4 w^3 - 54 w^2 - 36 w + 23, u, \\ 218 v - 162 w^3 + 3 w^2 + 160 w + 153, \\ 109 t^2 - 27 w^3 - 54 w^2 + 63 w + 80, \\ 1744 z + (-1458 w^3 + 27 w^2 + 1440 w + 505) t \end{array} \right\}, \\ \left\{ \begin{array}{l} 27 w^4 + 4 w^3 - 54 w^2 - 36 w + 23, u, \\ 218 v - 162 w^3 + 3 w^2 + 160 w + 153, \\ 109 t^2 - 27 w^3 - 54 w^2 + 63 w + 80, \\ 1308 z + 162 w^3 - 3 w^2 - 814 w - 153 \end{array} \right\}, \\ \left\{ \begin{array}{l} 729 w^4 + 972 w^3 - 1026 w^2 + 1684 w + 765, \\ 81 u^2 + 72 w^2 + 16 w - 72, \\ 702 v - 162 w^3 - 225 w^2 + 40 w - 99, \\ 11336 t + (324 w^3 - 603 w^2 - 1718 w - 1557) u, \\ 595003968 z^2 + ((-963325386 w^3 - 898607682 w^2 + \\ 1516286466 w - 3239166186) u - 1579048992 w^3 - \\ 1796454288 w^2 + 2428328160 w - 4368495024) z + \\ (9713133306 w^3 + 9678670317 w^2 - 16726834476 w + \\ 28144233593) u \end{array} \right\} \end{array} \right] \quad (36)$$

Type: List SquareFreeRegularTriangularSet(Integer, IndexedExponents  
OrderedVariableList [b1, x, y, z, t, v, u, w], OrderedVariableList [b1, x, y, z, t,  
v, u, w], NewSparseMultivariatePolynomial(Integer, OrderedVariableList  
[b1, x, y, z, t, v, u, w]))

Due to square-free factorization, we obtained now four zero-dimensional regular chains. Moreover, each of them is normalized (the initials are constant). Note that these zero-dimensional components may be investigated further with the `ZeroDimensionalSolvePackage` package constructor.

## 9.40 Library

To create a library, you supply a file name.

The Library domain provides a simple way to store AXIOM values in a file. This domain is similar to KeyedAccessFile but fewer declarations are needed and items of different types can be saved together in the same file.

```
stuff := library "/tmp/Neat.stuff"
"/tmp/Neat.stuff" (1)
```

Type: Library

Now values can be saved by key in the file. The keys should be mnemonic, just as the field names are for records. They can be given either as strings or symbols.

```
stuff.int := 32**2
1024 (2)
```

Type: PositiveInteger

```
stuff."poly" := x**2 + 1
 $x^2 + 1$  (3)
```

Type: Polynomial Integer

```
stuff.str := "Hello"
"Hello" (4)
```

Type: String

You obtain the set of available keys using the **keys** operation.

```
keys stuff
["str", "poly", "int"] (5)
```

Type: List String

You extract values by giving the desired key in this way.

```
stuff.poly
 $x^2 + 1$  (6)
```

Type: Polynomial Integer

```
stuff("poly")
 $x^2 + 1$  (7)
```

Type: Polynomial Integer

When the file is no longer needed, you should remove it from the file system.

```
)system rm -rf /tmp/Neat.stuff
```

For more information on related topics, see ‘File’ on page 420, ‘TextFile’ on page 588, and ‘KeyedAccessFile’ on page 460. Issue the system command **)show Library** to display the full list of operations defined by Library.



## 9.41 LinearOrdinary- Differential- Operator

LinearOrdinaryDifferentialOperator(A, diff) is the domain of linear ordinary differential operators with coefficients in a ring A with a given derivation. Issue the system command `)show LinearOrdinaryDifferentialOperator` to display the full list of operations defined by LinearOrdinaryDifferentialOperator.

### 9.41.1 Differential Operators with Series Coefficients

**Problem:** Find the first few coefficients of  $\exp(x)/x^{**i}$  of `Dop phi` where

```
Dop := D**3 + G/x**2 * D + H/x**3 - 1
phi := sum(s[i]*exp(x)/x**i, i = 0..)
```

**Solution:**

Define the differential.

```
Dx: LODO(EXPR INT, f +-> D(f, x))
```

Type: Void

```
Dx := D()
```

```
D
```

(2)

Type: LinearOrdinaryDifferentialOperator(Expression Integer, theMap NIL)

Now define the differential operator `Dop`.

```
Dop := Dx**3 + G/x**2*Dx + H/x**3 - 1
```

$$D^3 + \frac{G}{x^2} D + \frac{-x^3 + H}{x^3}$$

(3)

Type: LinearOrdinaryDifferentialOperator(Expression Integer, theMap NIL)

```
n == 3
```

Type: Void

```
phi == reduce(+,[subscript(s,[i])*exp(x)/x**i for i in
0..n])
```

Type: Void

```
phi1 == Dop(phi) / exp x
```

Type: Void

```
phi2 == phi1 *x**(n+3)
```

Type: Void

```
phi3 == retract(phi2)@(POLY INT)
```

Type: Void

```
pans == phi3 ::UP(x,POLY INT)
```

Type: Void

```
pans1 == [coefficient(pans, (n+3-i) :: NNI) for i in
2..n+1]
```

Type: Void

```
leq == solve(pans1,[subscript(s,[i]) for i in 1..n])
```

Type: Void

Evaluate this for several values  
of n.

```
leq
```

```
Compiling body of rule n to compute value of type
PositiveInteger
Compiling body of rule phi to compute value of type
Expression Integer
Compiling body of rule phi1 to compute value of type
Expression Integer
Compiling body of rule phi2 to compute value of type
Expression Integer
Compiling body of rule phi3 to compute value of type
Polynomial Integer
Compiling body of rule pans to compute value of type
UnivariatePolynomial(x,Polynomial Integer)
Compiling body of rule pans1 to compute value of type
List Polynomial Integer
Compiling body of rule leq to compute value of type
List List Equation Fraction Polynomial Integer
Compiling function G82300 with type Integer ->
Boolean
```

$$\left[ \begin{array}{l} s_1 = \frac{s_0 G}{3}, s_2 = \frac{3 s_0 H + s_0 G^2 + 6 s_0 G}{18}, \\ s_3 = \frac{(9 s_0 G + 54 s_0) H + s_0 G^3 + 18 s_0 G^2 + 72 s_0 G}{162} \end{array} \right] \quad (12)$$

Type: List List Equation Fraction Polynomial Integer

n==4

Compiled code for n has been cleared.  
 Compiled code for leq has been cleared.  
 Compiled code for pans1 has been cleared.  
 Compiled code for phi2 has been cleared.  
 Compiled code for phi has been cleared.  
 Compiled code for phi3 has been cleared.  
 Compiled code for phil has been cleared.  
 Compiled code for pans has been cleared.  
 1 old definition(s) deleted for function or rule n

Type: Void

leq

Compiling body of rule n to compute value of type  
 PositiveInteger  
 +++ |\*0;n;1;initial| redefined  
 Compiling body of rule phi to compute value of type  
 Expression Integer  
 +++ |\*0;phi;1;initial| redefined  
 Compiling body of rule phil to compute value of type  
 Expression Integer  
 +++ |\*0;phil;1;initial| redefined  
 Compiling body of rule phi2 to compute value of type  
 Expression Integer  
 +++ |\*0;phi2;1;initial| redefined  
 Compiling body of rule phi3 to compute value of type  
 Polynomial Integer  
 +++ |\*0;phi3;1;initial| redefined  
 Compiling body of rule pans to compute value of type  
 UnivariatePolynomial(x,Polynomial Integer)  
 +++ |\*0;pans;1;initial| redefined  
 Compiling body of rule pans1 to compute value of type  
 List Polynomial Integer  
 +++ |\*0;pans1;1;initial| redefined  
 Compiling body of rule leq to compute value of type  
 List List Equation Fraction Polynomial Integer  
 +++ |\*0;leq;1;initial| redefined

$$\left[ \begin{array}{l} s_1 = \frac{s_0 G}{3}, s_2 = \frac{3 s_0 H + s_0 G^2 + 6 s_0 G}{18}, \\ s_3 = \frac{(9 s_0 G + 54 s_0) H + s_0 G^3 + 18 s_0 G^2 + 72 s_0 G}{162}, \\ s_4 = \frac{\left( \frac{27 s_0 H^2 + (18 s_0 G^2 + 378 s_0 G + 1296 s_0) H + s_0 G^4 + 36 s_0 G^3 + 396 s_0 G^2 + 1296 s_0 G}{1944} \right)}{1944} \end{array} \right] \quad (14)$$

Type: List List Equation Fraction Polynomial Integer

```
n==7
Compiled code for n has been cleared.
Compiled code for leq has been cleared.
Compiled code for pans1 has been cleared.
Compiled code for phi2 has been cleared.
Compiled code for phi has been cleared.
Compiled code for phi3 has been cleared.
Compiled code for phil has been cleared.
Compiled code for pans has been cleared.
1 old definition(s) deleted for function or rule n
```

Type: Void

```

leq
Compiling body of rule n to compute value of type
  PositiveInteger
+++ |*0;n;1;initial| redefined
Compiling body of rule phi to compute value of type
  Expression Integer
+++ |*0;phi;1;initial| redefined
Compiling body of rule phil to compute value of type
  Expression Integer
+++ |*0;phil;1;initial| redefined
Compiling body of rule phi2 to compute value of type
  Expression Integer
+++ |*0;phi2;1;initial| redefined
Compiling body of rule phi3 to compute value of type
  Polynomial Integer
+++ |*0;phi3;1;initial| redefined
Compiling body of rule pans to compute value of type
  UnivariatePolynomial(x,Polynomial Integer)
+++ |*0;pans;1;initial| redefined
Compiling body of rule pans1 to compute value of type
  List Polynomial Integer
+++ |*0;pans1;1;initial| redefined
Compiling body of rule leq to compute value of type
  List List Equation Fraction Polynomial Integer
+++ |*0;leq;1;initial| redefined

```

$$\left[ \begin{array}{l}
s_1 = \frac{s_0 G}{3}, \quad s_2 = \frac{3 s_0 H + s_0 G^2 + 6 s_0 G}{18}, \\
s_3 = \frac{(9 s_0 G + 54 s_0) H + s_0 G^3 + 18 s_0 G^2 + 72 s_0 G}{162}, \\
s_4 = \frac{\left( 27 s_0 H^2 + (18 s_0 G^2 + 378 s_0 G + 1296 s_0) H + \right.}{s_0 G^4 + 36 s_0 G^3 + 396 s_0 G^2 + 1296 s_0 G} \\
\left. \right), \\
s_5 = \frac{\left( (135 s_0 G + 2268 s_0) H^2 + (30 s_0 G^3 + 1350 s_0 G^2 + \right.}{1188 s_0 G^3 + 9504 s_0 G^2 + 25920 s_0 G} \\
\left. 16416 s_0 G + 38880 s_0) H + s_0 G^5 + 60 s_0 G^4 + \right. \\
\left. \right), \\
s_6 = \frac{\left( 405 s_0 H^3 + (405 s_0 G^2 + 18468 s_0 G + 174960 s_0) H^2 \right.}{27864 s_0 G^3 + 90720 s_0 G^2} \\
\left. + (45 s_0 G^4 + 3510 s_0 G^3 + 88776 s_0 G^2 + 777600 s_0 G \right. \\
\left. + 1166400 s_0) H + s_0 G^6 + 90 s_0 G^5 + 2628 s_0 G^4 + \right. \\
\left. \right), \\
s_7 = \frac{\left( (2835 s_0 G + 91854 s_0) H^3 + (945 s_0 G^3 + 81648 s_0 G^2 \right.}{11022480} \\
\left. + 2082996 s_0 G + 14171760 s_0) H^2 + (63 s_0 G^5 + \right. \\
\left. 7560 s_0 G^4 + 317520 s_0 G^3 + 5554008 s_0 G^2 + \right. \\
\left. 34058880 s_0 G) H + s_0 G^7 + 126 s_0 G^6 + 4788 s_0 G^5 \right. \\
\left. + 25272 s_0 G^4 - 1744416 s_0 G^3 - 26827200 s_0 G^2 \right. \\
\left. - 97977600 s_0 G \right) }{11022480}
\end{array} \right] \quad (16)$$

Type: List List Equation Fraction Polynomial Integer

## 9.42 LinearOrdinary- Differential- Operator1

---

### 9.42.1 Differential Operators with Rational Function Coefficients

---

We begin by defining RFZ to be the rational functions in  $x$  with integer coefficients and  $Dx$  to be the differential operator for  $d/dx$ .

Operators are created using the usual arithmetic operations.

Operator multiplication corresponds to functional composition.

LinearOrdinaryDifferentialOperator1(A) is the domain of linear ordinary differential operators with coefficients in the differential ring A. Issue the system command `)show LinearOrdinaryDifferentialOperator1` to display the full list of operations defined by LinearOrdinaryDifferentialOperator1.

This example shows differential operators with rational function coefficients. In this case operator multiplication is non-commutative and, since the coefficients form a field, an operator division algorithm exists.

```
RFZ := Fraction UnivariatePolynomial('x, Integer)
Fraction UnivariatePolynomial (x , Integer )
```

(1)  
Type: Domain

```
x : RFZ := 'x
x
```

(2)  
Type: Fraction UnivariatePolynomial(x, Integer)

```
Dx : LOD01 RFZ := D()
D
```

(3)  
Type: LinearOrdinaryDifferentialOperator1 Fraction UnivariatePolynomial(x, Integer)

```
b : LOD01 RFZ := 3*x**2*Dx**2 + 2*Dx + 1/x
3 x^2 D^2 + 2 D + 1/x
```

(4)  
Type: LinearOrdinaryDifferentialOperator1 Fraction UnivariatePolynomial(x, Integer)

```
a : LOD01 RFZ := b*(5*x*Dx + 7)
15 x^3 D^3 + (51 x^2 + 10 x) D^2 + 29 D + 7/x
```

(5)  
Type: LinearOrdinaryDifferentialOperator1 Fraction UnivariatePolynomial(x, Integer)

```
p := x**2 + 1/x**2
x^4 + 1/x^2
```

(6)  
Type: Fraction UnivariatePolynomial(x, Integer)

Since operator coefficients depend on  $x$ , the multiplication is not commutative.

$$\frac{(a*b - b*a) \, p}{x^4} = \frac{-75 x^4 + 540 x - 75}{x^4} \quad (7)$$

Type: Fraction UnivariatePolynomial(x, Integer)

When the coefficients of operator polynomials come from a field, as in this case, it is possible to define operator division. Division on the left and division on the right yield different results when the multiplication is non-commutative.

The results of **leftDivide** and **rightDivide** are quotient-remainder pairs satisfying:

$$\begin{aligned} \text{leftDivide}(a,b) &= [q, r] \text{ such that } a = b*q + r \\ \text{rightDivide}(a,b) &= [q, r] \text{ such that } a = q*b + r \end{aligned}$$

In both cases, the **degree** of the remainder,  $r$ , is less than the degree of  $b$ .

$$\text{ld} := \text{leftDivide}(a,b) \quad [quotient = 5 x D + 7, remainder = 0] \quad (8)$$

Type: Record(quotient: LinearOrdinaryDifferentialOperator1 Fraction UnivariatePolynomial(x, Integer), remainder: LinearOrdinaryDifferentialOperator1 Fraction UnivariatePolynomial(x, Integer))

$$\begin{aligned} a &= b * \text{ld.quotient} + \text{ld.remainder} \\ 15 x^3 D^3 + (51 x^2 + 10 x) D^2 + 29 D + \frac{7}{x} &= 15 x^3 D^3 + \\ (51 x^2 + 10 x) D^2 + 29 D + \frac{7}{x} & \end{aligned} \quad (9)$$

Type: Equation LinearOrdinaryDifferentialOperator1 Fraction UnivariatePolynomial(x, Integer)

The operations of left and right division are so-called because the quotient is obtained by dividing  $a$  on that side by  $b$ .

$$\text{rd} := \text{rightDivide}(a,b) \quad \left[ quotient = 5 x D + 7, remainder = 10 D + \frac{5}{x} \right] \quad (10)$$

Type: Record(quotient: LinearOrdinaryDifferentialOperator1 Fraction UnivariatePolynomial(x, Integer), remainder: LinearOrdinaryDifferentialOperator1 Fraction UnivariatePolynomial(x, Integer))

$$\begin{aligned} a &= \text{rd.quotient} * b + \text{rd.remainder} \\ 15 x^3 D^3 + (51 x^2 + 10 x) D^2 + 29 D + \frac{7}{x} &= 15 x^3 D^3 + \\ (51 x^2 + 10 x) D^2 + 29 D + \frac{7}{x} & \end{aligned} \quad (11)$$

Type: Equation LinearOrdinaryDifferentialOperator1 Fraction UnivariatePolynomial(x, Integer)

Operations **rightQuotient** and **rightRemainder** are available if only one of the quotient or remainder are of interest to you. This is the quotient from right division.

This is the remainder from right division. The corresponding “left” functions **leftQuotient** and **leftRemainder** are also available.

For exact division, the operations **leftExactQuotient** and **rightExactQuotient** are supplied. These return the quotient but only if the remainder is zero. The call **rightExactQuotient(a,b)** would yield an error.

The division operations allow the computation of left and right greatest common divisors (**leftGcd** and **rightGcd**) via remainder sequences, and consequently the computation of left and right least common multiples (**rightLcm** and **leftLcm**).

Note that a greatest common divisor doesn’t necessarily divide **a** and **b** on both sides. Here the left greatest common divisor does not divide **a** on the right.

Similarly, a least common multiple is not necessarily divisible from both sides.

**rightQuotient(a,b)**

$$5 \ x \ D + 7 \quad (12)$$

Type: LinearOrdinaryDifferentialOperator1 Fraction UnivariatePolynomial(x, Integer)

**rightRemainder(a,b)**

$$10 \ D + \frac{5}{x} \quad (13)$$

Type: LinearOrdinaryDifferentialOperator1 Fraction UnivariatePolynomial(x, Integer)

**leftExactQuotient(a,b)**

$$5 \ x \ D + 7 \quad (14)$$

Type: Union(LinearOrdinaryDifferentialOperator1 Fraction UnivariatePolynomial(x, Integer), ...)

**e := leftGcd(a,b)**

$$3 \ x^2 \ D^2 + 2 \ D + \frac{1}{x} \quad (15)$$

Type: LinearOrdinaryDifferentialOperator1 Fraction UnivariatePolynomial(x, Integer)

**leftRemainder(a, e)**

$$0 \quad (16)$$

Type: LinearOrdinaryDifferentialOperator1 Fraction UnivariatePolynomial(x, Integer)

**rightRemainder(a, e)**

$$10 \ D + \frac{5}{x} \quad (17)$$

Type: LinearOrdinaryDifferentialOperator1 Fraction UnivariatePolynomial(x, Integer)

**f := rightLcm(a,b)**

$$15 \ x^3 \ D^3 + \left(51 \ x^2 + 10 \ x\right) \ D^2 + 29 \ D + \frac{7}{x} \quad (18)$$

Type: LinearOrdinaryDifferentialOperator1 Fraction UnivariatePolynomial(x, Integer)



`rightRemainder(f, b)`

$$10 D + \frac{5}{x} \quad (19)$$

Type: LinearOrdinaryDifferentialOperator1 Fraction UnivariatePolynomial(x, Integer)

`leftRemainder(f, b)`

$$0 \quad (20)$$

Type: LinearOrdinaryDifferentialOperator1 Fraction UnivariatePolynomial(x, Integer)

## 9.43 LinearOrdinary- Differential- Operator2

---

LinearOrdinaryDifferentialOperator2(A, M) is the domain of linear ordinary differential operators with coefficients in the differential ring A and operating on M, an A-module. This includes the cases of operators which are polynomials in D acting upon scalar or vector expressions of a single variable. The coefficients of the operator polynomials can be integers, rational functions, matrices or elements of other domains. Issue the system command `)show LinearOrdinaryDifferentialOperator2` to display the full list of operations defined by LinearOrdinaryDifferentialOperator2.

### 9.43.1 Differential Operators with Constant Coefficients

---

We begin by making type assignments so we can conveniently refer to univariate polynomials in  $x$  over the rationals.

```
Q := Fraction Integer
Fraction Integer
Type: Domain
```

```
PQ := UnivariatePolynomial('x, Q)
UnivariatePolynomial (x , Fraction Integer )
Type: Domain
```

```
x: PQ := 'x
x
Type: UnivariatePolynomial(x, Fraction Integer)
```

Now we assign  $Dx$  to be the differential operator  $D$  corresponding to  $d/dx$ .

```
Dx: LOD02(Q, PQ) := D()
D
Type: LinearOrdinaryDifferentialOperator2(Fraction Integer,
UnivariatePolynomial(x, Fraction Integer))
```

New operators are created as polynomials in  $D()$ .

```
a := Dx + 1
D + 1
Type: LinearOrdinaryDifferentialOperator2(Fraction Integer,
UnivariatePolynomial(x, Fraction Integer))
```

```
b := a + 1/2*Dx**2 - 1/2
1/2 D^2 + D + 1/2
Type: LinearOrdinaryDifferentialOperator2(Fraction Integer,
UnivariatePolynomial(x, Fraction Integer))
```

To apply the operator  $a$  to the value  $p$  the usual function call syntax is used.

$$p := 4x^2 + \frac{2}{3} \quad (7)$$

Type: UnivariatePolynomial(x, Fraction Integer)

$$a \ p$$

$$4x^2 + 8x + \frac{2}{3} \quad (8)$$

Type: UnivariatePolynomial(x, Fraction Integer)

Operator multiplication is defined by the identity  $(a*b) \ p = a(b(p))$

$$(a * b) \ p = a \ b \ p$$

$$2x^2 + 12x + \frac{37}{3} = 2x^2 + 12x + \frac{37}{3} \quad (9)$$

Type: Equation UnivariatePolynomial(x, Fraction Integer)

Exponentiation follows from multiplication.

$$c := (1/9)*b*(a + b)^2$$

$$\frac{1}{72} D^6 + \frac{5}{36} D^5 + \frac{13}{24} D^4 + \frac{19}{18} D^3 + \frac{79}{72} D^2 + \frac{7}{12} D + \frac{1}{8} \quad (10)$$

Type: LinearOrdinaryDifferentialOperator2(Fraction Integer, UnivariatePolynomial(x, Fraction Integer))

Finally, note that operator expressions may be applied directly.

$$(a^2 - 3/4*b + c) \ (p + 1)$$

$$3x^2 + \frac{44}{3}x + \frac{541}{36} \quad (11)$$

Type: UnivariatePolynomial(x, Fraction Integer)

### 9.43.2 Differential Operators with Matrix Coefficients Operating on Vectors

This is another example of linear ordinary differential operators with non-commutative multiplication. Unlike the rational function case, the differential ring of square matrices (of a given dimension) with univariate polynomial entries does not form a field. Thus the number of operations available is more limited.

In this section, the operators have three by three matrix coefficients with polynomial entries.

$$PZ := \text{UnivariatePolynomial}(x, \text{Integer})$$

$$\text{UnivariatePolynomial}(x, \text{Integer}) \quad (1)$$

Type: Domain

$$x:PZ := 'x$$

$$x \quad (2)$$

Type: UnivariatePolynomial(x, Integer)

$$\text{Mat} := \text{SquareMatrix}(3, PZ)$$

$$\text{SquareMatrix}(3, \text{UnivariatePolynomial}(x, \text{Integer})) \quad (3)$$

Type: Domain

The operators act on the vectors `Vect` := `DPM(3, PZ, Mat, PZ)`;  
considered as a `Mat`-module.

(4)

Type: Domain

`Modo` := `LOD02(Mat, Vect)`;

(5)

Type: Domain

The matrix `m` is used as a  
coefficient and the vectors `p` and  
`q` are operated upon.

`m:Mat` := `matrix`  $[[x^{**2}, 1, 0], [1, x^{**4}, 0], [0, 0, 4*x^{**2}]]$

$$\begin{bmatrix} x^2 & 1 & 0 \\ 1 & x^4 & 0 \\ 0 & 0 & 4x^2 \end{bmatrix} \quad (6)$$

Type: `SquareMatrix(3, UnivariatePolynomial(x, Integer))`

`p:Vect` := `directProduct`  $[3*x^{**2}+1, 2*x, 7*x^{**3}+2*x]$

$$\begin{bmatrix} 3x^2 + 1, 2x, 7x^3 + 2x \end{bmatrix} \quad (7)$$

Type: `DirectProductMatrixModule(3, UnivariatePolynomial(x, Integer),  
SquareMatrix(3, UnivariatePolynomial(x, Integer)), UnivariatePolynomial(x,  
Integer))`

`q: Vect` := `m * p`

$$\begin{bmatrix} 3x^4 + x^2 + 2x, 2x^5 + 3x^2 + 1, 28x^5 + 8x^3 \end{bmatrix} \quad (8)$$

Type: `DirectProductMatrixModule(3, UnivariatePolynomial(x, Integer),  
SquareMatrix(3, UnivariatePolynomial(x, Integer)), UnivariatePolynomial(x,  
Integer))`

Now form a few operators.

`Dx` : `Modo` := `D()`

$$D \quad (9)$$

Type: `LinearOrdinaryDifferentialOperator2(SquareMatrix(3,  
UnivariatePolynomial(x, Integer)), DirectProductMatrixModule(3,  
UnivariatePolynomial(x, Integer), SquareMatrix(3, UnivariatePolynomial(x,  
Integer)), UnivariatePolynomial(x, Integer)))`

`a` : `Modo` := `Dx + m`

$$D + \begin{bmatrix} x^2 & 1 & 0 \\ 1 & x^4 & 0 \\ 0 & 0 & 4x^2 \end{bmatrix} \quad (10)$$

Type: `LinearOrdinaryDifferentialOperator2(SquareMatrix(3,  
UnivariatePolynomial(x, Integer)), DirectProductMatrixModule(3,  
UnivariatePolynomial(x, Integer), SquareMatrix(3, UnivariatePolynomial(x,  
Integer)), UnivariatePolynomial(x, Integer)))`

b : Mod0 := m\*Dx + 1

$$\begin{bmatrix} x^2 & 1 & 0 \\ 1 & x^4 & 0 \\ 0 & 0 & 4x^2 \end{bmatrix} D + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (11)$$

Type: LinearOrdinaryDifferentialOperator2(SquareMatrix(3,  
UnivariatePolynomial(x, Integer)), DirectProductMatrixModule(3,  
UnivariatePolynomial(x, Integer), SquareMatrix(3, UnivariatePolynomial(x,  
Integer)), UnivariatePolynomial(x, Integer)))

c := a\*b

$$\begin{bmatrix} x^2 & 1 & 0 \\ 1 & x^4 & 0 \\ 0 & 0 & 4x^2 \end{bmatrix} D^2 + \begin{bmatrix} x^4 + 2x + 2 & x^4 + x^2 & 0 \\ x^4 + x^2 & x^8 + 4x^3 + 2 & 0 \\ 0 & 0 & 16x^4 + 8x + 1 \end{bmatrix} D + \begin{bmatrix} x^2 & 1 & 0 \\ 1 & x^4 & 0 \\ 0 & 0 & 4x^2 \end{bmatrix} \quad (12)$$

Type: LinearOrdinaryDifferentialOperator2(SquareMatrix(3,  
UnivariatePolynomial(x, Integer)), DirectProductMatrixModule(3,  
UnivariatePolynomial(x, Integer), SquareMatrix(3, UnivariatePolynomial(x,  
Integer)), UnivariatePolynomial(x, Integer)))

These operators can be applied  
to vector values.

a p

$$\begin{bmatrix} 3x^4 + x^2 + 8x, 2x^5 + 3x^2 + 3, 28x^5 + 8x^3 + 21x^2 + 2 \end{bmatrix} \quad (13)$$

Type: DirectProductMatrixModule(3, UnivariatePolynomial(x, Integer),  
SquareMatrix(3, UnivariatePolynomial(x, Integer)), UnivariatePolynomial(x,  
Integer))

b p

$$\begin{bmatrix} 6x^3 + 3x^2 + 3, 2x^4 + 8x, 84x^4 + 7x^3 + 8x^2 + 2x \end{bmatrix} \quad (14)$$

Type: DirectProductMatrixModule(3, UnivariatePolynomial(x, Integer),  
SquareMatrix(3, UnivariatePolynomial(x, Integer)), UnivariatePolynomial(x,  
Integer))

$$(a + b + c) (p + q)$$

$$\begin{aligned} & \left[ 10x^8 + 12x^7 + 16x^6 + 30x^5 + 85x^4 + 94x^3 + 40x^2 + \right. \\ & 40x + 17, \\ & 10x^{12} + 10x^9 + 12x^8 + 92x^7 + 6x^6 + 32x^5 + 72x^4 + \\ & 28x^3 + 49x^2 + 32x + 19, \\ & \left. 2240x^8 + 224x^7 + 1280x^6 + 3508x^5 + 492x^4 + 751x^3 + \right. \\ & \left. 98x^2 + 18x + 4 \right] \end{aligned} \quad (15)$$

Type: DirectProductMatrixModule(3, UnivariatePolynomial(x, Integer),  
 SquareMatrix(3, UnivariatePolynomial(x, Integer)), UnivariatePolynomial(x,  
 Integer))

## 9.44 List

---

A *list* is a finite collection of elements in a specified order that can contain duplicates. A list is a convenient structure to work with because it is easy to add or remove elements and the length need not be constant. There are many different kinds of lists in AXIOM, but the default types (and those used most often) are created by the List constructor. For example, there are objects of type List Integer, List Float and List Polynomial Fraction Integer. Indeed, you can even have List List List Boolean (that is, lists of lists of lists of Boolean values). You can have lists of any type of AXIOM object.

### 9.44.1 Creating Lists

---

The spaces after the commas are optional, but they do improve the readability.

```
[2, 4, 5, 6]
```

```
[2, 4, 5, 6]
```

(1)

Type: List PositiveInteger

To create a list with the single element 1, you can use either [1] or the operation list.

```
[1]
```

```
[1]
```

(2)

Type: List PositiveInteger

```
list(1)
```

```
[1]
```

(3)

Type: List PositiveInteger

Once created, two lists **k** and **m** can be concatenated by issuing **append(k,m)**. **append** does *not* physically join the lists, but rather produces a new list with the elements coming from the two arguments.

```
append([1,2,3],[5,6,7])
```

```
[1, 2, 3, 5, 6, 7]
```

(4)

Type: List PositiveInteger

Use **cons** to append an element onto the front of a list.

```
cons(10,[9,8,7])
```

```
[10, 9, 8, 7]
```

(5)

Type: List PositiveInteger

### 9.44.2 Accessing List Elements

---

To determine whether a list has any elements, use the operation **empty?**.

```
empty? [x+1]
false
```

(1)  
Type: Boolean

Alternatively, equality with the list constant **nil** can be tested.

```
([] = nil)@Boolean
true
```

(2)  
Type: Boolean

We'll use this in some of the following examples.

```
k := [4,3,7,3,8,5,9,2]
[4, 3, 7, 3, 8, 5, 9, 2]
```

(3)  
Type: List PositivelInteger

Each of the next four expressions extracts the **first** element of **k**.

```
first k
4
```

(4)  
Type: PositivelInteger

```
k.first
4
```

(5)  
Type: PositivelInteger

```
k.1
4
```

(6)  
Type: PositivelInteger

```
k(1)
4
```

(7)  
Type: PositivelInteger

The last two forms generalize to **k.i** and **k(i)**, respectively, where  $1 \leq i \leq n$  and **n** equals the length of **k**.

This length is calculated by “#”.

```
n := #k
8
```

(8)  
Type: PositivelInteger

Performing an operation such as **k.i** is sometimes referred to as *indexing into k* or *elting into k*. The latter phrase comes about because the name of the operation that extracts elements is called **elt**. That is, **k.3** is just alternative syntax for **elt(k,3)**. It is important to remember that list indices begin with 1. If we issue **k := [1,3,2,9,5]** then **k.4** returns 9. It is an error to use an index that is not in the range from 1 to the length of the list.



The last element of a list is extracted by any of the following three expressions.

```
last k
```

(9)

Type: PositiveInteger

```
k.last
```

(10)

Type: PositiveInteger

This form computes the index of the last element and then extracts the element from the list.

```
k.(#k)
```

(11)

Type: PositiveInteger

### 9.44.3 Changing List Elements

---

We'll use this in some of the following examples.

```
k := [4,3,7,3,8,5,9,2]
```

```
[4, 3, 7, 3, 8, 5, 9, 2]
```

(1)

Type: List PositiveInteger

List elements are reset by using the `k.i` form on the left-hand side of an assignment. This expression resets the first element of `k` to `999`.

```
k.1 := 999
```

```
999
```

(2)

Type: PositiveInteger

As with indexing into a list, it is an error to use an index that is not within the proper bounds. Here you see that `k` was modified.

```
k
```

```
[999, 3, 7, 3, 8, 5, 9, 2]
```

(3)

Type: List PositiveInteger

The operation that performs the assignment of an element to a particular position in a list is called **setelt**. This operation is *destructive* in that it changes the list. In the above example, the assignment returned the value `999` and `k` was modified. For this reason, lists are called *mutable* objects: it is possible to change part of a list (mutate it) rather than always returning a new list reflecting the intended modifications.

Moreover, since lists can share structure, changes to one list can sometimes affect others.

```
k := [1,2]
```

```
[1, 2]
```

(4)

Type: List PositiveInteger

	<code>m := cons(0,k)</code>		
	<code>[0, 1, 2]</code>	(5)	
		Type: List Integer	
Change the second element of m.	<code>m.2 := 99</code>		
	<code>99</code>	(6)	
		Type: PositiveInteger	
See, m was altered.	<code>m</code>		
	<code>[0, 99, 2]</code>	(7)	
		Type: List Integer	
But what about k? It changed too!	<code>k</code>		
	<code>[99, 2]</code>	(8)	
		Type: List PositiveInteger	

#### 9.44.4 Other Functions

An operation that is used frequently in list processing is that which returns all elements in a list after the first element.	<code>k := [1,2,3]</code>		
	<code>[1, 2, 3]</code>	(1)	
		Type: List PositiveInteger	
Use the <b>rest</b> operation to do this.	<code>rest k</code>		
	<code>[2, 3]</code>	(2)	
		Type: List PositiveInteger	
To remove duplicate elements in a list k, use <b>removeDuplicates</b> .	<code>removeDuplicates [4,3,4,3,5,3,4]</code>		
	<code>[4, 3, 5]</code>	(3)	
		Type: List PositiveInteger	
To get a list with elements in the order opposite to those in a list k, use <b>reverse</b> .	<code>reverse [1,2,3,4,5,6]</code>		
	<code>[6, 5, 4, 3, 2, 1]</code>	(4)	
		Type: List PositiveInteger	
To test whether an element is in a list, use <b>member?</b> : <b>member?(a,k)</b> returns <b>true</b> or <b>false</b> depending on whether a is in k or not.	<code>member?(1/2, [3/4,5/6,1/2])</code>		
	<code>true</code>	(5)	
		Type: Boolean	

```
member? (1/12, [3/4, 5/6, 1/2])
```

```
false
```

(6)

Type: Boolean

As an exercise, the reader should determine how to get a list containing all but the last of the elements in a given non-empty list *k*.<sup>4</sup>

### 9.44.5 Dot, Dot

Certain lists are used so often that AXIOM provides an easy way of constructing them. If *n* and *m* are integers, then **expand** [*n*..*m*] creates a list containing *n*, *n*+1, ... *m*. If *n* > *m* then the list is empty. It is actually permissible to leave off the *m* in the dot-dot construction (see below).

The dot-dot notation can be used more than once in a list construction and with specific elements being given. Items separated by dots are called *segments*.

```
[1..3, 10, 20..23]
```

```
[1..3, 10..10, 20..23]
```

(1)

Type: List Segment PositiveInteger

Segments can be expanded into the range of items between the endpoints by using **expand**.

```
expand [1..3, 10, 20..23]
```

```
[1, 2, 3, 10, 20, 21, 22, 23]
```

(2)

Type: List Integer

What happens if we leave off a number on the right-hand side of “..”?

```
expand [1..]
```

```
[1, 2, 3, 4, 5, 6, 7, ...]
```

(3)

Type: Stream Integer

What is created in this case is a Stream which is a generalization of a list. See ‘Stream’ on page 575 for more information.

<sup>4</sup>`reverse(rest(reverse(k)))` works.

## 9.45 MakeFunction

Suppose that you have obtained the following expression after several computations and that you now want to tabulate the numerical values of **f** for **x** between -1 and +1 with increment 0.1.

It is sometimes useful to be able to define a function given by the result of a calculation.

$$\begin{aligned} \text{expr} &:= (\mathbf{x} - \exp \mathbf{x} + 1)**2 * (\sin(\mathbf{x}**2) * \mathbf{x} + 1)**3 \\ &\left(x^3 e^{x^2} + (-2 x^4 - 2 x^3) e^x + x^5 + 2 x^4 + x^3\right) \sin(x^2)^3 + \\ &\left(3 x^2 e^{x^2} + (-6 x^3 - 6 x^2) e^x + 3 x^4 + 6 x^3 + 3 x^2\right) \sin(x^2)^2 + \\ &\left(3 x e^{x^2} + (-6 x^2 - 6 x) e^x + 3 x^3 + 6 x^2 + 3 x\right) \sin(x^2) + \\ &e^{x^2} + (-2 x - 2) e^x + x^2 + 2 x + 1 \end{aligned} \quad (1)$$

Type: Expression Integer

You could, of course, use the function **eval** within a loop and evaluate **expr** twenty-one times, but this would be quite slow. A better way is to create a numerical function **f** such that **f(x)** is defined by the expression **expr** above, but without retyping **expr**! The package **MakeFunction** provides the operation **function** which does exactly this.

Issue this to create the function **f(x)** given by **expr**.

$$\begin{aligned} &\text{function}(\text{expr}, \mathbf{f}, \mathbf{x}) \\ &\mathbf{f} \end{aligned} \quad (2)$$

Type: Symbol

To tabulate **expr**, we can now quickly evaluate **f** 21 times.

$$\begin{aligned} &\text{tbl} := [\mathbf{f}(0.1 * \mathbf{i} - 1) \text{ for } \mathbf{i} \text{ in } 0..20]; \\ &\text{Compiling function } \mathbf{f} \text{ with type Float } \rightarrow \text{Float} \end{aligned} \quad (2)$$

Type: List Float

Use the list **[x1,...,xn]** as the third argument to **function** to create a multivariate function **f(x1,...,xn)**.

$$\begin{aligned} \mathbf{e} &:= (\mathbf{x} - \mathbf{y} + 1)**2 * (\mathbf{x}**2 * \mathbf{y} + 1)**2 \\ &x^4 y^4 + (-2 x^5 - 2 x^4 + 2 x^2) y^3 + \\ &(x^6 + 2 x^5 + x^4 - 4 x^3 - 4 x^2 + 1) y^2 + \\ &(2 x^4 + 4 x^3 + 2 x^2 - 2 x - 2) y + x^2 + 2 x + 1 \end{aligned} \quad (4)$$

Type: Polynomial Integer

$$\begin{aligned} &\text{function}(\mathbf{e}, \mathbf{g}, [\mathbf{x}, \mathbf{y}]) \\ &\mathbf{g} \end{aligned} \quad (5)$$

Type: Symbol

In the case of just two variables, they can be given as arguments without making them into a list.

```
function(e, h, x, y)
h
```

(6)  
Type: Symbol

Note that the functions created by **function** are not limited to floating point numbers, but can be applied to any type for which they are defined.

```
m1 := squareMatrix [[1, 2], [3, 4]]
```

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

(7)  
Type: SquareMatrix(2, Integer)

```
m2 := squareMatrix [[1, 0], [-1, 1]]
```

$$\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

(8)  
Type: SquareMatrix(2, Integer)

```
h(m1, m2)
Compiling function h with type (SquareMatrix(2,
Integer), SquareMatrix(2, Integer)) -> SquareMatrix(
2, Integer)
```

$$\begin{bmatrix} -7836 & 8960 \\ -17132 & 19588 \end{bmatrix}$$

(9)  
Type: SquareMatrix(2, Integer)

For more information, see Section 6.14 on page 207. Issue the system command `)show MakeFunction` to display the full list of operations defined by `MakeFunction`.

## 9.46 Mapping- Package1

---

We begin by creating an example function that raises a rational number to an integer exponent.

Function are objects of type Mapping. In this section we demonstrate some library operations from the packages MappingPackage1, MappingPackage2, and MappingPackage3 that manipulate and create functions. Some terminology: a *nullary* function takes no arguments, a *unary* function takes one argument, and a *binary* function takes two arguments.

```
power(q: FRAC INT, n: INT): FRAC INT == q**n
```

```
Function declaration power : (Fraction Integer,
    Integer) -> Fraction Integer has been added to
workspace.
```

Type: Void

```
power(2,3)
```

```
Compiling function power with type (Fraction Integer,
    Integer) -> Fraction Integer
```

```
8
```

(2)

Type: Fraction Integer

The **twist** operation transposes the arguments of a binary function. Here **rewop(a, b)** is **power(b, a)**.

```
rewop := twist power
```

```
theMap (...)
```

(3)

Type: ((Integer, Fraction Integer) → Fraction Integer)

This is  $2^3$ .

```
rewop(3, 2)
```

```
8
```

(4)

Type: Fraction Integer

Now we define **square** in terms of **power**.

```
square: FRAC INT -> FRAC INT
```

Type: Void

The **curryRight** operation creates a unary function from a binary one by providing a constant argument on the right.

```
square:= curryRight(power, 2)
```

```
theMap (...)
```

(6)

Type: (Fraction Integer → Fraction Integer)

Likewise, the **curryLeft** operation provides a constant argument on the left.

```
square 4
```

```
16
```

(7)

Type: Fraction Integer

The **constantRight** operation creates (in a trivial way) a binary function from a unary one: **constantRight(f)** is the function **g** such that **g(a,b)=f(a)**.

```
squirrel:= constantRight(square)$MAPPKG3(FRAC INT,FRAC
    INT,FRAC INT)
```

```
theMap (...)
```

(8)

Type: ((Fraction Integer, Fraction Integer) → Fraction Integer)

Likewise, <code>constantLeft(f)</code> is the function <code>g</code> such that <code>g(a,b)= f(b)</code> .	<code>squirrel(1/2, 1/3)</code> $\frac{1}{4}$	(9)	Type: Fraction Integer
The <b>curry</b> operation makes a unary function nullary.	<code>sixteen := curry(square, 4/1)</code> <code>theMap (...)</code>	(10)	Type: $() \rightarrow$ Fraction Integer
	<code>sixteen()</code> 16	(11)	Type: Fraction Integer
The “*” operation constructs composed functions.	<code>square2:=square*square</code> <code>theMap (...)</code>	(12)	Type: $(\text{Fraction Integer} \rightarrow \text{Fraction Integer})$
	<code>square2 3</code> 81	(13)	Type: Fraction Integer
Use the “**” operation to create functions that are <code>n</code> -fold iterations of other functions.	<code>sc(x: FRAC INT): FRAC INT == x + 1</code> Function declaration <code>sc : Fraction Integer -&gt;</code> Fraction Integer has been added to workspace.		Type: Void
This is a list of Mapping objects.	<code>incfns := [sc**i for i in 0..10]</code> Compiling function <code>sc</code> with type Fraction Integer -> Fraction Integer <code>[theMap (...), theMap (...), theMap (...), theMap (...),</code> <code>theMap (...), theMap (...), theMap (...), theMap (...),</code> <code>theMap (...), theMap (...), theMap (...)]</code>	(15)	Type: List $(\text{Fraction Integer} \rightarrow \text{Fraction Integer})$
This is a list of applications of those functions.	<code>[f 4 for f in incfns]</code> [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]	(16)	Type: List Fraction Integer
Use the <b>recur</b> operation for recursion: <code>g := recur f</code> means <code>g(n,x) == f(n,f(n-1,...f(1,x)))</code> .	<code>times(n:NNI, i:INT):INT == n*i</code> Function declaration <code>times : (NonNegativeInteger, Integer) -&gt;</code> Integer has been added to workspace.		Type: Void

	<pre> r := recur(times) Compiling function times with type (   NonNegativeInteger,Integer) -&gt; Integer theMap (...) </pre>	(18)
		Type: ((NonNegativeInteger, Integer) → Integer)
This is a factorial function.	<pre> fact := curryRight(r, 1) theMap (...) </pre>	(19)
		Type: (NonNegativeInteger → Integer)
	<pre> fact 4 24 </pre>	(20)
		Type: PositiveInteger
Constructed functions can be used within other functions.	<pre> mto2ton(m, n) ==   raiser := square**n   raiser m </pre>	
		Type: Void
This is $3^{2^3}$ .	<pre> mto2ton(3, 3) Compiling function mto2ton with type (PositiveInteger   ,PositiveInteger) -&gt; Fraction Integer 6561 </pre>	(22)
		Type: Fraction Integer
Here <b>shiftfib</b> is a unary function that modifies its argument.	<pre> shiftfib(r: List INT) : INT ==   t := r.1   r.1 := r.2   r.2 := r.2 + t   t Function declaration shiftfib : List Integer -&gt;   Integer has been added to workspace. </pre>	
		Type: Void
By currying over the argument we get a function with private state.	<pre> fibinit: List INT := [0, 1] [0, 1] </pre>	(24)
		Type: List Integer
	<pre> fibs := curry(shiftfib, fibinit) Compiling function shiftfib with type List Integer   -&gt; Integer theMap (...) </pre>	(25)
		Type: () → Integer



```
[fibs() for i in 0..30]
```

[0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987,  
1597, 2584, 4181, 6765, 10946, 17711, 28657, (26)  
46368, 75025, 121393, 196418, 317811, 514229, 832040]

Type: List Integer

## 9.47 Matrix

The Matrix domain provides arithmetic operations on matrices and standard functions from linear algebra. This domain is similar to the TwoDimensionalArray domain, except that the entries for Matrix must belong to a Ring.

### 9.47.1 Creating Matrices

If the matrix has almost all items equal to the same value, use **new** to create a matrix filled with that value and then reset the entries that are different.

```
m : Matrix(Integer) := new(3,3,0)
```

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (1)$$

Type: Matrix Integer

To change the entry in the second row, third column to 5, use **setelt**.

```
setelt(m,2,3,5)
```

$$5 \quad (2)$$

Type: PositiveInteger

An alternative syntax is to use assignment.

```
m(1,2) := 10
```

$$10 \quad (3)$$

Type: PositiveInteger

The matrix was *destructively modified*.

```
m
```

$$\begin{bmatrix} 0 & 10 & 0 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix} \quad (4)$$

Type: Matrix Integer

If you already have the matrix entries as a list of lists, use **matrix**.

```
matrix [[1,2,3,4],[0,9,8,7]]
```

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 9 & 8 & 7 \end{bmatrix} \quad (5)$$

Type: Matrix Integer

If the matrix is diagonal, use **diagonalMatrix**.

```
dm := diagonalMatrix [1,x**2,x**3,x**4,x**5]
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & x^2 & 0 & 0 & 0 \\ 0 & 0 & x^3 & 0 & 0 \\ 0 & 0 & 0 & x^4 & 0 \\ 0 & 0 & 0 & 0 & x^5 \end{bmatrix} \quad (6)$$

Type: Matrix Polynomial Integer

Use **setRow!** and **setColumn!** to change a row or column of a matrix.

```
setRow!(dm,5,vector [1,1,1,1,1])
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & x^2 & 0 & 0 & 0 \\ 0 & 0 & x^3 & 0 & 0 \\ 0 & 0 & 0 & x^4 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \quad (7)$$

Type: Matrix Polynomial Integer

```
setColumn!(dm,2,vector [y,y,y,y,y])
```

$$\begin{bmatrix} 1 & y & 0 & 0 & 0 \\ 0 & y & 0 & 0 & 0 \\ 0 & y & x^3 & 0 & 0 \\ 0 & y & 0 & x^4 & 0 \\ 1 & y & 1 & 1 & 1 \end{bmatrix} \quad (8)$$

Type: Matrix Polynomial Integer

Use **copy** to make a copy of a matrix.

```
cdm := copy(dm)
```

$$\begin{bmatrix} 1 & y & 0 & 0 & 0 \\ 0 & y & 0 & 0 & 0 \\ 0 & y & x^3 & 0 & 0 \\ 0 & y & 0 & x^4 & 0 \\ 1 & y & 1 & 1 & 1 \end{bmatrix} \quad (9)$$

Type: Matrix Polynomial Integer

This is useful if you intend to modify a matrix destructively but want a copy of the original.

```
setelt(dm,4,1,1-x**7)
```

$$-x^7 + 1 \quad (10)$$

Type: Polynomial Integer

```
[dm,cdm]
```

$$\left[ \begin{bmatrix} 1 & y & 0 & 0 & 0 \\ 0 & y & 0 & 0 & 0 \\ 0 & y & x^3 & 0 & 0 \\ -x^7 + 1 & y & 0 & x^4 & 0 \\ 1 & y & 1 & 1 & 1 \end{bmatrix}, \right. \quad (11)$$

$$\left. \begin{bmatrix} 1 & y & 0 & 0 & 0 \\ 0 & y & 0 & 0 & 0 \\ 0 & y & x^3 & 0 & 0 \\ 0 & y & 0 & x^4 & 0 \\ 1 & y & 1 & 1 & 1 \end{bmatrix} \right]$$

Type: List Matrix Polynomial Integer

Use **subMatrix** to extract part of an existing matrix. The syntax is **subMatrix**(*m*, *firstrow*, *lastrow*, *firstcol*, *lastcol*).

$$\text{subMatrix}(\text{dm}, 2, 3, 2, 4)$$

$$\begin{bmatrix} y & 0 & 0 \\ y & x^3 & 0 \end{bmatrix} \quad (12)$$

Type: Matrix Polynomial Integer

To change a submatrix, use **setsubMatrix!**.

$$\text{d} := \text{diagonalMatrix} [1.2, -1.3, 1.4, -1.5]$$

$$\begin{bmatrix} 1.2 & 0.0 & 0.0 & 0.0 \\ 0.0 & -1.3 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.4 & 0.0 \\ 0.0 & 0.0 & 0.0 & -1.5 \end{bmatrix} \quad (13)$$

Type: Matrix Float

If **e** is too big to fit where you specify, an error message is displayed. Use **subMatrix** to extract part of **e**, if necessary.

$$\text{e} := \text{matrix} [[6.7, 9.11], [-31.33, 67.19]]$$

$$\begin{bmatrix} 6.7 & 9.11 \\ -31.33 & 67.19 \end{bmatrix} \quad (14)$$

Type: Matrix Float

This changes the submatrix of **d** whose upper left corner is at the first row and second column and whose size is that of **e**.

$$\text{setsubMatrix!}(\text{d}, 1, 2, \text{e})$$

$$\begin{bmatrix} 1.2 & 6.7 & 9.11 & 0.0 \\ 0.0 & -31.33 & 67.19 & 0.0 \\ 0.0 & 0.0 & 1.4 & 0.0 \\ 0.0 & 0.0 & 0.0 & -1.5 \end{bmatrix} \quad (15)$$

Type: Matrix Float

$$\text{d}$$

$$\begin{bmatrix} 1.2 & 6.7 & 9.11 & 0.0 \\ 0.0 & -31.33 & 67.19 & 0.0 \\ 0.0 & 0.0 & 1.4 & 0.0 \\ 0.0 & 0.0 & 0.0 & -1.5 \end{bmatrix} \quad (16)$$

Type: Matrix Float

Matrices can be joined either horizontally or vertically to make new matrices.

$$\text{a} := \text{matrix} [[1/2, 1/3, 1/4], [1/5, 1/6, 1/7]]$$

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{bmatrix} \quad (17)$$

Type: Matrix Fraction Integer

$$\text{b} := \text{matrix} [[3/5, 3/7, 3/11], [3/13, 3/17, 3/19]]$$

$$\begin{bmatrix} \frac{3}{5} & \frac{3}{7} & \frac{3}{11} \\ \frac{3}{13} & \frac{3}{17} & \frac{3}{19} \end{bmatrix} \quad (18)$$

Type: Matrix Fraction Integer

Use **horizConcat** to append them side to side. The two matrices must have the same number of rows.

$$\text{horizConcat}(a,b) \quad \left[ \begin{array}{ccc|ccc} \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{3}{5} & \frac{3}{7} & \frac{3}{11} \\ \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{3}{13} & \frac{3}{17} & \frac{3}{19} \end{array} \right] \quad (19)$$

Type: Matrix Fraction Integer

Use **vertConcat** to stack one upon the other. The two matrices must have the same number of columns.

$$\text{vab} := \text{vertConcat}(a,b) \quad \left[ \begin{array}{ccc} \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \\ \frac{3}{5} & \frac{3}{7} & \frac{3}{11} \\ \frac{3}{13} & \frac{3}{17} & \frac{3}{19} \end{array} \right] \quad (20)$$

Type: Matrix Fraction Integer

The operation **transpose** is used to create a new matrix by reflection across the main diagonal.

$$\text{transpose vab} \quad \left[ \begin{array}{ccc|ccc} \frac{1}{2} & \frac{1}{5} & \frac{3}{5} & \frac{3}{13} & \frac{3}{17} & \frac{3}{19} \\ \frac{1}{5} & \frac{1}{6} & \frac{3}{7} & \frac{3}{5} & \frac{3}{7} & \frac{3}{11} \\ \frac{3}{5} & \frac{3}{7} & \frac{3}{11} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \end{array} \right] \quad (21)$$

Type: Matrix Fraction Integer

## 9.47.2 Operations on Matrices

AXIOM provides both left and right scalar multiplication.

$$m := \text{matrix} \left[ [1,2], [3,4] \right] \quad \left[ \begin{array}{cc} 1 & 2 \\ 3 & 4 \end{array} \right] \quad (1)$$

Type: Matrix Integer

$$4 * m * (-5) \quad \left[ \begin{array}{cc} -20 & -40 \\ -60 & -80 \end{array} \right] \quad (2)$$

Type: Matrix Integer

You can add, subtract, and multiply matrices provided, of course, that the matrices have compatible dimensions. If not, an error message is displayed.

$$n := \text{matrix}([ [1,0,-2], [-3,5,1] ]) \quad \left[ \begin{array}{ccc} 1 & 0 & -2 \\ -3 & 5 & 1 \end{array} \right] \quad (3)$$

Type: Matrix Integer

This following product is defined but  $n * m$  is not.

$$m * n \quad \left[ \begin{array}{ccc} -5 & 10 & 0 \\ -9 & 20 & -2 \end{array} \right] \quad (4)$$

Type: Matrix Integer

The operations **nrows** and **ncols** return the number of rows and columns

of a matrix. You can extract a row or a column of a matrix using the operations **row** and **column**. The object returned is a Vector.

Here is the third column of the matrix **n**.

```
vec := column(n,3)
```

$$[-2, 1]$$
(5)

Type: Vector Integer

You can multiply a matrix on the left by a “row vector” and on the right by a “column vector.”

```
vec * m
```

$$[1, 0]$$
(6)

Type: Vector Integer

Of course, the dimensions of the vector and the matrix must be compatible or an error message is returned.

```
m * vec
```

$$[0, -2]$$
(7)

Type: Vector Integer

The operation **inverse** computes the inverse of a matrix if the matrix is invertible, and returns “failed” if not.

This Hilbert matrix is invertible.

```
hilb := matrix([[1/(i + j) for i in 1..3] for j in 1..3])
```

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \end{bmatrix}$$
(8)

Type: Matrix Fraction Integer

```
inverse(hilb)
```

$$\begin{bmatrix} 72 & -240 & 180 \\ -240 & 900 & -720 \\ 180 & -720 & 600 \end{bmatrix}$$
(9)

Type: Union(Matrix Fraction Integer, ...)

This matrix is not invertible.

```
mm := matrix([[1,2,3,4], [5,6,7,8], [9,10,11,12], [13,14,15,16]])
```

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$
(10)

Type: Matrix Integer

```
inverse(mm)
```

“failed”

(11)

Type: Union(“failed”, ...)

The operation **determinant** computes the determinant of a matrix provided that the entries of the matrix belong to a CommutativeRing.

The above matrix `mm` is not invertible and, hence, must have determinant 0.

`determinant(mm)`  
 0  
 (12)  
 Type: NonNegativeInteger

The operation **trace** computes the trace of a *square* matrix.

`trace(mm)`  
 34  
 (13)  
 Type: PositiveInteger

The operation **rank** computes the *rank* of a matrix: the maximal number of linearly independent rows or columns.

`rank(mm)`  
 2  
 (14)  
 Type: PositiveInteger

The operation **nullity** computes the *nullity* of a matrix: the dimension of its null space.

`nullity(mm)`  
 2  
 (15)  
 Type: PositiveInteger

The operation **nullSpace** returns a list containing a basis for the null space of a matrix. Note that the nullity is the number of elements in a basis for the null space.

`nullSpace(mm)`  
`[[1, -2, 1, 0], [2, -3, 0, 1]]`  
 (16)  
 Type: List Vector Integer

The operation **rowEchelon** returns the row echelon form of a matrix. It is easy to see that the rank of this matrix is two and that its nullity is also two.

`rowEchelon(mm)`  

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 4 & 8 & 12 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
  
 (17)  
 Type: Matrix Integer

For more information on related topics, see Section 1.7 on page 67, Section 8.4 on page 280, Section 9.27.4 on page 431, ‘Permanent’ on page 528, ‘Vector’ on page 601, ‘OneDimensionalArray’ on page 514, and ‘TwoDimensionalArray’ on page 590. Issue the system command `)show Matrix` to display the full list of operations defined by Matrix.

## 9.48 MultiSet

The domain `Multiset(R)` is similar to `Set(R)` except that multiplicities (counts of duplications) are maintained and displayed. Use the operation **multiset** to create multisets from lists. All the standard operations from sets are available for multisets. An element with multiplicity greater than one has the multiplicity displayed first, then a colon, and then the element.

Create a multiset of integers.

```
s := multiset [1,2,3,4,5,4,3,2,3,4,5,6,7,4,10]
{7, 2:5, 3:3, 1, 10, 6, 4:4, 2:2}
(1)
```

Type: Multiset PositivelInteger

The operation **insert!** adds an element to a multiset.

```
insert!(3,s)
{7, 2:5, 4:3, 1, 10, 6, 4:4, 2:2}
(2)
```

Type: Multiset PositivelInteger

Use **remove!** to remove an element. If a third argument is present, it specifies how many instances to remove. Otherwise all instances of the element are removed. Display the resulting multiset.

```
remove!(3,s,1); s
{7, 2:5, 3:3, 1, 10, 6, 4:4, 2:2}
(3)
```

Type: Multiset PositivelInteger

```
remove!(5,s); s
{7, 3:3, 1, 10, 6, 4:4, 2:2}
(4)
```

Type: Multiset PositivelInteger

The operation **count** returns the number of copies of a given value.

```
count(5,s)
0
(5)
```

Type: NonNegativeInteger

A second multiset.

```
t := multiset [2,2,2,-9]
{-9, 3:2}
(6)
```

Type: Multiset Integer

The **union** of two multisets is additive.

```
U := union(s,t)
{7, 3:3, 1, -9, 10, 6, 4:4, 5:2}
(7)
```

Type: Multiset Integer

The **intersect** operation gives the elements that are in common, with additive multiplicity.

```
I := intersect(s,t)
{5:2}
(8)
```

Type: Multiset Integer



The **difference** of **s** and **t** consists of the elements that **s** has but **t** does not. Elements are regarded as indistinguishable, so that if **s** and **t** have any element in common, the **difference** does not contain that element.

```
difference(s,t)
{7, 3:3, 1, 10, 6, 4:4}
```

(9)

Type: Multiset Integer

The **symmetricDifference** is the **union** of **difference(s, t)** and **difference(t, s)**.

```
S := symmetricDifference(s,t)
{7, 3:3, 1, -9, 10, 6, 4:4}
```

(10)

Type: Multiset Integer

Check that the **union** of the **symmetricDifference** and the **intersect** equals the **union** of the elements.

```
(U = union(S,I))@Boolean
true
```

(11)

Type: Boolean

Check some inclusion relations.

```
t1 := multiset [1,2,2,3]; [t1 < t, t1 < s, t < s, t1 <= s]
[false, true, false, true]
```

(12)

Type: List Boolean

## 9.49 Multivariate- Polynomial

The domain constructor `MultivariatePolynomial` is similar to `Polynomial` except that it specifies the variables to be used. Most functions available for `Polynomial` are available for `MultivariatePolynomial`. The abbreviation for `MultivariatePolynomial` is `MPOLY`. The type expressions

`MultivariatePolynomial([x,y],Integer)`    and    `MPOLY([x,y],INT)`

refer to the domain of multivariate polynomials in the variables `x` and `y` where the coefficients are restricted to be integers. The first variable specified is the main variable and the display of the polynomial reflects this.

This polynomial appears with terms in descending powers of the variable `x`.

`m : MPOLY([x,y],INT) := (x**2 - x*y**3 + 3*y)**2`  

$$x^4 - 2 y^3 x^3 + (y^6 + 6 y) x^2 - 6 y^4 x + 9 y^2$$
(1)  
 Type: MultivariatePolynomial([x, y], Integer)

It is easy to see a different variable ordering by doing a conversion.

`m :: MPOLY([y,x],INT)`  

$$x^2 y^6 - 6 x y^4 - 2 x^3 y^3 + 9 y^2 + 6 x^2 y + x^4$$
(2)  
 Type: MultivariatePolynomial([y, x], Integer)

You can use other, unspecified variables, by using `Polynomial` in the coefficient type of `MPOLY`.

`p : MPOLY([x,y],POLY INT)`  
Type: Void  
`p := (a**2*x - b*y**2 + 1)**2`  

$$a^4 x^2 + (-2 a^2 b y^2 + 2 a^2) x + b^2 y^4 - 2 b y^2 + 1$$
(4)  
 Type: MultivariatePolynomial([x, y], Polynomial Integer)

Conversions can be used to re-express such polynomials in terms of the other variables. For example, you can first push all the variables into a polynomial with integer coefficients.

`p :: POLY INT`  

$$b^2 y^4 + (-2 a^2 b x - 2 b) y^2 + a^4 x^2 + 2 a^2 x + 1$$
(5)  
 Type: Polynomial Integer

Now pull out the variables of interest.

`% :: MPOLY([a,b],POLY INT)`  

$$x^2 a^4 + (-2 x y^2 b + 2 x) a^2 + y^4 b^2 - 2 y^2 b + 1$$
(6)  
 Type: MultivariatePolynomial([a, b], Polynomial Integer)

### Restriction:

AXIOM does not allow you to create types where `MultivariatePolynomial` is contained in the coefficient type of `Polynomial`. Therefore, `MPOLY([x,y],POLY INT)` is legal but `POLY MPOLY([x,y],INT)` is not.

Multivariate polynomials may be combined with univariate polynomials to create types with special structures.

`q : UP(x, FRAC MPOLY([y,z],INT))`

Type: Void

This is a polynomial in `x` whose coefficients are quotients of polynomials in `y` and `z`.

`q := (x**2 - x*(z+1)/y + 2)**2`

$$x^4 + \frac{-2z - 2}{y} x^3 + \frac{4y^2 + z^2 + 2z + 1}{y^2} x^2 + \frac{-4z - 4}{y} x + 4 \quad (8)$$

Type: UnivariatePolynomial(x, Fraction MultivariatePolynomial([y, z], Integer))

Use conversions for structural rearrangements. `z` does not appear in a denominator and so it can be made the main variable.

`q :: UP(z, FRAC MPOLY([x,y],INT))`

$$\frac{x^2}{y^2} z^2 + \frac{-2y x^3 + 2x^2 - 4yx}{y^2} z + \frac{y^2 x^4 - 2yx^3 + (4y^2 + 1)x^2 - 4yx + 4y^2}{y^2} \quad (9)$$

Type: UnivariatePolynomial(z, Fraction MultivariatePolynomial([x, y], Integer))

Or you can make a multivariate polynomial in `x` and `z` whose coefficients are fractions in polynomials in `y`.

`q :: MPOLY([x,z], FRAC UP(y,INT))`

$$x^4 + \left(-\frac{2}{y}z - \frac{2}{y}\right)x^3 + \left(\frac{1}{y^2}z^2 + \frac{2}{y^2}z + \frac{4y^2 + 1}{y^2}\right)x^2 + \left(-\frac{4}{y}z - \frac{4}{y}\right)x + 4 \quad (10)$$

Type: MultivariatePolynomial([x, z], Fraction UnivariatePolynomial(y, Integer))

A conversion like `q :: MPOLY([x,y], FRAC UP(z,INT))` is not possible in this example because `y` appears in the denominator of a fraction. As you can see, AXIOM provides extraordinary flexibility in the manipulation and display of expressions via its conversion facility.

For more information on related topics, see ‘Polynomial’ on page 529, ‘UnivariatePolynomial’ on page 594, and ‘DistributedMultivariatePolynomial’ on page 402. Issue the system command `)show MultivariatePolynomial` to display the full list of operations defined by MultivariatePolynomial.

## 9.50 None

---

The None domain is not very useful for interactive work but it is provided nevertheless for completeness of the AXIOM type system.

Probably the only place you will ever see it is if you enter an empty list with no type information.

```
[]
```

(1)

Type: List None

Such an empty list can be converted into an empty list of any other type.

```
[] :: List Float
```

```
[]
```

(2)

Type: List Float

If you wish to produce an empty list of a particular type directly, such as List NonNegativeInteger, do it this way.

```
[] $List (NonNegativeInteger)
```

```
[]
```

(3)

Type: List NonNegativeInteger

## 9.51 Octonion

As Octonion creates an eight-dimensional algebra, you have to give eight components to construct an octonion.

The Octonions, also called the Cayley-Dixon algebra, defined over a commutative ring are an eight-dimensional non-associative algebra. Their construction from quaternions is similar to the construction of quaternions from complex numbers (see ‘Quaternion’ on page 535).

```
oci1 := octon(1,2,3,4,5,6,7,8)
1 + 2 i + 3 j + 4 k + 5 E + 6 I + 7 J + 8 K
```

(1)

Type: Octonion Integer

```
oci2 := octon(7,2,3,-4,5,6,-7,0)
7 + 2 i + 3 j - 4 k + 5 E + 6 I - 7 J
```

(2)

Type: Octonion Integer

Or you can use two quaternions to create an octonion.

```
oci3 := octon(quatern(-7,-12,3,-10), quatern(5,6,9,0))
-7 - 12 i + 3 j - 10 k + 5 E + 6 I + 9 J
```

(3)

Type: Octonion Integer

You can easily demonstrate the non-associativity of multiplication.

```
(oci1 * oci2) * oci3 - oci1 * (oci2 * oci3)
2696 i - 2928 j - 4072 k + 16 E - 1192 I + 832 J + 2616 K
```

(4)

Type: Octonion Integer

As with the quaternions, we have a real part, the imaginary parts *i*, *j*, *k*, and four additional imaginary parts *E*, *I*, *J* and *K*. These parts correspond to the canonical basis (1,*i*,*j*,*k*,*E*,*I*,*J*,*K*).

For each basis element there is a component operation to extract the coefficient of the basis element for a given octonion.

```
[real oci1, imagi oci1, imagj oci1, imagk oci1,
  imagE oci1, imagI oci1, imagJ oci1, imagK oci1]
[1, 2, 3, 4, 5, 6, 7, 8]
```

(5)

Type: List PositiveInteger

A basis with respect to the quaternions is given by (1,*E*). However, you might ask, what then are the commuting rules? To answer this, we create some generic elements.

We do this in AXIOM by simply changing the ground ring from Integer to Polynomial Integer.

```
q : Quaternion Polynomial Integer := quatern(q1, qi, qj,
  qk)
q1 + qi i + qj j + qk k
```

(6)

Type: Quaternion Polynomial Integer

```
E : Octonion Polynomial Integer := octon(0,0,0,0,1,0,0,0)
E
```

(7)

Type: Octonion Polynomial Integer

Note that quaternions are automatically converted to octonions in the obvious way.

$$q * E = q1 E + qi I + qj J + qk K \quad (8)$$

Type: Octonion Polynomial Integer

$$E * q = q1 E - qi I - qj J - qk K \quad (9)$$

Type: Octonion Polynomial Integer

$$q * 1\$(Octonion Polynomial Integer) = q1 + qi i + qj j + qk k \quad (10)$$

Type: Octonion Polynomial Integer

$$1\$(Octonion Polynomial Integer) * q = q1 + qi i + qj j + qk k \quad (11)$$

Type: Octonion Polynomial Integer

Finally, we check that the **norm**, defined as the sum of the squares of the coefficients, is a multiplicative map.

$$\begin{aligned} o : \text{Octonion Polynomial Integer} &:= \text{octon}(o1, oi, oj, ok, \\ &\quad oE, oI, oJ, oK) \\ o1 + oi i + oj j + ok k + oE E + oI I + oJ J + oK K \end{aligned} \quad (12)$$

Type: Octonion Polynomial Integer

$$\begin{aligned} \text{norm } o &= \\ ok^2 + oj^2 + oi^2 + oK^2 + oJ^2 + oI^2 + oE^2 + o1^2 \end{aligned} \quad (13)$$

Type: Polynomial Integer

$$\begin{aligned} p : \text{Octonion Polynomial Integer} &:= \text{octon}(p1, pi, pj, pk, \\ &\quad pE, pI, pJ, pK) \\ p1 + pi i + pj j + pk k + pE E + pI I + pJ J + pK K \end{aligned} \quad (14)$$

Type: Octonion Polynomial Integer

Since the result is 0, the norm is multiplicative.

$$\begin{aligned}
& \text{norm}(o * p) - \text{norm}(p) * \text{norm}(p) \\
& -pk^4 + \left( \frac{-2pj^2 - 2pi^2 - 2pK^2 - 2pJ^2 - 2pI^2 - 2pE^2 - 2p1^2 + ok^2 + oj^2 + oi^2 + oK^2 + oJ^2 + oI^2 + oE^2 + o1^2}{2pI^2 - 2pE^2 - 2p1^2 + ok^2 + oj^2 + oi^2 + oK^2 + oJ^2 + oI^2 + oE^2 + o1^2} \right) pk^2 \\
& -pj^4 + \left( \frac{-2pi^2 - 2pK^2 - 2pJ^2 - 2pI^2 - 2pE^2 - 2p1^2 + ok^2 + oj^2 + oi^2 + oK^2 + oJ^2 + oI^2 + oE^2 + o1^2}{-2pE^2 - 2p1^2 + ok^2 + oj^2 + oi^2 + oK^2 + oJ^2 + oI^2 + oE^2 + o1^2} \right) pj^2 \\
& -pi^4 + \left( \frac{-2pK^2 - 2pJ^2 - 2pI^2 - 2pE^2 - 2p1^2 + ok^2 + oj^2 + oi^2 + oK^2 + oJ^2 + oI^2 + oE^2 + o1^2}{-2p1^2 + ok^2 + oj^2 + oi^2 + oK^2 + oJ^2 + oI^2 + oE^2 + o1^2} \right) pi^2 \\
& -pK^4 + \left( \frac{-2pJ^2 - 2pI^2 - 2pE^2 - 2p1^2 + ok^2 + oj^2 + oi^2 + oK^2 + oJ^2 + oI^2 + oE^2 + o1^2}{+ok^2 + oj^2 + oi^2 + oK^2 + oJ^2 + oI^2 + oE^2 + o1^2} \right) pK^2 \\
& -pJ^4 + \left( \frac{-2pI^2 - 2pE^2 - 2p1^2 + ok^2 + oj^2 + oi^2 + oK^2 + oJ^2 + oI^2 + oE^2 + o1^2}{+oK^2 + oJ^2 + oI^2 + oE^2 + o1^2} \right) pJ^2 \\
& -pI^4 + \left( \frac{-2pE^2 - 2p1^2 + ok^2 + oj^2 + oi^2 + oK^2 + oJ^2 + oI^2 + oE^2 + o1^2}{+oJ^2 + oI^2 + oE^2 + o1^2} \right) pI^2 \\
& -pE^4 + \left( \frac{-2p1^2 + ok^2 + oj^2 + oi^2 + oK^2 + oJ^2 + oI^2 + oE^2 + o1^2}{+oI^2 + oE^2 + o1^2} \right) pE^2 \\
& -p1^4 + \left( ok^2 + oj^2 + oi^2 + oK^2 + oJ^2 + oI^2 + oE^2 + o1^2 \right) p1^2
\end{aligned} \tag{15}$$

Type: Polynomial Integer

Issue the system command `)show Octonion` to display the full list of operations defined by Octonion.

## 9.52 OneDimensional- Array

---

The OneDimensionalArray domain is used for storing data in a one-dimensional indexed data structure. Such an array is a homogeneous data structure in that all the entries of the array must belong to the same AXIOM domain. Each array has a fixed length specified by the user and arrays are not extensible. The indexing of one-dimensional arrays is one-based. This means that the “first” element of an array is given the index 1. See also ‘Vector’ on page 601 and ‘FlexibleArray’ on page 425.

To create a one-dimensional array, apply the operation **oneDimensionalArray** to a list.

```
oneDimensionalArray [i**2 for i in 1..10]
[1, 4, 9, 16, 25, 36, 49, 64, 81, 100] (1)
Type: OneDimensionalArray PositiveInteger
```

Another approach is to first create **a**, a one-dimensional array of 10 0's. OneDimensionalArray has the convenient abbreviation **ARRAY1**.

```
a : ARRAY1 INT := new(10,0)
[0, 0, 0, 0, 0, 0, 0, 0, 0, 0] (2)
Type: OneDimensionalArray Integer
```

Set each *i*th element to *i*, then display the result.

```
for i in 1..10 repeat a.i := i; a
[1, 2, 3, 4, 5, 6, 7, 8, 9, 10] (3)
Type: OneDimensionalArray Integer
```

Square each element by mapping the function  $i \mapsto i^2$  onto each element.

```
map!(i +-> i ** 2, a); a
[1, 4, 9, 16, 25, 36, 49, 64, 81, 100] (4)
Type: OneDimensionalArray Integer
```

Reverse the elements in place.

```
reverse! a
[100, 81, 64, 49, 36, 25, 16, 9, 4, 1] (5)
Type: OneDimensionalArray Integer
```

Swap the 4th and 5th element.

```
swap!(a,4,5); a
[100, 81, 64, 36, 49, 25, 16, 9, 4, 1] (6)
Type: OneDimensionalArray Integer
```

Sort the elements in place.

```
sort! a
[1, 4, 9, 16, 25, 36, 49, 64, 81, 100] (7)
Type: OneDimensionalArray Integer
```

Create a new one-dimensional array **b** containing the last 5 elements of **a**.

```
b := a(6..10)
[36, 49, 64, 81, 100] (8)
Type: OneDimensionalArray Integer
```



Replace the first 5 elements of **a**  
with those of **b**.

```
copyInto!(a,b,1)
```

```
[36, 49, 64, 81, 100, 36, 49, 64, 81, 100]
```

(9)

Type: OneDimensionalArray Integer

## 9.53 Operator

Given any ring  $R$ , the ring of the Integer-linear operators over  $R$  is called  $\text{Operator}(R)$ . To create an operator over  $R$ , first create a basic operator using the operation **operator**, and then convert it to  $\text{Operator}(R)$  for the  $R$  you want.

We choose  $R$  to be the two by two matrices over the integers.

```
R := SQMATRIX(2, INT)
SquareMatrix (2, Integer )
```

(1)

Type: Domain

Create the operator `tilde` on  $R$ .

```
t := operator("tilde") :: OP(R)
tilde
```

(2)

Type: Operator SquareMatrix(2, Integer)

Since `Operator` is unexposed we must either package-call operations from it, or expose it explicitly. For convenience we will do the latter.

Expose `Operator`.

```
)set expose add constructor Operator
Operator is now explicitly ex-
posed in frame initial
```

To attach an evaluation function (from  $R$  to  $R$ ) to an operator over  $R$ , use `evaluate(op, f)` where `op` is an operator over  $R$  and `f` is a function  $R \rightarrow R$ . This needs to be done only once when the operator is defined. Note that `f` must be Integer-linear (that is,  $f(ax+y) = a f(x) + f(y)$  for any integer  $a$ , and any  $x$  and  $y$  in  $R$ ).

We now attach the transpose map to the above operator `t`.

```
evaluate(t, m +-> transpose m)
tilde
```

(3)

Type: Operator SquareMatrix(2, Integer)

Operators can be manipulated formally as in any ring: “+” is the pointwise addition and “\*” is composition. Any element  $x$  of  $R$  can be converted to an operator  $\text{op}_x$  over  $R$ , and the evaluation function of  $\text{op}_x$  is left-multiplication by  $x$ .

Multiplying on the left by this matrix swaps the two rows.

```
s : R := matrix [[0, 1], [1, 0]]
[ 0  1 ]
[ 1  0 ]
```

(4)

Type: SquareMatrix(2, Integer)

Can you guess what is the action of the following operator?

```
rho := t * s
tilde [ 0  1 ]
      [ 1  0 ]
```

(5)

Type: Operator SquareMatrix(2, Integer)

Hint: applying `rho` four times gives the identity, so `rho**4-1` should return 0 when applied to any two by two matrix.

$$\begin{aligned} z &:= \text{rho}^{**4} - 1 \\ &\quad \text{tilde} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{tilde} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{tilde} \cdot \\ &\quad -1 + \\ &\quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{tilde} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{aligned} \tag{6}$$

Type: Operator SquareMatrix(2, Integer)

Now check with this matrix.

$$\begin{aligned} m:R &:= \text{matrix} \ [[1, 2], [3, 4]] \\ &\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \end{aligned} \tag{7}$$

Type: SquareMatrix(2, Integer)

$$\begin{aligned} z \ m \\ &\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned} \tag{8}$$

Type: SquareMatrix(2, Integer)

As you have probably guessed by now, `rho` acts on matrices by rotating the elements clockwise.

$$\begin{aligned} \text{rho} \ m \\ &\begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} \end{aligned} \tag{9}$$

Type: SquareMatrix(2, Integer)

$$\begin{aligned} \text{rho} \ \text{rho} \ m \\ &\begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} \end{aligned} \tag{10}$$

Type: SquareMatrix(2, Integer)

$$\begin{aligned} (\text{rho}^{**3}) \ m \\ &\begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix} \end{aligned} \tag{11}$$

Type: SquareMatrix(2, Integer)

Do the swapping of rows and transposition commute? We can check by computing their bracket.

$$\begin{aligned} b &:= t * s - s * t \\ &= - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{tilde} + \text{tilde} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{aligned} \tag{12}$$

Type: Operator SquareMatrix(2, Integer)

Now apply it to `m`.

$$\begin{aligned} b \ m \\ &\begin{bmatrix} 1 & -3 \\ 3 & -1 \end{bmatrix} \end{aligned} \tag{13}$$

Type: SquareMatrix(2, Integer)

Next we demonstrate how to define a differential operator on a polynomial ring.

This is the recursive definition of the  $n$ -th Legendre polynomial.

```
L n ==
  n = 0 => 1
  n = 1 => x
  (2*n-1)/n * x * L(n-1) - (n-1)/n * L(n-2)
```

Type: Void

Create the differential operator  $\frac{d}{dx}$  on polynomials in  $x$  over the rational numbers.

```
dx := operator("D") :: OP(POLY FRAC INT)
D
```

(15)

Type: Operator Polynomial Fraction Integer

Now attach the map to it.

```
evaluate(dx, p +-> D(p, 'x))
D
```

(16)

Type: Operator Polynomial Fraction Integer

This is the differential equation satisfied by the  $n$ -th Legendre polynomial.

```
E n == (1 - x**2) * dx**2 - 2 * x * dx + n*(n+1)
```

Type: Void

Now we verify this for  $n = 15$ . Here is the polynomial.

```
L 15
Compiling function L with type Integer -> Polynomial
Fraction Integer
Compiling function L as a recurrence relation.
```

$$\begin{aligned} & \frac{9694845}{2048} x^{15} - \frac{35102025}{2048} x^{13} + \frac{50702925}{2048} x^{11} - \frac{37182145}{2048} x^9 + \\ & \frac{14549535}{2048} x^7 - \frac{2909907}{2048} x^5 + \frac{255255}{2048} x^3 - \frac{6435}{2048} x \end{aligned}$$

(18)

Type: Polynomial Fraction Integer

Here is the operator.

```
E 15
Compiling function E with type PositiveInteger ->
Operator Polynomial Fraction Integer
```

$$240 - 2 x D - (x^2 - 1) D^2$$

(19)

Type: Operator Polynomial Fraction Integer

Here is the evaluation.

```
(E 15) (L 15)
```

```
0
```

(20)

Type: Polynomial Fraction Integer

## 9.54 OrderedVariableList

This is a sample ordering of three symbols.

The domain OrderedVariableList provides symbols which are restricted to a particular list and have a definite ordering. Those two features are specified by a List Symbol object that is the argument to the domain.

```
ls:List Symbol:=['x','a','z']
[x, a, z] (1)
```

Type: List Symbol

Let's build the domain

```
Z:=OVAR ls
OrderedVariableList [ x , a , z ] (2)
```

Type: Domain

How many variables does it have?

```
size()$Z
3 (3)
```

Type: NonNegativeInteger

They are (in the imposed order)

```
lv:=[index(i::PI)$Z for i in 1..size()$Z]
Compiling function G82211 with type Integer ->
Boolean
Compiling function G82223 with type
NonNegativeInteger -> Boolean
[x, a, z] (4)
```

Type: List OrderedVariableList [x, a, z]

Check that the ordering is right

```
sorted?(>,lv)
true (5)
```

Type: Boolean

## 9.55 Orderly- Differential- Polynomial

---

Many systems of differential equations may be transformed to equivalent systems of ordinary differential equations where the equations are expressed polynomially in terms of the unknown functions. In AXIOM, the domain constructors `OrderlyDifferentialPolynomial` (abbreviated `ODPOL`) and `SequentialDifferentialPolynomial` (abbreviation `SDPOL`) implement two domains of ordinary differential polynomials over any differential ring. In the simplest case, this differential ring is usually either the ring of integers, or the field of rational numbers. However, AXIOM can handle ordinary differential polynomials over a field of rational functions in a single indeterminate.

The two domains `ODPOL` and `SDPOL` are almost identical, the only difference being the choice of a different ranking, which is an ordering of the derivatives of the indeterminates. The first domain uses an orderly ranking, that is, derivatives of higher order are ranked higher, and derivatives of the same order are ranked alphabetically. The second domain uses a sequential ranking, where derivatives are ordered first alphabetically by the differential indeterminates, and then by order. A more general domain constructor, `DifferentialSparseMultivariatePolynomial` (abbreviation `DSMP`) allows both a user-provided list of differential indeterminates as well as a user-defined ranking. We shall illustrate `ODPOL(FRAC INT)`, which constructs a domain of ordinary differential polynomials in an arbitrary number of differential indeterminates with rational numbers as coefficients.

```
dpol := ODPOL(FRAC INT)
```

```
OrderlyDifferentialPolynomial Fraction Integer (1)
```

Type: Domain

A differential indeterminate `w` may be viewed as an infinite sequence of algebraic indeterminates, which are the derivatives of `w`. To facilitate referencing these, AXIOM provides the operation **makeVariable** to convert an element of type `Symbol` to a map from the natural numbers to the differential polynomial ring.

```
w := makeVariable('w)$dpol
```

```
theMap (...) (2)
```

Type: (NonNegativeInteger → OrderlyDifferentialPolynomial Fraction Integer)

```
z := makeVariable('z)$dpol
```

```
theMap (...) (3)
```

Type: (NonNegativeInteger → OrderlyDifferentialPolynomial Fraction Integer)

The fifth derivative of **w** can be obtained by applying the map **w** to the number 5. Note that the order of differentiation is given as a subscript (except when the order is 0).

$$\begin{aligned} & \mathbf{w} . 5 \\ & w_5 \end{aligned} \tag{4}$$

Type: OrderlyDifferentialPolynomial Fraction Integer

$$\begin{aligned} & \mathbf{w} . 0 \\ & w \end{aligned} \tag{5}$$

Type: OrderlyDifferentialPolynomial Fraction Integer

The first five derivatives of **z** can be generated by a list.

$$\begin{aligned} & [\mathbf{z} . i \text{ for } i \text{ in } 1..5] \\ & [z_1, z_2, z_3, z_4, z_5] \end{aligned} \tag{6}$$

Type: List OrderlyDifferentialPolynomial Fraction Integer

The usual arithmetic can be used to form a differential polynomial from the derivatives.

$$\begin{aligned} & \mathbf{f} := \mathbf{w} . 4 - \mathbf{w} . 1 * \mathbf{w} . 1 * \mathbf{z} . 3 \\ & w_4 - w_1^2 z_3 \end{aligned} \tag{7}$$

Type: OrderlyDifferentialPolynomial Fraction Integer

$$\begin{aligned} & \mathbf{g} := (\mathbf{z} . 1)^{**3} * (\mathbf{z} . 2)^{**2} - \mathbf{w} . 2 \\ & z_1^3 z_2^2 - w_2 \end{aligned} \tag{8}$$

Type: OrderlyDifferentialPolynomial Fraction Integer

The operation **D** computes the derivative of any differential polynomial.

$$\begin{aligned} & \mathbf{D}(\mathbf{f}) \\ & w_5 - w_1^2 z_4 - 2 w_1 w_2 z_3 \end{aligned} \tag{9}$$

Type: OrderlyDifferentialPolynomial Fraction Integer

The same operation can compute higher derivatives, like the fourth derivative.

$$\begin{aligned} & \mathbf{D}(\mathbf{f}, 4) \\ & w_8 - w_1^2 z_7 - 8 w_1 w_2 z_6 + (-12 w_1 w_3 - 12 w_2^2) z_5 - 2 w_1 z_3 w_5 + \\ & (-8 w_1 w_4 - 24 w_2 w_3) z_4 - 8 w_2 z_3 w_4 - 6 w_3^2 z_3 \end{aligned} \tag{10}$$

Type: OrderlyDifferentialPolynomial Fraction Integer

The operation **makeVariable** creates a map to facilitate referencing the derivatives of **f**, similar to the map **w**.

$$\begin{aligned} & \mathbf{df} := \mathbf{makeVariable}(\mathbf{f}) \$ \mathbf{dpol} \\ & \mathbf{theMap}(\dots) \end{aligned} \tag{11}$$

Type: (NonNegativeInteger → OrderlyDifferentialPolynomial Fraction Integer)

The fourth derivative of **f** may be referenced easily.

$$\begin{aligned} & \mathbf{df} . 4 \\ & w_8 - w_1^2 z_7 - 8 w_1 w_2 z_6 + (-12 w_1 w_3 - 12 w_2^2) z_5 - 2 w_1 z_3 w_5 + \\ & (-8 w_1 w_4 - 24 w_2 w_3) z_4 - 8 w_2 z_3 w_4 - 6 w_3^2 z_3 \end{aligned} \tag{12}$$

Type: OrderlyDifferentialPolynomial Fraction Integer

The operation <b>order</b> returns the order of a differential polynomial, or the order in a specified differential indeterminate.	<code>order(g)</code>		
	2	(13)	Type: PositiveInteger
	<code>order(g, 'w)</code>		
	2	(14)	Type: PositiveInteger
The operation <b>differentialVariables</b> returns a list of differential indeterminates occurring in a differential polynomial.	<code>differentialVariables(g)</code>		
	$[z, w]$	(15)	Type: List Symbol
The operation <b>degree</b> returns the degree, or the degree in the differential indeterminate specified.	<code>degree(g)</code>		
	$z_2^2 z_1^3$	(16)	Type: IndexedExponents OrderlyDifferentialVariable Symbol
	<code>degree(g, 'w)</code>		
	1	(17)	Type: PositiveInteger
The operation <b>weights</b> returns a list of weights of differential monomials appearing in differential polynomial, or a list of weights in a specified differential indeterminate.	<code>weights(g)</code>		
	$[7, 2]$	(18)	Type: List NonNegativeInteger
	<code>weights(g, 'w)</code>		
	$[2]$	(19)	Type: List NonNegativeInteger
The operation <b>weight</b> returns the maximum weight of all differential monomials appearing in the differential polynomial.	<code>weight(g)</code>		
	7	(20)	Type: PositiveInteger
A differential polynomial is <i>isobaric</i> if the weights of all differential monomials appearing in it are equal.	<code>isobaric?(g)</code>		
	false	(21)	Type: Boolean



To substitute *differentially*, use **eval**. Note that we must coerce 'w' to Symbol, since in ODPOL, differential indeterminates belong to the domain Symbol. Compare this result to the next, which substitutes *algebraically* (no substitution is done since w.0 does not appear in g).

$$\text{eval}(g, ['w::\text{Symbol}], [f])$$

$$-w_6 + w_1^2 z_5 + 4 w_1 w_2 z_4 + (2 w_1 w_3 + 2 w_2^2) z_3 + z_1^3 z_2^2 \quad (22)$$

Type: OrderlyDifferentialPolynomial Fraction Integer

$$\text{eval}(g, \text{variables}(w.0), [f])$$

$$z_1^3 z_2^2 - w_2 \quad (23)$$

Type: OrderlyDifferentialPolynomial Fraction Integer

Since OrderlyDifferentialPolynomial belongs to PolynomialCategory, all the operations defined in the latter category, or in packages for the latter category, are available.

$$\text{monomials}(g)$$

$$[z_1^3 z_2^2, -w_2] \quad (24)$$

Type: List OrderlyDifferentialPolynomial Fraction Integer

$$\text{variables}(g)$$

$$[z_2, w_2, z_1] \quad (25)$$

Type: List OrderlyDifferentialVariable Symbol

$$\text{gcd}(f, g)$$

$$1 \quad (26)$$

Type: OrderlyDifferentialPolynomial Fraction Integer

$$\text{groebner}([f, g])$$

$$[w_4 - w_1^2 z_3, z_1^3 z_2^2 - w_2] \quad (27)$$

Type: List OrderlyDifferentialPolynomial Fraction Integer

The next three operations are essential for elimination procedures in differential polynomial rings. The operation **leader** returns the leader of a differential polynomial, which is the highest ranked derivative of the differential indeterminates that occurs.

$$\text{lg}:=\text{leader}(g)$$

$$z_2 \quad (28)$$

Type: OrderlyDifferentialVariable Symbol

The operation **separant** returns the separant of a differential polynomial, which is the partial derivative with respect to the leader.

$$\text{sg}:=\text{separant}(g)$$

$$2 z_1^3 z_2 \quad (29)$$

Type: OrderlyDifferentialPolynomial Fraction Integer

The operation **initial** returns the initial, which is the leading coefficient when the given differential polynomial is expressed as a polynomial in the leader.

$$\text{ig} := \text{initial}(g) \quad z_1^3 \quad (30)$$

Type: OrderlyDifferentialPolynomial Fraction Integer

Using these three operations, it is possible to reduce **f** modulo the differential ideal generated by **g**. The general scheme is to first reduce the order, then reduce the degree in the leader. First, eliminate **z.3** using the derivative of **g**.

$$g1 := D \ g \quad 2 \ z_1^3 \ z_2 \ z_3 - w_3 + 3 \ z_1^2 \ z_2^3 \quad (31)$$

Type: OrderlyDifferentialPolynomial Fraction Integer

Find its leader.

$$lg1 := \text{leader } g1 \quad z_3 \quad (32)$$

Type: OrderlyDifferentialVariable Symbol

Differentiate **f** partially with respect to this leader.

$$\text{pdf} := D(f, lg1) \quad -w_1^2 \quad (33)$$

Type: OrderlyDifferentialPolynomial Fraction Integer

Compute the partial remainder of **f** with respect to **g**.

$$\text{prf} := sg * f - \text{pdf} * g1 \quad 2 \ z_1^3 \ z_2 \ w_4 - w_1^2 \ w_3 + 3 \ w_1^2 \ z_1^2 \ z_2^3 \quad (34)$$

Type: OrderlyDifferentialPolynomial Fraction Integer

Note that high powers of **lg** still appear in **prf**. Compute the leading coefficient of **prf** as a polynomial in the leader of **g**.

$$\text{lcf} := \text{leadingCoefficient univariate}(\text{prf}, lg) \quad 3 \ w_1^2 \ z_1^2 \quad (35)$$

Type: OrderlyDifferentialPolynomial Fraction Integer

Finally, continue eliminating the high powers of **lg** appearing in **prf** to obtain the (pseudo) remainder of **f** modulo **g** and its derivatives.

$$\text{ig} * \text{prf} - \text{lcf} * g * lg \quad 2 \ z_1^6 \ z_2 \ w_4 - w_1^2 \ z_1^3 \ w_3 + 3 \ w_1^2 \ z_1^2 \ w_2 \ z_2 \quad (36)$$

Type: OrderlyDifferentialPolynomial Fraction Integer

Issue the system command `)show OrderlyDifferentialPolynomial` to display the full list of operations defined by OrderlyDifferentialPolynomial. Issue the system command `)show SequentialDifferentialPolynomial` to display the full list of operations defined by SequentialDifferentialPolynomial.

## 9.56 PartialFraction

A *partial fraction* is a decomposition of a quotient into a sum of quotients where the denominators of the summands are powers of primes.<sup>5</sup> For example, the rational number  $1/6$  is decomposed into  $1/2 - 1/3$ . You can compute partial fractions of quotients of objects from domains belonging to the category `EuclideanDomain`. For example, `Integer`, `Complex Integer`, and `UnivariatePolynomial(x, Fraction Integer)` all belong to `EuclideanDomain`. In the examples following, we demonstrate how to decompose quotients of each of these kinds of object into partial fractions. Issue the system command `)show PartialFraction` to display the full list of operations defined by `PartialFraction`.

It is necessary that we know how to factor the denominator when we want to compute a partial fraction. Although the interpreter can often do this automatically, it may be necessary for you to include a call to **factor**. In these examples, it is not necessary to factor the denominators explicitly.

The main operation for computing partial fractions is called **partialFraction** and we use this to compute a decomposition of  $1 / 10!$ . The first argument to **partialFraction** is the numerator of the quotient and the second argument is the factored denominator.

Since the denominators are powers of primes, it may be possible to expand the numerators further with respect to those primes. Use the operation **padicFraction** to do this.

The operation **compactFraction** returns an expanded fraction into the usual form. The compacted version is used internally for computational efficiency.

```
partialFraction(1,factorial 10)
```

$$\frac{159}{2^8} - \frac{23}{3^4} - \frac{12}{5^2} + \frac{1}{7} \quad (1)$$

Type: PartialFraction Integer

```
f := padicFraction(%)
```

$$\frac{1}{2} + \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6} + \frac{1}{2^7} + \frac{1}{2^8} - \frac{2}{3^2} - \frac{1}{3^3} - \frac{2}{3^4} - \frac{2}{5} - \frac{2}{5^2} + \frac{1}{7} \quad (2)$$

Type: PartialFraction Integer

```
compactFraction(f)
```

$$\frac{159}{2^8} - \frac{23}{3^4} - \frac{12}{5^2} + \frac{1}{7} \quad (3)$$

Type: PartialFraction Integer

---

<sup>5</sup>Most people first encounter partial fractions when they are learning integral calculus. For a technical discussion of partial fractions, see, for example, Lang's *Algebra*.

You can add, subtract, multiply and divide partial fractions. In addition, you can extract the parts of the decomposition.

**numberOfFractionalTerms** computes the number of terms in the fractional part. This does not include the whole part of the fraction, which you get by calling **wholePart**. In this example, the whole part is just 0.

The operation

**nthFractionalTerm** returns the individual terms in the decomposition. Notice that the object returned is a partial fraction itself. **firstNumer** and **firstDenom** extract the numerator and denominator of the first term of the fraction.

Given two gaussian integers (see ‘Complex’ on page 383), you can decompose their quotient into a partial fraction.

To convert back to a quotient, simply use a conversion.

```
numberOfFractionalTerms(f)
```

```
12
```

(4)

Type: PositiveInteger

```
nthFractionalTerm(f,3)
```

$$\frac{1}{2^5}$$

(5)

Type: PartialFraction Integer

```
partialFraction(1,- 13 + 14 * %i)
```

$$-\frac{1}{1+2i} + \frac{4}{3+8i}$$

(6)

Type: PartialFraction Complex Integer

```
% :: Fraction Complex Integer
```

$$-\frac{i}{14+13i}$$

(7)

Type: Fraction Complex Integer

To conclude this section, we compute the decomposition of

$$\frac{1}{(x+1)(x+2)^2(x+3)^3(x+4)^4}$$

The polynomials in this object have type UnivariatePolynomial(x, Fraction Integer).

We use the **primeFactor** operation (see ‘Factored’ on page 414) to create the denominator in factored form directly.

```
u : FR UP(x, FRAC INT) := reduce(*,[primeFactor(x+i,i) for
i in 1..4])
```

$$(x+1)(x+2)^2(x+3)^3(x+4)^4$$

(8)

Type: Factored UnivariatePolynomial(x, Fraction Integer)

These are the compact and expanded partial fractions for the quotient.

`partialFraction(1,u)`

$$\frac{\frac{1}{648}}{x+1} + \frac{\frac{1}{4}x + \frac{7}{16}}{(x+2)^2} + \frac{-\frac{17}{8}x^2 - 12x - \frac{139}{8}}{(x+3)^3} + \frac{\frac{607}{324}x^3 + \frac{10115}{432}x^2 + \frac{391}{4}x + \frac{44179}{324}}{(x+4)^4} \quad (9)$$

Type: PartialFraction UnivariatePolynomial(x, Fraction Integer)

`padicFraction %`

$$\frac{\frac{1}{648}}{x+1} + \frac{\frac{1}{4}}{x+2} - \frac{\frac{1}{16}}{(x+2)^2} - \frac{\frac{17}{8}}{x+3} + \frac{\frac{3}{4}}{(x+3)^2} - \frac{\frac{1}{2}}{(x+3)^3} + \frac{\frac{607}{324}}{x+4} + \frac{\frac{403}{432}}{(x+4)^2} + \frac{\frac{13}{36}}{(x+4)^3} + \frac{\frac{1}{12}}{(x+4)^4} \quad (10)$$

Type: PartialFraction UnivariatePolynomial(x, Fraction Integer)

All see ‘FullPartialFractionExpansion’ on page 435 for examples of factor-free conversion of quotients to full partial fractions.

## 9.57 Permanent

Consider an  $n$  by  $n$  matrix with entries 0 on the diagonal and 1 elsewhere. Think of the rows as one-half of each couple (for example, the males) and the columns the other half. The permanent of such a matrix gives the desired derangement number.

Here are some derangement numbers, which you see grow quite fast.

The package Permanent provides the function **permanent** for square matrices. The **permanent** of a square matrix can be computed in the same way as the determinant by expansion of minors except that for the permanent the sign for each element is 1, rather than being 1 if the row plus column indices is positive and -1 otherwise. This function is much more difficult to compute efficiently than the **determinant**. An example of the use of **permanent** is the calculation of the  $n^{\text{th}}$  derangement number, defined to be the number of different possibilities for  $n$  couples to dance but never with their own spouse.

```
kn n ==
  r : MATRIX INT := new(n,n,1)
  for i in 1..n repeat
    r.i.i := 0
  r
```

Type: Void

```
permanent(kn(5) :: SQMATRIX(5,INT))
```

```
Compiling function kn with type PositiveInteger ->
  Matrix Integer
```

```
44
```

(2)

Type: PositiveInteger

```
[permanent(kn(n) :: SQMATRIX(n,INT)) for n in 1..13]
```

```
Cannot compile conversion for types involving local
  variables. In particular, could not compile the
  expression involving :: SQMATRIX(n,INT)
AXIOM will attempt to step through and interpret the
  code.
```

```
[0, 1, 2, 9, 44, 265, 1854, 14833, 133496, 1334961, 14684570,
  176214841, 2290792932]
```

(3)

Type: List NonNegativeInteger

## 9.58 Polynomial

It is used to create the default polynomial domains in AXIOM. Here the coefficients are integers.

$$\begin{array}{l} \mathbf{x} + 1 \\ x + 1 \end{array} \quad (1)$$

Type: Polynomial Integer

Here the coefficients have type Float.

$$\begin{array}{l} z - 2.3 \\ z - 2.3 \end{array} \quad (2)$$

Type: Polynomial Float

And here we have a polynomial in two variables with coefficients which have type Fraction Integer.

$$\begin{array}{l} y^{**2} - z + 3/4 \\ -z + y^2 + \frac{3}{4} \end{array} \quad (3)$$

Type: Polynomial Fraction Integer

The representation of objects of domains created by Polynomial is that of recursive univariate polynomials.<sup>6</sup>

This recursive structure is sometimes obvious from the display of a polynomial.

$$\begin{array}{l} y^{**2} + \mathbf{x} * y + y \\ y^2 + (x + 1) y \end{array} \quad (4)$$

Type: Polynomial Integer

In this example, you see that the polynomial is stored as a polynomial in  $y$  with coefficients that are polynomials in  $x$  with integer coefficients. In fact, you really don't need to worry about the representation unless you are working on an advanced application where it is critical. The polynomial types created from DistributedMultivariatePolynomial and NewDistributedMultivariatePolynomial (discussed in 'DistributedMultivariatePolynomial' on page 402) are stored and displayed in a non-recursive manner.

You see a "flat" display of the above polynomial by converting to one of those types.

$$\begin{array}{l} \% :: \text{DMP}([y, x], \text{INT}) \\ y^2 + y x + y \end{array} \quad (5)$$

Type: DistributedMultivariatePolynomial([y, x], Integer)

We will demonstrate many of the polynomial facilities by using two polynomials with integer coefficients.

By default, the interpreter expands polynomial expressions, even if they are written in a factored format.

$$\begin{array}{l} p := (y-1)^{**2} * x * z \\ (x y^2 - 2 x y + x) z \end{array} \quad (6)$$

Type: Polynomial Integer

<sup>6</sup>The term **univariate** means "one variable." **multivariate** means "possibly more than one variable."

See ‘Factored’ on page 414 to see how to create objects in factored form directly.

$$q := (y-1) * x * (z+5) \\ (x \ y - x) \ z + 5 \ x \ y - 5 \ x \quad (7)$$

Type: Polynomial Integer

The fully factored form can be recovered by using **factor**.

$$\text{factor}(q) \\ x \ (y - 1) \ (z + 5) \quad (8)$$

Type: Factored Polynomial Integer

This is the same name used for the operation to factor integers. Such reuse of names is called *overloading* and makes it much easier to think of solving problems in general ways. AXIOM facilities for factoring polynomials created with Polynomial are currently restricted to the integer and rational number coefficient cases. There are more complete facilities for factoring univariate polynomials: see Section 8.2 on page 274.

The standard arithmetic operations are available for polynomials.

$$p - q^{**2} \\ \left(-x^2 \ y^2 + 2 \ x^2 \ y - x^2\right) \ z^2 + \\ \left(\left(-10 \ x^2 + x\right) \ y^2 + \left(20 \ x^2 - 2 \ x\right) \ y - 10 \ x^2 + x\right) \ z - 25 \ x^2 \ y^2 + \\ 50 \ x^2 \ y - 25 \ x^2 \quad (9)$$

Type: Polynomial Integer

The operation **gcd** is used to compute the greatest common divisor of two polynomials.

$$\text{gcd}(p, q) \\ x \ y - x \quad (10)$$

Type: Polynomial Integer

In the case of **p** and **q**, the gcd is obvious from their definitions. We factor the gcd to show this relationship better.

$$\text{factor} \% \\ x \ (y - 1) \quad (11)$$

Type: Factored Polynomial Integer

The least common multiple is computed by using **lcm**.

$$\text{lcm}(p, q) \\ \left(x \ y^2 - 2 \ x \ y + x\right) \ z^2 + \left(5 \ x \ y^2 - 10 \ x \ y + 5 \ x\right) \ z \quad (12)$$

Type: Polynomial Integer

Use **content** to compute the greatest common divisor of the coefficients of the polynomial.

$$\text{content} \ p \\ 1 \quad (13)$$

Type: PositiveInteger

Many of the operations on polynomials require you to specify a variable. For example, **resultant** requires you to give the variable in which the polynomials should be expressed.



This computes the resultant of the values of <b>p</b> and <b>q</b> , considering them as polynomials in the variable <b>z</b> . They do not share a root when thought of as polynomials in <b>z</b> .	<code>resultant(p,q,z)</code> $5 x^2 y^3 - 15 x^2 y^2 + 15 x^2 y - 5 x^2$	(14)	Type: Polynomial Integer
This value is 0 because as polynomials in <b>x</b> the polynomials have a common root.	<code>resultant(p,q,x)</code> 0	(15)	Type: Polynomial Integer
	The data type used for the variables created by Polynomial is Symbol. As mentioned above, the representation used by Polynomial is recursive and so there is a main variable for nonconstant polynomials.		
The operation <b>mainVariable</b> returns this variable. The return type is actually a union of Symbol and "failed".	<code>mainVariable p</code> <i>z</i>	(16)	Type: Union(Symbol, ...)
The latter branch of the union is be used if the polynomial has no variables, that is, is a constant.	<code>mainVariable(1 :: POLY INT)</code> "failed"	(17)	Type: Union("failed", ...)
You can also use the predicate <b>ground?</b> to test whether a polynomial is in fact a member of its ground ring.	<code>ground? p</code> false	(18)	Type: Boolean
	<code>ground?(1 :: POLY INT)</code> true	(19)	Type: Boolean
The complete list of variables actually used in a particular polynomial is returned by <b>variables</b> . For constant polynomials, this list is empty.	<code>variables p</code> <i>[z, y, x]</i>	(20)	Type: List Symbol
The <b>degree</b> operation returns the degree of a polynomial in a specific variable.	<code>degree(p,x)</code> 1	(21)	Type: PositiveInteger
	<code>degree(p,y)</code> 2	(22)	Type: PositiveInteger

	<code>degree(p,z)</code>		
	1	(23)	Type: PositiveInteger
If you give a list of variables for the second argument, a list of the degrees in those variables is returned.	<code>degree(p,[x,y,z])</code>		
	[1, 2, 1]	(24)	Type: List NonNegativeInteger
The minimum degree of a variable in a polynomial is computed using <b>minimumDegree</b> .	<code>minimumDegree(p,z)</code>		
	1	(25)	Type: PositiveInteger
The total degree of a polynomial is returned by <b>totalDegree</b> .	<code>totalDegree p</code>		
	4	(26)	Type: PositiveInteger
It is often convenient to think of a polynomial as a leading monomial plus the remaining terms.	<code>leadingMonomial p</code>		
	$x^2 y z$	(27)	Type: Polynomial Integer
The <b>reductum</b> operation returns a polynomial consisting of the sum of the monomials after the first.	<code>reductum p</code>		
	$(-2 x y + x) z$	(28)	Type: Polynomial Integer
These have the obvious relationship that the original polynomial is equal to the leading monomial plus the reductum.	<code>p - leadingMonomial p - reductum p</code>		
	0	(29)	Type: Polynomial Integer
The value returned by <b>leadingMonomial</b> includes the coefficient of that term. This is extracted by using <b>leadingCoefficient</b> on the original polynomial.	<code>leadingCoefficient p</code>		
	1	(30)	Type: PositiveInteger
The operation <b>eval</b> is used to substitute a value for a variable in a polynomial.	<code>p</code>		
	$(x^2 y - 2 x y + x) z$	(31)	Type: Polynomial Integer
This value may be another variable, a constant or a polynomial.	<code>eval(p,x,w)</code>		
	$(w^2 y - 2 w y + w) z$	(32)	Type: Polynomial Integer

$$\text{eval}(\mathbf{p}, \mathbf{x}, 1) \\ (y^2 - 2y + 1)z \quad (33)$$

Type: Polynomial Integer

Actually, all the things being substituted are just polynomials, some more trivial than others.

$$\text{eval}(\mathbf{p}, \mathbf{x}, y^{**2} - 1) \\ (y^4 - 2y^3 + 2y - 1)z \quad (34)$$

Type: Polynomial Integer

Derivatives are computed using the **D** operation.

$$\mathbf{D}(\mathbf{p}, \mathbf{x}) \\ (y^2 - 2y + 1)z \quad (35)$$

Type: Polynomial Integer

The first argument is the polynomial and the second is the variable.

$$\mathbf{D}(\mathbf{p}, y) \\ (2xy - 2x)z \quad (36)$$

Type: Polynomial Integer

Even if the polynomial has only one variable, you must specify it.

$$\mathbf{D}(\mathbf{p}, z) \\ xy^2 - 2xy + x \quad (37)$$

Type: Polynomial Integer

Integration of polynomials is similar and the **integrate** operation is used.

Integration requires that the coefficients support division. Consequently, AXIOM converts polynomials over the integers to polynomials over the rational numbers before integrating them.

$$\text{integrate}(\mathbf{p}, y) \\ \left(\frac{1}{3}xy^3 - xy^2 + xy\right)z \quad (38)$$

Type: Polynomial Fraction Integer

It is not possible, in general, to divide two polynomials. In our example using polynomials over the integers, the operation **monicDivide** divides a polynomial by a monic polynomial (that is, a polynomial with leading coefficient equal to 1). The result is a record of the quotient and remainder of the division.

You must specify the variable in which to express the polynomial.

$$\mathbf{qr} := \text{monicDivide}(\mathbf{p}, \mathbf{x}+1, \mathbf{x}) \\ \left[ \text{quotient} = (y^2 - 2y + 1)z, \text{remainder} = (-y^2 + 2y - 1)z \right] \quad (39)$$

Type: Record(quotient: Polynomial Integer, remainder: Polynomial Integer)

The selectors of the components of the record are **quotient** and **remainder**. Issue this to extract the remainder.

$$\mathbf{qr}.\text{remainder} \\ (-y^2 + 2y - 1)z \quad (40)$$

Type: Polynomial Integer

Now that we can extract the components, we can demonstrate the relationship among them and the arguments to our original expression `qr := monicDivide(p,x+1,x)`.

$$p - ((x+1) * qr.quotient + qr.remainder) = 0$$

(41)  
Type: Polynomial Integer

If the “/” operator is used with polynomials, a fraction object is created. In this example, the result is an object of type Fraction Polynomial Integer.

$$\frac{p}{q} = \frac{(y-1)z}{z+5}$$

(42)  
Type: Fraction Polynomial Integer

If you use rational numbers as polynomial coefficients, the resulting object is of type Polynomial Fraction Integer.

$$(2/3) * x^{**2} - y + 4/5 = -y + \frac{2}{3}x^2 + \frac{4}{5}$$

(43)  
Type: Polynomial Fraction Integer

This can be converted to a fraction of polynomials and back again, if required.

$$\% :: \text{FRAC POLY INT} = \frac{-15y + 10x^2 + 12}{15}$$

(44)  
Type: Fraction Polynomial Integer

$$\% :: \text{POLY FRAC INT} = -y + \frac{2}{3}x^2 + \frac{4}{5}$$

(45)  
Type: Polynomial Fraction Integer

To convert the coefficients to floating point, map the **numeric** operation on the coefficients of the polynomial.

$$\text{map}(\text{numeric},\%) = -1.0y + 0.6666666666666667x^2 + 0.8$$

(46)  
Type: Polynomial Float

For more information on related topics, see ‘UnivariatePolynomial’ on page 594, ‘MultivariatePolynomial’ on page 508, and ‘DistributedMultivariatePolynomial’ on page 402. You can also issue the system command `)show Polynomial` to display the full list of operations defined by Polynomial.

## 9.59 Quaternion

The basic operation for creating quaternions is **quatern**. This is a quaternion over the rational numbers.

The four arguments are the real part, the **i** imaginary part, the **j** imaginary part, and the **k** imaginary part, respectively.

Because **q** is over the rationals (and nonzero), you can invert it.

The usual arithmetic (ring) operations are available

In general, multiplication is not commutative.

There are no predefined constants for the imaginary **i**, **j**, and **k** parts, but you can easily define them.

These satisfy the normal identities.

The domain constructor Quaternion implements quaternions over commutative rings. For information on related topics, see ‘Complex’ on page 383 and ‘Octonion’ on page 511. You can also issue the system command **)show Quaternion** to display the full list of operations defined by Quaternion.

$$q := \text{quatern}(2/11, -8, 3/4, 1)$$

$$\frac{2}{11} - 8i + \frac{3}{4}j + k \quad (1)$$

Type: Quaternion Fraction Integer

$$[\text{real } q, \text{imagI } q, \text{imagJ } q, \text{imagK } q]$$

$$\left[ \frac{2}{11}, -8, \frac{3}{4}, 1 \right] \quad (2)$$

Type: List Fraction Integer

$$\text{inv } q$$

$$\frac{352}{126993} + \frac{15488}{126993}i - \frac{484}{42331}j - \frac{1936}{126993}k \quad (3)$$

Type: Quaternion Fraction Integer

$$q^{**6}$$

$$-\frac{2029490709319345}{7256313856} - \frac{48251690851}{1288408}i + \frac{144755072553}{41229056}j + \frac{48251690851}{10307264}k \quad (4)$$

Type: Quaternion Fraction Integer

$$r := \text{quatern}(-2, 3, 23/9, -89); q + r$$

$$-\frac{20}{11} - 5i + \frac{119}{36}j - 88k \quad (5)$$

Type: Quaternion Fraction Integer

$$q * r - r * q$$

$$-\frac{2495}{18}i - 1418j - \frac{817}{18}k \quad (6)$$

Type: Quaternion Fraction Integer

$$i := \text{quatern}(0, 1, 0, 0); j := \text{quatern}(0, 0, 1, 0);$$

$$k := \text{quatern}(0, 0, 0, 1)$$

$$k \quad (7)$$

Type: Quaternion Integer

$$[i*i, j*j, k*k, i*j, j*k, k*i, q*i]$$

$$\left[ -1, -1, -1, k, i, j, 8 + \frac{2}{11}i + j - \frac{3}{4}k \right] \quad (8)$$

Type: List Quaternion Fraction Integer

The norm is the quaternion times its conjugate.

`norm q`
$$\frac{126993}{1936}$$

(9)

Type: Fraction Integer

`conjugate q`
$$\frac{2}{11} + 8\,i - \frac{3}{4}\,j - k$$

(10)

Type: Quaternion Fraction Integer

`q * %`
$$\frac{126993}{1936}$$

(11)

Type: Quaternion Fraction Integer

## 9.60 RadixExpansion

It possible to expand numbers in general bases.

Here we expand 111 in base 5.  
This means  
 $10^2 + 10^1 + 10^0 = 4 \cdot 5^2 + 2 \cdot 5^1 + 5^0$ .

111::RadixExpansion(5)

421

(1)

Type: RadixExpansion 5

You can expand fractions to  
form repeating expansions.

(5/24)::RadixExpansion(2)

0.00110

(2)

Type: RadixExpansion 2

(5/24)::RadixExpansion(3)

0.012

(3)

Type: RadixExpansion 3

(5/24)::RadixExpansion(8)

0.152

(4)

Type: RadixExpansion 8

(5/24)::RadixExpansion(10)

0.2083

(5)

Type: RadixExpansion 10

For bases from 11 to 36 the  
letters A through Z are used.

(5/24)::RadixExpansion(12)

0.26

(6)

Type: RadixExpansion 12

(5/24)::RadixExpansion(16)

0.35

(7)

Type: RadixExpansion 16

(5/24)::RadixExpansion(36)

0.7I

(8)

Type: RadixExpansion 36

For bases greater than 36, the  
ragits are separated by blanks.

(5/24)::RadixExpansion(38)

0 . 7 34 31 25 12

(9)

Type: RadixExpansion 38

The RadixExpansion type  
provides operations to obtain  
the individual ragits. Here is a  
rational number in base 8.

a := (76543/210)::RadixExpansion(8)

554.37307

(10)

Type: RadixExpansion 8

The operation **wholeRagits** returns a list of the ragits for the integral part of the number.

$$w := \text{wholeRagits } a$$

$$[5, 5, 4] \quad (11)$$

Type: List Integer

The operations **prefixRagits** and **cycleRagits** return lists of the initial and repeating ragits in the fractional part of the number.

$$f0 := \text{prefixRagits } a$$

$$[3] \quad (12)$$

Type: List Integer

$$f1 := \text{cycleRagits } a$$

$$[7, 3, 0, 7] \quad (13)$$

Type: List Integer

You can construct any radix expansion by giving the whole, prefix and cycle parts. The declaration is necessary to let AXIOM know the base of the ragits.

$$u:\text{RadixExpansion}(8):=\text{wholeRadix}(w)+\text{fractRadix}(f0,f1)$$

$$554.3\overline{7307} \quad (14)$$

Type: RadixExpansion 8

If there is no repeating part, then the list [0] should be used.

$$v:\text{RadixExpansion}(12) := \text{fractRadix}([1,2,3,11], [0])$$

$$0.123B\overline{0} \quad (15)$$

Type: RadixExpansion 12

If you are not interested in the repeating nature of the expansion, an infinite stream of ragits can be obtained using **fractRagits**.

$$\text{fractRagits}(u)$$

$$[3, 7, 3, 0, 7, 7] \quad (16)$$

Type: Stream Integer

Of course, it's possible to recover the fraction representation:

$$a :: \text{Fraction}(\text{Integer})$$

$$\frac{76543}{210} \quad (17)$$

Type: Fraction Integer

Issue the system command `)show RadixExpansion` to display the full list of operations defined by RadixExpansion. More examples of expansions are available in 'DecimalExpansion' on page 401, 'BinaryExpansion' on page 359, and 'HexadecimalExpansion' on page 444.



## 9.61 RealClosure

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The Real Closure 1.0 package provided by Renaud Rioboo ( Renaud.Rioboo@lip6.fr ) consists of different packages, categories and domains :

- The package `RealPolynomialUtilitiesPackage` which needs a Field  $F$  and a `UnivariatePolynomialCategory` domain with coefficients in  $F$ . It computes some simple functions such as Sturm and Sylvester sequences (`“sturmSequence”`, `“sylvesterSequence”`).
- The category `RealRootCharacterizationCategory` provides abstract functions to work with ”real roots” of univariate polynomials. These resemble variables with some functionality needed to compute important operations.
- The category `RealClosedField` provides common operations available over real closed fields. These include finding all the roots of a univariate polynomial, taking square (and higher) roots, ...
- The domain `RightOpenIntervalRootCharacterization` is the main code that provides the functionality of `RealRootCharacterizationCategory` for the case of archimedean fields. Abstract roots are encoded with a left closed right open interval containing the root together with a defining polynomial for the root.
- The `RealClosure` domain is the end-user code. It provides usual arithmetic with real algebraic numbers, along with the functionality of a real closed field. It also provides functions to approximate a real algebraic number by an element of the base field. This approximation may either be absolute (`“approximate”`) or relative (`“relativeApprox”`).

### CAVEATS

Since real algebraic expressions are stored as depending on ”real roots” which are managed like variables, there is an ordering on these. This ordering is dynamical in the sense that any new algebraic takes precedence over older ones. In particular every creation function raises a new ”real root”. This has the effect that when you type something like `sqrt(2) + sqrt(2)` you have two new variables which happen to be equal. To avoid this name the expression such as in `s2 := sqrt(2) ; s2 + s2`

Also note that computing times depend strongly on the ordering you implicitly provide. Please provide algebraics in the order which seems most natural to you.

### LIMITATIONS

This packages uses algorithms which are published in [1] and [2] which are based on field arithmetics, in particular for polynomial gcd related algorithms. This can be quite slow for high degree polynomials and sub-resultants methods usually work best. Beta versions of the package try to use these techniques in a better way and work significantly faster. These

are mostly based on unpublished algorithms and cannot be distributed. Please contact the author if you have a particular problem to solve or want to use these versions.

Be aware that approximations behave as post-processing and that all computations are done exactly. They can thus be quite time consuming when depending on several "real roots".

#### REFERENCES

- [1] R. Rioboo : Real Algebraic Closure of an ordered Field : Implementation in Axiom. In proceedings of the ISSAC'92 Conference, Berkeley 1992 pp. 206-215.
- [2] Z. Ligatsikas, R. Rioboo, M. F. Roy : Generic computation of the real closure of an ordered field. In Mathematics and Computers in Simulation Volume 42, Issue 4-6, November 1996.

#### EXAMPLES

We shall work with the real closure of the ordered field of rational numbers.

```
Ran := RECLOS(FRAC INT)
```

```
RealClosure Fraction Integer
```

(1)

Type: Domain

Some simple signs for square roots, these correspond to an extension of degree 16 of the rational numbers. Examples provided by J. Abbot.

```
fourSquares(a:Ran,b:Ran,c:Ran,d:Ran):Ran ==  
  sqrt(a)+sqrt(b) - sqrt(c)-sqrt(d)
```

```
Function declaration fourSquares : (RealClosure  
  Fraction Integer,RealClosure Fraction Integer,  
  RealClosure Fraction Integer,RealClosure Fraction  
  Integer) -> RealClosure Fraction Integer has been  
  added to workspace.
```

Type: Void

These produce values very close to zero.

```
squareDiff1 := fourSquares(73,548,60,586)
```

```
Compiling function fourSquares with type (RealClosure  
  Fraction Integer,RealClosure Fraction Integer,  
  RealClosure Fraction Integer,RealClosure Fraction  
  Integer) -> RealClosure Fraction Integer
```

$$-\sqrt{586} - \sqrt{60} + \sqrt{548} + \sqrt{73}$$

(3)

Type: RealClosure Fraction Integer

```
recip(squareDiff1)
```

$$\left( (54602 \sqrt{548} + 149602 \sqrt{73}) \sqrt{60} + 49502 \sqrt{73} \sqrt{548} + 9900895 \sqrt{586} + (154702 \sqrt{73} \sqrt{548} + 30941947) \sqrt{60} + 10238421 \sqrt{548} + 28051871 \sqrt{73} \right)$$

(4)

Type: Union(RealClosure Fraction Integer, ...)

sign(squareDiff1)

1 (5)

Type: PositiveInteger

squareDiff2 := fourSquares(165,778,86,990)

$-\sqrt{990} - \sqrt{86} + \sqrt{778} + \sqrt{165}$  (6)

Type: RealClosure Fraction Integer

recip(squareDiff2)

$$\left( (556778 \sqrt{778} + 1209010 \sqrt{165}) \sqrt{86} + 401966 \sqrt{165} \sqrt{778} + \right. \\ \left. 144019431 \sqrt{990} + (1363822 \sqrt{165} \sqrt{778} + 488640503) \sqrt{86} + \right. \\ \left. 162460913 \sqrt{778} + 352774119 \sqrt{165} \right) \quad (7)$$

Type: Union(RealClosure Fraction Integer, ...)

sign(squareDiff2)

1 (8)

Type: PositiveInteger

squareDiff3 := fourSquares(217,708,226,692)

$-\sqrt{692} - \sqrt{226} + \sqrt{708} + \sqrt{217}$  (9)

Type: RealClosure Fraction Integer

recip(squareDiff3)

$$\left( (-34102 \sqrt{708} - 61598 \sqrt{217}) \sqrt{226} - 34802 \sqrt{217} \sqrt{708} - \right. \\ \left. 13641141 \sqrt{692} + (-60898 \sqrt{217} \sqrt{708} - 23869841) \sqrt{226} - \right. \\ \left. 13486123 \sqrt{708} - 24359809 \sqrt{217} \right) \quad (10)$$

Type: Union(RealClosure Fraction Integer, ...)

sign(squareDiff3)

-1 (11)

Type: Integer

squareDiff4 := fourSquares(155,836,162,820)

$-\sqrt{820} - \sqrt{162} + \sqrt{836} + \sqrt{155}$  (12)

Type: RealClosure Fraction Integer

$$\begin{aligned} &\text{recip}(\text{squareDiff4}) \\ &\left( \left( -37078 \sqrt{836} - 86110 \sqrt{155} \right) \sqrt{162} - 37906 \sqrt{155} \sqrt{836} - \right. \\ &13645107) \sqrt{820} + \left( -85282 \sqrt{155} \sqrt{836} - 30699151 \right) \sqrt{162} - \\ &13513901 \sqrt{836} - 31384703 \sqrt{155} \end{aligned} \quad (13)$$

Type: Union(RealClosure Fraction Integer, ...)

$$\begin{aligned} &\text{sign}(\text{squareDiff4}) \\ &-1 \end{aligned} \quad (14)$$

Type: Integer

$$\begin{aligned} &\text{squareDiff5} := \text{fourSquares}(591, 772, 552, 818) \\ &-\sqrt{818} - \sqrt{552} + \sqrt{772} + \sqrt{591} \end{aligned} \quad (15)$$

Type: RealClosure Fraction Integer

$$\begin{aligned} &\text{recip}(\text{squareDiff5}) \\ &\left( \left( 70922 \sqrt{772} + 81058 \sqrt{591} \right) \sqrt{552} + 68542 \sqrt{591} \sqrt{772} + \right. \\ &46297673) \sqrt{818} + \left( 83438 \sqrt{591} \sqrt{772} + 56359389 \right) \sqrt{552} + \\ &47657051 \sqrt{772} + 54468081 \sqrt{591} \end{aligned} \quad (16)$$

Type: Union(RealClosure Fraction Integer, ...)

$$\begin{aligned} &\text{sign}(\text{squareDiff5}) \\ &1 \end{aligned} \quad (17)$$

Type: PositiveInteger

$$\begin{aligned} &\text{squareDiff6} := \text{fourSquares}(434, 1053, 412, 1088) \\ &-\sqrt{1088} - \sqrt{412} + \sqrt{1053} + \sqrt{434} \end{aligned} \quad (18)$$

Type: RealClosure Fraction Integer

$$\begin{aligned} &\text{recip}(\text{squareDiff6}) \\ &\left( \left( 115442 \sqrt{1053} + 179818 \sqrt{434} \right) \sqrt{412} + 112478 \sqrt{434} \sqrt{1053} + \right. \\ &76037291) \sqrt{1088} + \left( 182782 \sqrt{434} \sqrt{1053} + 123564147 \right) \sqrt{412} + \\ &77290639 \sqrt{1053} + 120391609 \sqrt{434} \end{aligned} \quad (19)$$

Type: Union(RealClosure Fraction Integer, ...)

$$\begin{aligned} &\text{sign}(\text{squareDiff6}) \\ &1 \end{aligned} \quad (20)$$

Type: PositiveInteger

$$\text{squareDiff7} := \text{fourSquares}(514, 1049, 446, 1152) \\ -\sqrt{1152} - \sqrt{446} + \sqrt{1049} + \sqrt{514} \quad (21)$$

Type: RealClosure Fraction Integer

$$\text{recip}(\text{squareDiff7}) \\ \left( (349522 \sqrt{1049} + 499322 \sqrt{514}) \sqrt{446} + 325582 \sqrt{514} \sqrt{1049} + \right. \\ \left. 239072537 \sqrt{1152} + (523262 \sqrt{514} \sqrt{1049} + 384227549) \sqrt{446} + \right. \\ \left. 250534873 \sqrt{1049} + 357910443 \sqrt{514} \right) \quad (22)$$

Type: Union(RealClosure Fraction Integer, ...)

$$\text{sign}(\text{squareDiff7}) \\ 1 \quad (23)$$

Type: PositiveInteger

$$\text{squareDiff8} := \text{fourSquares}(190, 1751, 208, 1698) \\ -\sqrt{1698} - \sqrt{208} + \sqrt{1751} + \sqrt{190} \quad (24)$$

Type: RealClosure Fraction Integer

$$\text{recip}(\text{squareDiff8}) \\ \left( (-214702 \sqrt{1751} - 651782 \sqrt{190}) \sqrt{208} - 224642 \sqrt{190} \sqrt{1751} \right. \\ \left. - 129571901 \sqrt{1698} + (-641842 \sqrt{190} \sqrt{1751} - 370209881) \sqrt{208} \right. \\ \left. - 127595865 \sqrt{1751} - 387349387 \sqrt{190} \right) \quad (25)$$

Type: Union(RealClosure Fraction Integer, ...)

$$\text{sign}(\text{squareDiff8}) \\ -1 \quad (26)$$

Type: Integer

This should give three digits of precision

$$\text{relativeApprox}(\text{squareDiff8}, 10^{**(-3)}) :: \text{Float} \\ -0.23405277715937700123E - 10 \quad (27)$$

Type: Float

The sum of these 4 roots is 0

$$1 := \text{allRootsOf}((x^{**2} - 2)^{**2} - 2) \$ \text{Ran} \\ [\%R33, \%R34, \%R35, \%R36] \quad (28)$$

Type: List RealClosure Fraction Integer

Check that they are all roots of the same polynomial

$$\text{removeDuplicates map}(\text{mainDefiningPolynomial}, 1) \\ \left[ ?^4 - 4 ?^2 + 2 \right] \quad (29)$$

Type: List Union(SparseUnivariatePolynomial RealClosure Fraction Integer, "failed")

We can see at a glance that they are separate roots

$$\text{map}(\text{mainCharacterization}, 1) \\ [[-2, -1[, [-1, 0[, [0, 1[, [1, 2[ \quad (30)$$

Type: List Union(RightOpenIntervalRootCharacterization(RealClosure Fraction Integer, SparseUnivariatePolynomial RealClosure Fraction Integer), "failed")

Check the sum and product

$$[\text{reduce}(+, 1), \text{reduce}(*, 1) - 2] \\ [0, 0] \quad (31)$$

Type: List RealClosure Fraction Integer

A more complicated test that involve an extension of degree 256. This is a way of checking nested radical identities.

$$(s2, s5, s10) := (\text{sqrt}(2)\$Ran, \text{sqrt}(5)\$Ran, \text{sqrt}(10)\$Ran) \\ \sqrt{10} \quad (32)$$

Type: RealClosure Fraction Integer

$$\text{eq1} := \text{sqrt}(s10+3) * \text{sqrt}(s5+2) - \text{sqrt}(s10-3) * \text{sqrt}(s5-2) = \\ \text{sqrt}(10*s2+10) \\ -\sqrt{\sqrt{10}-3} \sqrt{\sqrt{5}-2} + \sqrt{\sqrt{10}+3} \sqrt{\sqrt{5}+2} = \sqrt{10 \sqrt{2}+10} \quad (33)$$

Type: Equation RealClosure Fraction Integer

$$\text{eq1}::\text{Boolean} \\ \text{true} \quad (34)$$

Type: Boolean

$$\text{eq2} := \text{sqrt}(s5+2) * \text{sqrt}(s2+1) - \text{sqrt}(s5-2) * \text{sqrt}(s2-1) = \\ \text{sqrt}(2*s10+2) \\ -\sqrt{\sqrt{5}-2} \sqrt{\sqrt{2}-1} + \sqrt{\sqrt{5}+2} \sqrt{\sqrt{2}+1} = \sqrt{2 \sqrt{10}+2} \quad (35)$$

Type: Equation RealClosure Fraction Integer

$$\text{eq2}::\text{Boolean} \\ \text{true} \quad (36)$$

Type: Boolean

Some more examples from J. M. Arnaudies

$$s3 := \text{sqrt}(3)\$Ran \\ \sqrt{3} \quad (37)$$

Type: RealClosure Fraction Integer

$$s7 := \text{sqrt}(7)\$Ran \\ \sqrt{7} \quad (38)$$

Type: RealClosure Fraction Integer

$$e1 := \text{sqrt}(2*s7-3*s3, 3) \\ \sqrt[3]{2 \sqrt{7}-3 \sqrt{3}} \quad (39)$$

Type: RealClosure Fraction Integer

	$e2 := \text{sqrt}(2*s7+3*s3,3)$ $\sqrt[3]{2\sqrt{7}+3\sqrt{3}}$ <div style="text-align: right;">(40)</div>	Type: RealClosure Fraction Integer
This should be null	$e2-e1-s3$ $0$ <div style="text-align: right;">(41)</div>	Type: RealClosure Fraction Integer
A quartic polynomial	$\text{pol} : \text{UP}(\mathbf{x}, \text{Ran}) := \mathbf{x}^{**4} + (7/3) * \mathbf{x}^{**2} + 30 * \mathbf{x} - (100/3)$ $x^4 + \frac{7}{3} x^2 + 30 x - \frac{100}{3}$ <div style="text-align: right;">(42)</div>	Type: UnivariatePolynomial(x, RealClosure Fraction Integer)
Add some cubic roots	$r1 := \text{sqrt}(7633)\$Ran$ $\sqrt{7633}$ <div style="text-align: right;">(43)</div>	Type: RealClosure Fraction Integer
	$\alpha := \text{sqrt}(5*r1-436,3)/3$ $\frac{1}{3} \sqrt[3]{5\sqrt{7633}-436}$ <div style="text-align: right;">(44)</div>	Type: RealClosure Fraction Integer
	$\beta := -\text{sqrt}(5*r1+436,3)/3$ $-\frac{1}{3} \sqrt[3]{5\sqrt{7633}+436}$ <div style="text-align: right;">(45)</div>	Type: RealClosure Fraction Integer
this should be null	$\text{pol} . (\alpha + \beta - 1/3)$ $0$ <div style="text-align: right;">(46)</div>	Type: RealClosure Fraction Integer
A quintic polynomial	$\text{qol} : \text{UP}(\mathbf{x}, \text{Ran}) := \mathbf{x}^{**5} + 10 * \mathbf{x}^{**3} + 20 * \mathbf{x} + 22$ $x^5 + 10 x^3 + 20 x + 22$ <div style="text-align: right;">(47)</div>	Type: UnivariatePolynomial(x, RealClosure Fraction Integer)
Add some cubic roots	$r2 := \text{sqrt}(153)\$Ran$ $\sqrt{153}$ <div style="text-align: right;">(48)</div>	Type: RealClosure Fraction Integer
	$\alpha2 := \text{sqrt}(r2-11,5)$ $\sqrt[5]{\sqrt{153}-11}$ <div style="text-align: right;">(49)</div>	Type: RealClosure Fraction Integer

```
beta2 := -sqrt(r2+11,5)
```

$$-\sqrt[5]{\sqrt{153} + 11} \quad (50)$$

Type: RealClosure Fraction Integer

this should be null

```
qol(alpha2+beta2)
```

$$0 \quad (51)$$

Type: RealClosure Fraction Integer

Finally, some examples from the book Computer Algebra by Davenport, Siret and Tournier (page 77). The last one is due to Ramanujan.

```
dst1:=sqrt(9+4*s2)=1+2*s2
```

$$\sqrt{4\sqrt{2} + 9} = 2\sqrt{2} + 1 \quad (52)$$

Type: Equation RealClosure Fraction Integer

```
dst1::Boolean
```

$$\text{true} \quad (53)$$

Type: Boolean

```
s6:Ran:=sqrt 6
```

$$\sqrt{6} \quad (54)$$

Type: RealClosure Fraction Integer

```
dst2:=sqrt(5+2*s6)+sqrt(5-2*s6) = 2*s3
```

$$\sqrt{-2\sqrt{6} + 5} + \sqrt{2\sqrt{6} + 5} = 2\sqrt{3} \quad (55)$$

Type: Equation RealClosure Fraction Integer

```
dst2::Boolean
```

$$\text{true} \quad (56)$$

Type: Boolean

```
s29:Ran:=sqrt 29
```

$$\sqrt{29} \quad (57)$$

Type: RealClosure Fraction Integer

```
dst4:=sqrt(16-2*s29+2*sqrt(55-10*s29)) = sqrt(22+2*s5) -  
sqrt(11+2*s29)+s5
```

$$\sqrt{2\sqrt{-10\sqrt{29} + 55} - 2\sqrt{29} + 16} = -\sqrt{2\sqrt{29} + 11} + \sqrt{2\sqrt{5} + 22} + \sqrt{5} \quad (58)$$

Type: Equation RealClosure Fraction Integer



dst4::Boolean

true (59)

Type: Boolean

dst6:=sqrt((112+70\*s2)+(46+34\*s2)\*s5) = (5+4\*s2)+(3+s2)\*s5

$$\sqrt{(34\sqrt{2}+46)\sqrt{5}+70\sqrt{2}+112} = (\sqrt{2}+3)\sqrt{5}+4\sqrt{2}+5 \quad (60)$$

Type: Equation RealClosure Fraction Integer

dst6::Boolean

true (61)

Type: Boolean

f3:Ran:=sqrt(3,5)

$$\sqrt[5]{3} \quad (62)$$

Type: RealClosure Fraction Integer

f25:Ran:=sqrt(1/25,5)

$$\sqrt[5]{\frac{1}{25}} \quad (63)$$

Type: RealClosure Fraction Integer

f32:Ran:=sqrt(32/5,5)

$$\sqrt[5]{\frac{32}{5}} \quad (64)$$

Type: RealClosure Fraction Integer

f27:Ran:=sqrt(27/5,5)

$$\sqrt[5]{\frac{27}{5}} \quad (65)$$

Type: RealClosure Fraction Integer

dst5:=sqrt((f32-f27,3)) = f25\*(1+f3-f3\*\*2)

$$\sqrt[3]{-\sqrt[5]{\frac{27}{5}}+\sqrt[5]{\frac{32}{5}}} = (-\sqrt[5]{3^2}+\sqrt[5]{3}+1)\sqrt[5]{\frac{1}{25}} \quad (66)$$

Type: Equation RealClosure Fraction Integer

dst5::Boolean

true (67)

Type: Boolean

## 9.62 Regular- TriangularSet

---

The `RegularTriangularSet` domain constructor implements regular triangular sets. These particular triangular sets were introduced by M. Kalkbrener (1991) in his PhD Thesis under the name regular chains. Regular chains and their related concepts are presented in the paper "On the Theories of Triangular sets" By P. Aubry, D. Lazard and M. Moreno Maza (to appear in the Journal of Symbolic Computation). This constructor also provides a new method (by the third author) for solving polynomial system by means of regular chains. This method has two ways of solving. One has the same specifications as Kalkbrener's algorithm (1991) and the other is closer to Lazard's method (Discr. App. Math, 1991). Moreover, this new method removes redundant component from the decompositions when this is not *too much expensive*. This is always the case with square-free regular chains. So if you want to obtain decompositions without redundant components just use the `SquareFreeRegularTriangularSet` domain constructor or the `LazardSetSolvingPackage` package constructor. See also the `ZeroDimensionalSolvePackage` for the case of algebraic systems with a finite number of (complex) solutions.

One of the main features of regular triangular sets is that they naturally define towers of simple extensions of a field. This allows to perform with multivariate polynomials the same kind of operations as one can do in an `EuclideanDomain`.

We shall explain now how to use the constructor `RegularTriangularSet` and how the decomposition of a polynomial system by means of regular sets has to be understood.

This constructor takes four arguments. The first one, **R**, is the coefficient ring of the polynomials; it must belong to the category `GcdDomain`. The second one, **E**, is the exponent monoid of the polynomials; it must belong to the category `OrderedAbelianMonoidSup`. the third one, **V**, is the ordered set of variables; it must belong to the category `OrderedSet`. The last one is the polynomial ring; it must belong to the category `RecursivePolynomialCategory(R,E,V)`. The abbreviation for `RegularTriangularSet` is `REGSET`. See also the constructor `RegularChain` which only takes two arguments, the coefficient ring and the ordered set of variables; in that case, polynomials are necessarily built with the `NewSparseMultivariatePolynomial` domain constructor.

Let us illustrate the facilities of the `REGSET` constructor by some examples. We start with an easy example (Donati-Traverso) in order to understand its two ways of solving polynomial systems.

Define the coefficient ring.	$R := \text{Integer}$ <i>Integer</i>	(1)
		Type: Domain
Define the list of variables,	$ls : \text{List Symbol} := [x, y, z, t]$ $[x, y, z, t]$	(2)
		Type: List Symbol
and make it an ordered set;	$V := \text{OVAR}(ls)$ $\text{OrderedVariableList } [x, y, z, t]$	(3)
		Type: Domain
then define the exponent monoid.	$E := \text{IndexedExponents } V$ $\text{IndexedExponents OrderedVariableList } [x, y, z, t]$	(4)
		Type: Domain
Define the polynomial ring.	$P := \text{NSMP}(R, V)$ $\text{NewSparseMultivariatePolynomial}(\text{Integer}, \text{OrderedVariableList } [x, y, z, t])$	(5)
		Type: Domain
Let the variables be polynomial.	$x : P := 'x$ $x$	(6)
	Type: $\text{NewSparseMultivariatePolynomial}(\text{Integer}, \text{OrderedVariableList } [x, y, z, t])$	
	$y : P := 'y$ $y$	(7)
	Type: $\text{NewSparseMultivariatePolynomial}(\text{Integer}, \text{OrderedVariableList } [x, y, z, t])$	
	$z : P := 'z$ $z$	(8)
	Type: $\text{NewSparseMultivariatePolynomial}(\text{Integer}, \text{OrderedVariableList } [x, y, z, t])$	
	$t : P := 't$ $t$	(9)
	Type: $\text{NewSparseMultivariatePolynomial}(\text{Integer}, \text{OrderedVariableList } [x, y, z, t])$	
Now call the RegularTriangularSet domain constructor.	$T := \text{REGSET}(R, E, V, P)$ $\text{RegularTriangularSet}(\text{Integer}, \text{IndexedExponents}$ $\text{OrderedVariableList } [x, y, z, t], \text{OrderedVariableList } [x, y, z, t],$ $\text{NewSparseMultivariatePolynomial}(\text{Integer}, \text{OrderedVariableList } [x,$ $y, z, t]))$	(10)
		Type: Domain

Define a polynomial system.

$$\begin{aligned} \text{p1} &:= x^{31} - x^6 - x - y \\ x^{31} - x^6 - x - y \end{aligned} \quad (11)$$

Type: NewSparseMultivariatePolynomial(Integer, OrderedVariableList [x, y, z, t])

$$\begin{aligned} \text{p2} &:= x^8 - z \\ x^8 - z \end{aligned} \quad (12)$$

Type: NewSparseMultivariatePolynomial(Integer, OrderedVariableList [x, y, z, t])

$$\begin{aligned} \text{p3} &:= x^{10} - t \\ x^{10} - t \end{aligned} \quad (13)$$

Type: NewSparseMultivariatePolynomial(Integer, OrderedVariableList [x, y, z, t])

$$\begin{aligned} \text{lp} &:= [\text{p1}, \text{p2}, \text{p3}] \\ [x^{31} - x^6 - x - y, x^8 - z, x^{10} - t] \end{aligned} \quad (14)$$

Type: List NewSparseMultivariatePolynomial(Integer, OrderedVariableList [x, y, z, t])

First of all, let us solve this system in the sense of Kalkbrener.

$$\begin{aligned} \text{zeroSetSplit}(\text{lp})\$T \\ \left[ \left\{ \begin{aligned} &z^5 - t^4, t z y^2 + 2 z^3 y - t^8 + 2 t^5 + t^3 - t^2, \\ &(t^4 - t) x - t y - z^2 \end{aligned} \right\} \right] \end{aligned} \quad (15)$$

Type: List RegularTriangularSet(Integer, IndexedExponents OrderedVariableList [x, y, z, t], OrderedVariableList [x, y, z, t], NewSparseMultivariatePolynomial(Integer, OrderedVariableList [x, y, z, t]))

And now in the sense of Lazard (or Wu and other authors).

$$\begin{aligned} \text{lts} &:= \text{zeroSetSplit}(\text{lp}, \text{false})\$T \\ \left[ \begin{aligned} &\left\{ z^5 - t^4, t z y^2 + 2 z^3 y - t^8 + 2 t^5 + t^3 - t^2, (t^4 - t) x - t y - z^2 \right\}, \\ &\left\{ t^3 - 1, z^5 - t, t z y^2 + 2 z^3 y + 1, z x^2 - t \right\}, \\ &\{t, z, y, x\} \end{aligned} \right] \end{aligned} \quad (16)$$

Type: List RegularTriangularSet(Integer, IndexedExponents OrderedVariableList [x, y, z, t], OrderedVariableList [x, y, z, t], NewSparseMultivariatePolynomial(Integer, OrderedVariableList [x, y, z, t]))

We can see that the first decomposition is a subset of the second. So how can both be correct ?

Recall first that polynomials from a domain of the category RecursivePolynomialCategory are regarded as univariate polynomials in their main variable. For instance the second polynomial in the first set of each decomposition has main variable **y** and its initial (i.e. its leading coefficient w.r.t. its main variable) is **t z**.

Now let us explain how to read the second decomposition. Note that the non-constant initials of the first set are  $t^4 - t$  and  $tz$ . Then the solu-

tions described by this first set are the common zeros of its polynomials that do not cancel the polynomials  $t^4 - t$  and  $tyz$ . Now the solutions of the input system **lp** satisfying these equations are described by the second and the third sets of the decomposition. Thus, in some sense, they can be considered as degenerated solutions. The solutions given by the first set are called the generic points of the system; they give the general form of the solutions. The first decomposition only provides these generic points. This latter decomposition is useful when they are many degenerated solutions (which is sometimes hard to compute) and when one is only interested in general informations, like the dimension of the input system.

We can get the dimensions of each component of a decomposition as follows.

```
[coHeight(ts) for ts in lts]
```

```
[1, 0, 0]
```

(17)

Type: List NonNegativeInteger

Thus the first set has dimension one. Indeed **t** can take any value, except **0** or any third root of **1**, whereas **z** is completely determined from **t**, **y** is given by **z** and **t**, and finally **x** is given by the other three variables. In the second and the third sets of the second decomposition the four variables are completely determined and thus these sets have dimension zero.

We give now the precise specifications of each decomposition. This assume some mathematical knowledge. However, for the non-expert user, the above explanations will be sufficient to understand the other features of the RSEGSET constructor.

The input system **lp** is decomposed in the sense of Kalkbrener as finitely many regular sets **T1,...,Ts** such that the radical ideal generated by **lp** is the intersection of the radicals of the saturated ideals of **T1,...,Ts**. In other words, the affine variety associated with **lp** is the union of the closures (w.r.t. Zarisky topology) of the regular zeros sets of **T1,...,Ts**.

**N. B.** The prime ideals associated with the radical of the saturated ideal of a regular triangular set have all the same dimension; moreover these prime ideals can be given by characteristic sets with the same main variables. Thus a decomposition in the sense of Kalkbrener is unmixed dimensional. Then it can be viewed as a *lazy* decomposition into prime ideals (some of these prime ideals being merged into unmixed dimensional ideals).

Now we explain the other way of solving by means of regular triangular sets. The input system **lp** is decomposed in the sense of Lazard as finitely many regular triangular sets **T1,...,Ts** such that the affine variety associated with **lp** is the union of the regular zeros sets of **T1,...,Ts**. Thus a decomposition in the sense of Lazard is also a decomposition in the sense

of Kalkbrener; the converse is false as we have seen before.

When the input system has a finite number of solutions, both ways of solving provide similar decompositions as we shall see with this second example (Caprasse).

Define a polynomial system.

$$\begin{aligned}
 f1 &:= y^{**2}z + 2*x*y*t - 2*x - z \\
 (2\ t\ y - 2)\ x + z\ y^2 - z & \quad (18) \\
 \text{Type: NewSparseMultivariatePolynomial(Integer, OrderedVariableList [x, y, z, t])} \\
 f2 &:= -x^{**3}z + 4*x*y^{**2}z + 4*x^{**2}*y*t + 2*y^{**3}*t + 4*x^{**2} - \\
 10*y^{**2} + 4*x*z - 10*y*t + 2 \\
 -z\ x^3 + (4\ t\ y + 4)\ x^2 + (4\ z\ y^2 + 4\ z)\ x + 2\ t\ y^3 - 10\ y^2 - 10\ t\ y + 2 & \quad (19) \\
 \text{Type: NewSparseMultivariatePolynomial(Integer, OrderedVariableList [x, y, z, t])} \\
 f3 &:= 2*y*z*t + x*t^{**2} - x - 2*z \\
 (t^2 - 1)\ x + 2\ t\ z\ y - 2\ z & \quad (20) \\
 \text{Type: NewSparseMultivariatePolynomial(Integer, OrderedVariableList [x, y, z, t])} \\
 f4 &:= -x*z^{**3} + 4*y*z^{**2}*t + 4*x*z*t^{**2} + 2*y*t^{**3} + 4*x*z + \\
 4*z^{**2} - 10*y*t - 10*t^{**2} + 2 \\
 (-z^3 + (4\ t^2 + 4)\ z)\ x + (4\ t\ z^2 + 2\ t^3 - 10\ t)\ y + 4\ z^2 - 10\ t^2 + 2 & \quad (21) \\
 \text{Type: NewSparseMultivariatePolynomial(Integer, OrderedVariableList [x, y, z, t])} \\
 lf &:= [f1, f2, f3, f4] \\
 \left[ \begin{aligned} &(2\ t\ y - 2)\ x + z\ y^2 - z, \\ &-z\ x^3 + (4\ t\ y + 4)\ x^2 + (4\ z\ y^2 + 4\ z)\ x + 2\ t\ y^3 - 10\ y^2 - 10\ t\ y + 2, \\ &(t^2 - 1)\ x + 2\ t\ z\ y - 2\ z, \\ &(-z^3 + (4\ t^2 + 4)\ z)\ x + (4\ t\ z^2 + 2\ t^3 - 10\ t)\ y + 4\ z^2 - 10\ t^2 + 2 \end{aligned} \right] & \quad (22) \\
 \text{Type: List NewSparseMultivariatePolynomial(Integer, OrderedVariableList [x, y, z, t])}
 \end{aligned}$$

First of all, let us solve this system in the sense of Kalkbrener.

$$\begin{aligned}
 \text{zeroSetSplit}(lf)\$T \\
 \left[ \begin{aligned} &\{t^2 - 1, z^8 - 16\ z^6 + 256\ z^2 - 256, t\ y - 1, (z^3 - 8\ z)\ x - 8\ z^2 + 16\}, \\ &\{3\ t^2 + 1, z^2 - 7\ t^2 - 1, y + t, x + z\}, \\ &\{t^8 - 10\ t^6 + 10\ t^2 - 1, z, (t^3 - 5\ t)\ y - 5\ t^2 + 1, x\}, \\ &\{t^2 + 3, z^2 - 4, y + t, x - z\} \end{aligned} \right] & \quad (23) \\
 \text{Type: List RegularTriangularSet(Integer, IndexedExponents OrderedVariableList [x, y, z, t], OrderedVariableList [x, y, z, t], NewSparseMultivariatePolynomial(Integer, OrderedVariableList [x, y, z, t]))}
 \end{aligned}$$

And now in the sense of Lazard  
(or Wu and other authors).

$$\text{1ts2} := \text{zeroSetSplit}(\text{1f}, \text{false})\$T \left[ \begin{array}{l} \{t^8 - 10t^6 + 10t^2 - 1, z, (t^3 - 5t)y - 5t^2 + 1, x\}, \\ \{t^2 - 1, z^8 - 16z^6 + 256z^2 - 256, ty - 1, (z^3 - 8z)x - 8z^2 + 16\}, \\ \{3t^2 + 1, z^2 - 7t^2 - 1, y + t, x + z\}, \\ \{t^2 + 3, z^2 - 4, y + t, x - z\} \end{array} \right] \quad (24)$$

Type: List RegularTriangularSet(Integer, IndexedExponents OrderedVariableList [x, y, z, t], OrderedVariableList [x, y, z, t], NewSparseMultivariatePolynomial(Integer, OrderedVariableList [x, y, z, t]))

Up to the ordering of the components, both decompositions are identical.

Let us check that each  
component has a finite number  
of solutions.

$$[\text{coHeight}(\text{ts}) \text{ for } \text{ts} \text{ in } \text{1ts2}]$$

$$[0, 0, 0, 0] \quad (25)$$

Type: List NonNegativeInteger

Let us count the degrees of each  
component,

$$\text{degrees} := [\text{degree}(\text{ts}) \text{ for } \text{ts} \text{ in } \text{1ts2}]$$

$$[8, 16, 4, 4] \quad (26)$$

Type: List NonNegativeInteger

and compute their sum.

$$\text{reduce}(+, \text{degrees})$$

$$32 \quad (27)$$

Type: PositiveInteger

We study now the options of the **zeroSetSplit** operation. As we have seen yet, there is an optional second argument which is a boolean value. If this value is true (this is the default) then the decomposition is computed in the sense of Kalkbrener, otherwise it is computed in the sense of Lazard.

There is a second boolean optional argument that can be used (in that case the first optional argument must be present). This second option allows you to get some information during the computations.

Therefore, we need to understand a little what is going on during the computations. An important feature of the algorithm is that the intermediate computations are managed in some sense like the processes of a Unix system. Indeed, each intermediate computation may generate other intermediate computations and the management of all these computations is a crucial task for the efficiency. Thus any intermediate computation may be suspended, killed or resumed, depending on algebraic considerations that determine priorities for these processes. The goal is of course to go as fast as possible towards the final decomposition which means to avoid as much as possible unnecessary computations.

To follow the computations, one needs to set to **true** the second argument.

Then a lot of numbers and letters are displayed. Between a [ and a ] one has the state of the processes at a given time. Just after [ one can see the number of processes. Then each process is represented by two numbers between < and >. A process consists of a list of polynomial **ps** and a triangular set **ts**; its goal is to compute the common zeros of **ps** that belong to the regular zeros set of **ts**. After the processes, the number between pipes gives the total number of polynomials in all the sets **ps**. Finally, the number between braces gives the number of components of a decomposition that are already computed. This number may decrease.

Let us take a third example (Czapor-Geddes-Wang) to see how these informations are displayed.

Define a polynomial system.

u : R := 2

2

(28)

Type: Integer

$$\begin{aligned} q1 := & 2*(u-1)**2 + 2*(x-z*x+z**2) + y**2*(x-1)**2 - 2*u*x + \\ & 2*y*t*(1-x)*(x-z) + 2*u*z*t*(t-y) + u**2*t**2*(1-2*z) + \\ & 2*u*t**2*(z-x) + 2*u*t*y*(z-1) + 2*u*z*x*(y+1) + (u**2 - \\ & 2*u)*z**2*t**2 + 2*u**2*z**2 + 4*u*(1-u)*z + t**2*(z-x)**2 \\ & (y^2 - 2 t y + t^2) x^2 + \\ & (-2 y^2 + ((2 t + 4) z + 2 t) y + (-2 t^2 + 2) z - 4 t^2 - 2) x + \\ & y^2 + (-2 t z - 4 t) y + (t^2 + 10) z^2 - 8 z + 4 t^2 + 2 \end{aligned} \quad (29)$$

Type: NewSparseMultivariatePolynomial(Integer, OrderedVariableList [x, y, z, t])

$$\begin{aligned} q2 := & t*(2*z+1)*(x-z) + y*(z+2)*(1-x) + u*(u-2)*t + u*(1- \\ & 2*u)*z*t + u*y*(x+u-z*x-1) + u*(u+1)*z**2*t \\ & (-3 z y + 2 t z + t) x + (z + 4) y + 4 t z^2 - 7 t z \end{aligned} \quad (30)$$

Type: NewSparseMultivariatePolynomial(Integer, OrderedVariableList [x, y, z, t])

$$\begin{aligned} q3 := & -u**2*(z-1)**2 + 2*z*(z-x) - 2*(x-1) \\ & (-2 z - 2) x - 2 z^2 + 8 z - 2 \end{aligned} \quad (31)$$

Type: NewSparseMultivariatePolynomial(Integer, OrderedVariableList [x, y, z, t])

$$\begin{aligned} q4 := & u**2 + 4*(z-x**2) + 3*y**2*(x-1)**2 - 3*t**2*(z-x)**2 \\ & + 3*u**2*t**2*(z-1)**2 + u**2*z*(z-2) + 6*u*t*y*(z+x+z*x-1) \\ & (3 y^2 - 3 t^2 - 4) x^2 + (-6 y^2 + (12 t z + 12 t) y + 6 t^2 z) x + \\ & 3 y^2 + (12 t z - 12 t) y + (9 t^2 + 4) z^2 + (-24 t^2 - 4) z + \\ & 12 t^2 + 4 \end{aligned} \quad (32)$$

Type: NewSparseMultivariatePolynomial(Integer, OrderedVariableList [x, y, z, t])



$$lq := [q1, q2, q3, q4]$$

$$\left[ \begin{array}{l} (y^2 - 2 t y + t^2) x^2 + (-2 y^2 + ((2 t + 4) z + 2 t) y + \\ (-2 t^2 + 2) z - 4 t^2 - 2) x + y^2 + (-2 t z - 4 t) y + \\ (t^2 + 10) z^2 - 8 z + 4 t^2 + 2, \\ (-3 z y + 2 t z + t) x + (z + 4) y + 4 t z^2 - 7 t z, \\ (-2 z - 2) x - 2 z^2 + 8 z - 2, \\ (3 y^2 - 3 t^2 - 4) x^2 + (-6 y^2 + (12 t z + 12 t) y + 6 t^2 z) x \\ + 3 y^2 + (12 t z - 12 t) y + (9 t^2 + 4) z^2 + (-24 t^2 - 4) z + \\ 12 t^2 + 4 \end{array} \right] \quad (33)$$

Type: List NewSparseMultivariatePolynomial(Integer, OrderedVariableList [x, y, z, t])

Let us try the information option. N.B. The timing should be between 1 and 10 minutes, depending on your machine.

zeroSetSplit(lq,true,true)\$T

\*\*\* QCMPACK Statistics \*\*\*

Table size: 36  
Entries reused: 255

\*\*\* REGSETGCD: Gcd Statistics \*\*\*

Table size: 125  
Entries reused: 0

\*\*\* REGSETGCD: Inv Set Statistics \*\*\*

Table size: 30  
Entries reused: 0

[{ 960725655771966 t<sup>24</sup> + 386820897948702 t<sup>23</sup> + 8906817198608181 t<sup>22</sup> + 27049668  
(26604210869491302385515265737052082361668474181372891857784 t<sup>23</sup> + 4431043  
(3 z<sup>3</sup> - 11 z<sup>2</sup> + 8 z + 4) y + 2 t z<sup>3</sup> + 4 t z<sup>2</sup> - 5 t z - t,  
(z + 1) x + z<sup>2</sup> - 4 z + 1 }]

Type: List RegularTriangularSet(Integer, IndexedExponents OrderedVariableList [x, y, z, t], OrderedVariableList [x, y, z, t], NewSparseMultivariatePolynomial(Integer, OrderedVariableList [x, y, z, t]))

Between a sequence of processes, thus between a ] and a [ you can see capital letters W, G, I and lower case letters i, w. Each time a capital letter appears a non-trivial computation has be performed and its result is put in a hash-table. Each time a lower case letter appears a needed result has been found in an hash-table. The use of these hash-tables generally speed up the computations. However, on very large systems, it may happen that these hash-tables become too big to be handle by your AXIOM configuration. Then in these exceptional cases, you may prefer getting a result (even if it takes a long time) than getting nothing. Hence you need to know how to prevent the RSEGSET constructor from using

these hash-tables. In that case you will be using the **zeroSetSplit** with five arguments. The first one is the input system **lp** as above. The second one is a boolean value **hash?** which is **true** iff you want to use hash-tables. The third one is boolean value **clos?** which is **true** iff you want to solve your system in the sense of Kalkbrener, the other way remaining that of Lazard. The fourth argument is boolean value **info?** which is **true** iff you want to display information during the computations. The last one is boolean value **prep?** which is **true** iff you want to use some heuristics that are performed on the input system before starting the real algorithm. The value of this flag is **true** when you are using **zeroSetSplit** with less than five arguments. Note that there is no available signature for **zeroSetSplit** with four arguments.

We finish this section by some remarks about both ways of solving, in the sense of Kalkbrener or in the sense of Lazard. For problems with a finite number of solutions, there are theoretically equivalent and the resulting decompositions are identical, up to the ordering of the components. However, when solving in the sense of Lazard, the algorithm behaves differently. In that case, it becomes more incremental than in the sense of Kalkbrener. That means the polynomials of the input system are considered one after another whereas in the sense of Kalkbrener the input system is treated more globally.

This makes an important difference in positive dimension. Indeed when solving in the sense of Kalkbrener, the *Primeidealkettensatz* of Krull is used. That means any regular triangular containing more polynomials than the input system can be deleted. This is not possible when solving in the sense of Lazard. This explains why Kalkbrener's decompositions usually contain less components than those of Lazard. However, it may happen with some examples that the incremental process (that cannot be used when solving in the sense of Kalkbrener) provide a more efficient way of solving than the global one even if the *Primeidealkettensatz* is used. Thus just try both, with the various options, before concluding that you cannot solve your favorite system with **zeroSetSplit**. There exist more options at the development level that are not currently available in this public version. So you are welcome to contact *marc@nag.co.uk* for more information and help.

## 9.63 RomanNumeral

For example, let  $f$  be a symbolic operator.

The Roman numeral package was added to AXIOM in MCMLXXXVI for use in denoting higher order derivatives.

```
f := operator 'f
f
```

(1)

Type: BasicOperator

This is the seventh derivative of  $f$  with respect to  $x$ .

```
D(f x,x,7)
f(vii)(x)
```

(2)

Type: Expression Integer

You can have integers printed as Roman numerals by declaring variables to be of type RomanNumeral (abbreviation ROMAN).

```
a := roman(1978 - 1965)
XIII
```

(3)

Type: RomanNumeral

This package now has a small but devoted group of followers that claim this domain has shown its efficacy in many other contexts. They claim that Roman numerals are every bit as useful as ordinary integers.

In a sense, they are correct, because Roman numerals form a ring and you can therefore construct polynomials with Roman numeral coefficients, matrices over Roman numerals, etc..

```
x : UTS(ROMAN,'x,0) := x
x
```

(4)

Type: UnivariateTaylorSeries(RomanNumeral, x, 0)

Was Fibonacci Italian or ROMAN?

```
recip(1 - x - x**2)
I + x + II x2 + III x3 + V x4 + VIII x5 + XIII x6 + XXI x7 + O(x8)
```

(5)

Type: Union(UnivariateTaylorSeries(RomanNumeral, x, 0), ...)

You can also construct fractions with Roman numeral numerators and denominators, as this matrix Hilberticus illustrates.

```
m : MATRIX FRAC ROMAN
```

Type: Void

```
m := matrix [[1/(i + j) for i in 1..3] for j in 1..3]
[ [ I/I, I/II, I/III ]
  [ I/II, II/II, II/III ]
  [ I/III, II/III, III/III ] ]
```

(7)

Type: Matrix Fraction RomanNumeral

Note that the inverse of the matrix has integral ROME entries.

inverse m

$$\begin{bmatrix} \text{LXXII} & -\text{CCXL} & \text{CLXXX} \\ -\text{CCXL} & CM & -\text{DCCXX} \\ \text{CLXXX} & -\text{DCCXX} & DC \end{bmatrix} \quad (8)$$

Type: Union(Matrix Fraction RomanNumeral, ...)

Unfortunately, the spoil-sports say that the fun stops when the numbers get big—mostly because the Romans didn't establish conventions about representing very large numbers.

```
y := factorial 10
```

$$3628800 \tag{9}$$

Type: PositiveInteger

You work it out!

roman y

$$\begin{array}{l} (((((I)))) (((((I)))) (((((I)))) (((((I)))) (((((I)))) (((((I)))) (((((I)))) (((((I)))) (((((I)))) (((((I)))) \\ ((I)) ((I)) \text{MMMMMMMMMDCCC} \end{array} \quad (10)$$

Type: RomanNumeral

Issue the system command `)show RomanNumeral` to display the full list of operations defined by RomanNumeral.

## 9.64 Segment

The Segment domain provides a generalized interval type.

Segments are created using the “..” construct by indicating the (included) end points.

```
s := 3..10
```

(1)

Type: Segment PositiveInteger

The first end point is called the **lo** and the second is called **hi**.

```
lo s
3
```

(2)

Type: PositiveInteger

These names are used even though the end points might belong to an unordered set.

```
hi s
10
```

(3)

Type: PositiveInteger

In addition to the end points, each segment has an integer “increment.” An increment can be specified using the “by” construct.

```
t := 10..3 by -2
10..3 by -2
```

(4)

Type: Segment PositiveInteger

This part can be obtained using the **incr** function.

```
incr s
1
```

(5)

Type: PositiveInteger

Unless otherwise specified, the increment is 1.

```
incr t
-2
```

(6)

Type: Integer

A single value can be converted to a segment with equal end points. This happens if segments and single values are mixed in a list.

```
l := [1..3, 5, 9, 15..11 by -1]
[1..3, 5..5, 9..9, 15..11 by -1]
```

(7)

Type: List Segment PositiveInteger

If the underlying type is an ordered ring, it is possible to perform additional operations. The **expand** operation creates a list of points in a segment.

```
expand s
[3, 4, 5, 6, 7, 8, 9, 10]
```

(8)

Type: List Integer

If  $k > 0$ , then **expand**( $l..h$  by  $k$ ) creates the list  $[l, l+k, \dots, h]$  where  $lN \leq h < lN+k$ . If  $k < 0$ , then  $lN \geq h > lN+k$ .

```
expand t
[10, 8, 6, 4]
```

(9)

Type: List Integer

It is also possible to expand a list of segments. This is equivalent to appending lists obtained by expanding each segment individually.

```
expand 1
```

```
[1, 2, 3, 5, 9, 15, 14, 13, 12, 11]
```

(10)

Type: List Integer

For more information on related topics, see ‘SegmentBinding’ on page 561 and ‘UniversalSegment’ on page 599. Issue the system command `)show Segment` to display the full list of operations defined by Segment.

## 9.65 SegmentBinding

First give the symbol, then an “=” and finally a segment of values.

This is used to provide a convenient syntax for arguments to certain operations.

The **draw** operation uses a SegmentBinding argument as a range of coordinates. This is an example of a two-dimensional parametrized plot; other **draw** options use more than one SegmentBinding argument.

The SegmentBinding type is used to indicate a range for a named symbol.

`x = a..b`

$x = a..b$

(1)

Type: SegmentBinding Symbol

`sum(i**2, i = 0..n)`

$$\frac{2n^3 + 3n^2 + n}{6}$$

(2)

Type: Fraction Polynomial Integer

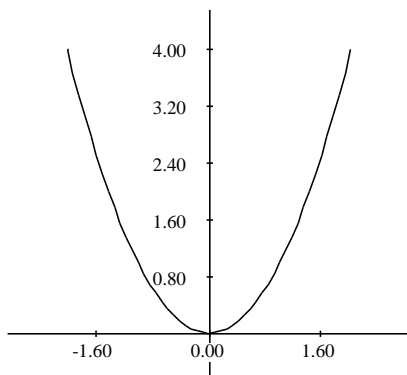
`draw(x**2, x = -2..2)`

Compiling function %B with type DoubleFloat -> Double-Float  
Graph data being transmitted to the viewport manager...  
AXIOM2D data being transmitted to the viewport manager...

TwoDimensionalViewport: "x\*x"

(3)

Type: TwoDimensionalViewport



The left-hand side must be of type Symbol but the right-hand side can be a segment over any type.

`sb := y = 1/2..3/2`

$y = \left(\frac{1}{2}\right)..\left(\frac{3}{2}\right)$

(4)

Type: SegmentBinding Fraction Integer

The left- and right-hand sides can be obtained using the **variable** and **segment** operations.

`variable(sb)`

`y`

(5)

Type: Symbol

`segment (sb)`

$$\left(\frac{1}{2}\right)..\left(\frac{3}{2}\right) \quad (6)$$

Type: Segment Fraction Integer

For more information on related topics, see ‘Segment’ on page 559 and ‘UniversalSegment’ on page 599. Issue the system command `)show SegmentBinding` to display the full list of operations defined by SegmentBinding.



## 9.66 Set

Sets can be created by giving a fixed set of values ...

The Set domain allows one to represent explicit finite sets of values. These are similar to lists, but duplicate elements are not allowed.

```
s := set [x**2-1, y**2-1, z**2-1]
{ x^2 - 1, y^2 - 1, z^2 - 1 }
```

(1)

Type: Set Polynomial Integer

or by using a collect form, just as for lists. In either case, the set is formed from a finite collection of values.

```
t := set [x**i - i+1 for i in 2..10 | prime? i]
{ x^2 - 1, x^3 - 2, x^5 - 4, x^7 - 6 }
```

(2)

Type: Set Polynomial Integer

The basic operations on sets are **intersect**, **union**, **difference**, and **symmetricDifference**.

```
i := intersect(s,t)
{ x^2 - 1 }
```

(3)

Type: Set Polynomial Integer

```
u := union(s,t)
{ x^2 - 1, x^3 - 2, x^5 - 4, x^7 - 6, y^2 - 1, z^2 - 1 }
```

(4)

Type: Set Polynomial Integer

The set **difference(s,t)** contains those members of **s** which are not in **t**.

```
difference(s,t)
{ y^2 - 1, z^2 - 1 }
```

(5)

Type: Set Polynomial Integer

The set **symmetricDifference(s,t)** contains those elements which are in **s** or **t** but not in both.

```
symmetricDifference(s,t)
{ x^3 - 2, x^5 - 4, x^7 - 6, y^2 - 1, z^2 - 1 }
```

(6)

Type: Set Polynomial Integer

Set membership is tested using the **member?** operation.

```
member?(y, s)
false
```

(7)

Type: Boolean

```
member?((y+1)*(y-1), s)
true
```

(8)

Type: Boolean

The **subset?** function determines whether one set is a subset of another.

```
subset?(i, s)
true
```

(9)

Type: Boolean

```
subset?(u, s)
false
```

(10)

Type: Boolean

When the base type is finite, the absolute complement of a set is defined. This finds the set of all multiplicative generators of PrimeField 11—the integers mod 11.

```
gs := set [g for i in 1..11 | primitive?(g := i::PF 11)]
{2, 6, 7, 8}
(11)
Type: Set PrimeField 11
```

The following values are not generators.

```
complement gs
{1, 3, 4, 5, 9, 10, 0}
(12)
Type: Set PrimeField 11
```

Often the members of a set are computed individually; in addition, values can be inserted or removed from a set over the course of a computation.

There are two ways to do this:

```
a := set [i**2 for i in 1..5]
{1, 4, 9, 16, 25}
(13)
Type: Set PositiveInteger
```

One is to view a set as a data structure and to apply updating operations.

```
insert!(32, a)
{1, 4, 9, 16, 25, 32}
(14)
Type: Set PositiveInteger
```

```
remove!(25, a)
{1, 4, 9, 16, 32}
(15)
Type: Set PositiveInteger
```

```
a
{1, 4, 9, 16, 32}
(16)
Type: Set PositiveInteger
```

The other way is to view a set as a mathematical entity and to create new sets from old.

```
b := b0 := set [i**2 for i in 1..5]
{1, 4, 9, 16, 25}
(17)
Type: Set PositiveInteger
```

```
b := union(b, {32})
{1, 4, 9, 16, 25, 32}
(18)
Type: Set PositiveInteger
```

```
b := difference(b, {25})
{1, 4, 9, 16, 32}
(19)
Type: Set PositiveInteger
```

```
b0
{1, 4, 9, 16, 25}
(20)
Type: Set PositiveInteger
```

For more information about lists, see ‘List’ on page 489. Issue the system command `)show Set` to display the full list of operations defined by Set.

## 9.67 SingleInteger

The SingleInteger domain is intended to provide support in AXIOM for machine integer arithmetic. It is generally much faster than (bignum) Integer arithmetic but suffers from a limited range of values. Since AXIOM can be implemented on top of various dialects of Lisp, the actual representation of small integers may not correspond exactly to the host machines integer representation.

In the CCL implementation of AXIOM (Release 2.1 onwards) the underlying representation of SingleInteger is the same as Integer. The underlying Lisp primitives treat machine-word sized computations specially.

You can discover the minimum and maximum values in your implementation by using **min** and **max**.

```
min()$SingleInteger
-134217728
```

(1)  
Type: SingleInteger

```
max()$SingleInteger
134217727
```

(2)  
Type: SingleInteger

To avoid confusion with Integer, which is the default type for integers, you usually need to work with declared variables (Section 2.3 on page 103) ...

```
a := 1234 :: SingleInteger
1234
```

(3)  
Type: SingleInteger

or use package calling (Section 2.9 on page 119).

```
b := 124$SingleInteger
124
```

(4)  
Type: SingleInteger

You can add, multiply and subtract SingleInteger objects, and ask for the greatest common divisor (**gcd**).

```
gcd(a,b)
2
```

(5)  
Type: SingleInteger

The least common multiple (**lcm**) is also available.

```
lcm(a,b)
76508
```

(6)  
Type: SingleInteger

Operations **mulmod**, **addmod**, **submod**, and **invmod** are similar—they provide arithmetic modulo a given small integer. Here is  $5 * 6 \bmod 13$ .

```
mulmod(5,6,13)$SingleInteger
4
```

(7)  
Type: SingleInteger

To reduce a small integer modulo a prime, use **positiveRemainder**.

```
positiveRemainder(37,13)$SingleInteger
11
```

(8)  
Type: SingleInteger

Operations **And**, **Or**, **xor**, and **Not** provide bit level operations on small integers.

```
And(3,4)$SingleInteger
```

```
0
```

(9)

Type: SingleInteger

Use `shift(int,numToShift)` to shift bits, where `i` is shifted left if `numToShift` is positive, right if negative.

```
shift(1,4)$SingleInteger
```

```
16
```

(10)

Type: SingleInteger

```
shift(31,-1)$SingleInteger
```

```
15
```

(11)

Type: SingleInteger

Many other operations are available for small integers, including many of those provided for `Integer`. To see the other operations, use the Browse HyperDoc facility (Section 14 on page 699). Issue the system command `)show SingleInteger` to display the full list of operations defined by `SingleInteger`..

## 9.68 SparseTable

Here we create a table to save strings under integer keys. The value "Try again!" is returned if no other value has been stored for a key.

Entries can be stored in the table.

These values can be retrieved as usual, but if a look up fails the default entry will be returned.

To see which values are explicitly stored, the **keys** and **entries** functions can be used.

The SparseTable domain provides a general purpose table type with default entries.

```
t: SparseTable(Integer, String, "Try again!") := table()
table()
Type: SparseTable(Integer, String, Try again!)
(1)
```

```
t.3 := "Number three"
"Number three"
Type: String
(2)
```

```
t.4 := "Number four"
"Number four"
Type: String
(3)
```

```
t.3
"Number three"
Type: String
(4)
```

```
t.2
"Try again!"
Type: String
(5)
```

```
keys t
[4, 3]
Type: List Integer
(6)
```

```
entries t
["Number four", "Number three"]
Type: List String
(7)
```

If a specific table representation is required, the GeneralSparseTable constructor should be used. The domain SparseTable(K, E, dflt) is equivalent to GeneralSparseTable(K,E, Table(K,E), dflt). For more information, see 'Table' on page 585 and 'GeneralSparseTable' on page 439. Issue the system command `)show SparseTable` to display the full list of operations defined by SparseTable.

## 9.69 SquareMatrix

The top level matrix type in AXIOM is `Matrix` (see ‘`Matrix`’ on page 500), which provides basic arithmetic and linear algebra functions. However, since the matrices can be of any size it is not true that any pair can be added or multiplied. Thus `Matrix` has little algebraic structure.

Sometimes you want to use matrices as coefficients for polynomials or in other algebraic contexts. In this case, `SquareMatrix` should be used. The domain `SquareMatrix(n,R)` gives the ring of `n` by `n` square matrices over `R`.

Since `SquareMatrix` is not normally exposed at the top level, you must expose it before it can be used.

```
)set expose add constructor SquareMatrix
```

`SquareMatrix` is now explicitly exposed in frame

Once `SQMATRIX` has been exposed, values can be created using the `squareMatrix` function.

```
m := squareMatrix [[1, -%i], [%i, 4]]
```

$$\begin{bmatrix} 1 & -i \\ i & 4 \end{bmatrix} \quad (1)$$

Type: `SquareMatrix(2, Complex Integer)`

The usual arithmetic operations are available.

```
m*m - m
```

$$\begin{bmatrix} 1 & -4i \\ 4i & 13 \end{bmatrix} \quad (2)$$

Type: `SquareMatrix(2, Complex Integer)`

Square matrices can be used where ring elements are required. For example, here is a matrix with matrix entries.

```
mm := squareMatrix [[m, 1], [1-m, m**2]]
```

$$\begin{bmatrix} \begin{bmatrix} 1 & -i \\ i & 4 \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \begin{bmatrix} 0 & i \\ -i & -3 \end{bmatrix} & \begin{bmatrix} 2 & -5i \\ 5i & 17 \end{bmatrix} \end{bmatrix} \quad (3)$$

Type: `SquareMatrix(2, SquareMatrix(2, Complex Integer))`

Or you can construct a polynomial with square matrix coefficients.

```
p := (x + m)**2
```

$$x^2 + \begin{bmatrix} 2 & -2i \\ 2i & 8 \end{bmatrix} x + \begin{bmatrix} 2 & -5i \\ 5i & 17 \end{bmatrix} \quad (4)$$

Type: `Polynomial SquareMatrix(2, Complex Integer)`

This value can be converted to a square matrix with polynomial coefficients.

```
p::SquareMatrix(2, ?)
```

$$\begin{bmatrix} x^2 + 2x + 2 & -2ix - 5i \\ 2ix + 5i & x^2 + 8x + 17 \end{bmatrix} \quad (5)$$

Type: `SquareMatrix(2, Polynomial Complex Integer)`

For more information on related topics, see Section 2.2.4 on page 100, Section 2.11 on page 124, and ‘`Matrix`’ on page 500. Issue the system command `)show SquareMatrix` to display the full list of operations defined by `SquareMatrix`.

## 9.70 SquareFree- Regular- TriangularSet

---

The SquareFreeRegularTriangularSet domain constructor implements square-free regular triangular sets. See the RegularTriangularSet domain constructor for general regular triangular sets. Let  $T$  be a regular triangular set consisting of polynomials  $t_1, \dots, t_m$  ordered by increasing main variables. The regular triangular set  $T$  is square-free if  $T$  is empty or if the polynomial  $t_m$  is square-free as a univariate polynomial with coefficients in the tower of simple extensions associated with  $t_1, \dots, t_{m-1}$ .

The main interest of square-free regular triangular sets is that their associated towers of simple extensions are product of fields. Consequently, the saturated ideal of a square-free regular triangular set is radical. This property simplifies some of the operations related to regular triangular sets. However, building square-free regular triangular sets is generally more expensive than building general regular triangular sets.

As the RegularTriangularSet domain constructor, the SquareFreeRegularTriangularSet domain constructor also implements a method for solving polynomial systems by means of regular triangular sets. This is in fact the same method with some adaptations to take into account the fact that the computed regular chains are square-free. Note that it is also possible to pass from a decomposition into general regular triangular sets to a decomposition into square-free regular triangular sets. This conversion is used internally in the LazardSetSolvingPackage package constructor.

**N.B.** When solving polynomial systems with the SquareFreeRegularTriangularSet domain constructor or the LazardSetSolvingPackage package constructor, decompositions have no redundant components. See also the ZeroDimensionalSolvePackage for the case of algebraic systems with a finite number of (complex) solutions.

We shall explain now how to use the constructor SquareFreeRegularTriangularSet.

This constructor takes four arguments. The first one,  $\mathbf{R}$ , is the coefficient ring of the polynomials; it must belong to the category GcdDomain. The second one,  $\mathbf{E}$ , is the exponent monoid of the polynomials; it must belong to the category OrderedAbelianMonoidSup. the third one,  $\mathbf{V}$ , is the ordered set of variables; it must belong to the category OrderedSet. The last one is the polynomial ring; it must belong to the category RecursivePolynomialCategory( $\mathbf{R}, \mathbf{E}, \mathbf{V}$ ). The abbreviation for SquareFreeRegularTriangularSet is SREGSET.

Let us illustrate the use of this constructor with one example (Donati-Traverso). Define the coefficient ring.

```
R := Integer
```

```
Integer
```

(1)

Type: Domain



Define the list of variables,  $ls : \text{List Symbol} := [x, y, z, t]$  (2)  
 $[x, y, z, t]$  Type: List Symbol

and make it an ordered set;  $V := \text{OVAR}(ls)$  (3)  
 $\text{OrderedVariableList } [x, y, z, t]$  Type: Domain

then define the exponent monoid.  $E := \text{IndexedExponents } V$  (4)  
 $\text{IndexedExponents OrderedVariableList } [x, y, z, t]$  Type: Domain

Define the polynomial ring.  $P := \text{NSMP}(R, V)$  (5)  
 $\text{NewSparseMultivariatePolynomial (Integer, OrderedVariableList } [x, y, z, t] \text{)}$  Type: Domain

Let the variables be polynomial.  $x : P := 'x$  (6)  
 $x$  Type: NewSparseMultivariatePolynomial(Integer, OrderedVariableList [x, y, z, t])

$y : P := 'y$  (7)  
 $y$  Type: NewSparseMultivariatePolynomial(Integer, OrderedVariableList [x, y, z, t])

$z : P := 'z$  (8)  
 $z$  Type: NewSparseMultivariatePolynomial(Integer, OrderedVariableList [x, y, z, t])

$t : P := 't$  (9)  
 $t$  Type: NewSparseMultivariatePolynomial(Integer, OrderedVariableList [x, y, z, t])

Now call the  $\text{SquareFreeRegularTriangularSet}$  domain constructor.  $ST := \text{SREGSET}(R, E, V, P)$  (10)  
 $\text{SquareFreeRegularTriangularSet ( Integer, IndexedExponents OrderedVariableList } [x, y, z, t], \text{ OrderedVariableList } [x, y, z, t], \text{ NewSparseMultivariatePolynomial ( Integer, OrderedVariableList } [x, y, z, t] \text{) )}$  Type: Domain

Define a polynomial system.  $p1 := x^{31} - x^6 - x - y$  (11)  
 $x^{31} - x^6 - x - y$  Type: NewSparseMultivariatePolynomial(Integer, OrderedVariableList [x, y, z, t])

$$p2 := x^{**8} - z$$

$$x^8 - z \quad (12)$$

Type: NewSparseMultivariatePolynomial(Integer, OrderedVariableList [x, y, z, t])

$$p3 := x^{**10} - t$$

$$x^{10} - t \quad (13)$$

Type: NewSparseMultivariatePolynomial(Integer, OrderedVariableList [x, y, z, t])

$$lp := [p1, p2, p3]$$

$$\left[ x^{31} - x^6 - x - y, x^8 - z, x^{10} - t \right] \quad (14)$$

Type: List NewSparseMultivariatePolynomial(Integer, OrderedVariableList [x, y, z, t])

First of all, let us solve this system in the sense of Kalkbrener.

$$\text{zeroSetSplit}(lp)\$ST$$

$$\left\{ z^5 - t^4, t z y^2 + 2 z^3 y - t^8 + 2 t^5 + t^3 - t^2, (t^4 - t) x - t y - z^2 \right\} \quad (15)$$

Type: List SquareFreeRegularTriangularSet(Integer, IndexedExponents OrderedVariableList [x, y, z, t], OrderedVariableList [x, y, z, t], NewSparseMultivariatePolynomial(Integer, OrderedVariableList [x, y, z, t]))

And now in the sense of Lazard (or Wu and other authors).

$$\text{zeroSetSplit}(lp, \text{false})\$ST$$

$$\left\{ z^5 - t^4, t z y^2 + 2 z^3 y - t^8 + 2 t^5 + t^3 - t^2, (t^4 - t) x - t y - z^2 \right\}, \quad (16)$$

$$\left\{ t^3 - 1, z^5 - t, t y + z^2, z x^2 - t \right\}, \{t, z, y, x\}$$

Type: List SquareFreeRegularTriangularSet(Integer, IndexedExponents OrderedVariableList [x, y, z, t], OrderedVariableList [x, y, z, t], NewSparseMultivariatePolynomial(Integer, OrderedVariableList [x, y, z, t]))

Now to see the difference with the RegularTriangularSet domain constructor,

we define:

$$T := \text{REGSET}(R, E, V, P)$$

$$\text{RegularTriangularSet}(\text{Integer}, \text{IndexedExponents OrderedVariableList [x, y, z, t]}, \text{OrderedVariableList [x, y, z, t]}, \text{NewSparseMultivariatePolynomial}(\text{Integer}, \text{OrderedVariableList [x, y, z, t]})) \quad (17)$$

Type: Domain

and compute:

$$\begin{aligned} \text{lhs} &:= \text{zeroSetSplit}(\text{lp}, \text{false})\$T \\ &\left[ \left\{ z^5 - t^4, t z y^2 + 2 z^3 y - t^8 + 2 t^5 + t^3 - t^2, (t^4 - t) x - t y - z^2 \right\}, \right. \\ &\left. \left\{ t^3 - 1, z^5 - t, t z y^2 + 2 z^3 y + 1, z x^2 - t \right\}, \{t, z, y, x\} \right] \end{aligned} \quad (18)$$

Type: List RegularTriangularSet(Integer, IndexedExponents OrderedVariableList [x, y, z, t], OrderedVariableList [x, y, z, t], NewSparseMultivariatePolynomial(Integer, OrderedVariableList [x, y, z, t]))

If you look at the second set in both decompositions in the sense of Lazard, you will see that the polynomial with main variable **y** is not the same.

Let us understand what has happened.

We define:

$$\text{ts} := \text{lhs}.2 \quad \left\{ t^3 - 1, z^5 - t, t z y^2 + 2 z^3 y + 1, z x^2 - t \right\} \quad (19)$$

Type: RegularTriangularSet(Integer, IndexedExponents OrderedVariableList [x, y, z, t], OrderedVariableList [x, y, z, t], NewSparseMultivariatePolynomial(Integer, OrderedVariableList [x, y, z, t]))

$$\text{pol} := \text{select}(\text{ts}, 'y')\$T \quad t z y^2 + 2 z^3 y + 1 \quad (20)$$

Type: Union(NewSparseMultivariatePolynomial(Integer, OrderedVariableList [x, y, z, t]), ...)

$$\text{tower} := \text{collectUnder}(\text{ts}, 'y')\$T \quad \left\{ t^3 - 1, z^5 - t \right\} \quad (21)$$

Type: RegularTriangularSet(Integer, IndexedExponents OrderedVariableList [x, y, z, t], OrderedVariableList [x, y, z, t], NewSparseMultivariatePolynomial(Integer, OrderedVariableList [x, y, z, t]))

$$\begin{aligned} \text{pack} &:= \text{RegularTriangularSetGcdPackage}(\text{R}, \text{E}, \text{V}, \text{P}, \text{T}) \\ &\text{RegularTriangularSetGcdPackage} ( \text{Integer} , \text{IndexedExponents} \\ &\text{OrderedVariableList} [ \text{x} , \text{y} , \text{z} , \text{t} ] , \text{OrderedVariableList} [ \text{x} , \text{y} , \text{z} , \text{t} ] , \\ &\text{NewSparseMultivariatePolynomial} ( \text{Integer} , \text{OrderedVariableList} [ \text{x} , \text{y} , \text{z} , \text{t} ] ) , \text{RegularTriangularSet} ( \text{Integer} , \text{IndexedExponents} \\ &\text{OrderedVariableList} [ \text{x} , \text{y} , \text{z} , \text{t} ] , \text{OrderedVariableList} [ \text{x} , \text{y} , \text{z} , \text{t} ] , \\ &\text{NewSparseMultivariatePolynomial} ( \text{Integer} , \text{OrderedVariableList} [ \text{x} , \text{y} , \text{z} , \text{t} ] ) ) \end{aligned} \quad (22)$$

Type: Domain

Then we compute:

$$\begin{aligned} & \text{toseSquareFreePart}(\text{pol}, \text{tower}) \$\text{pack} \\ & \left[ \left[ \text{val} = t y + z^2, \text{tower} = \{t^3 - 1, z^5 - t\} \right] \right] \end{aligned} \quad (23)$$

Type: List Record(val: NewSparseMultivariatePolynomial(Integer, OrderedVariableList [x, y, z, t]), tower: RegularTriangularSet(Integer, IndexedExponents OrderedVariableList [x, y, z, t], OrderedVariableList [x, y, z, t], NewSparseMultivariatePolynomial(Integer, OrderedVariableList [x, y, z, t])))

## 9.71 Stream

Let `ints` be the infinite stream of non-negative integers.

A Stream object is represented as a list whose last element contains the wherewithal to create the next element, should it ever be required.

```
ints := [i for i in 0..]
[0, 1, 2, 3, 4, 5, 6, ...] (1)
```

Type: Stream NonNegativeInteger

By default, ten stream elements are calculated. This number may be changed to something else by the system command `)set streams calculate`. For the display purposes of this book, we have chosen a smaller value.

More generally, you can construct a stream by specifying its initial value and a function which, when given an element, creates the next element.

```
f : List INT -> List INT
Type: Void
```

```
f x == [x.1 + x.2, x.1]
```

Type: Void

```
fibs := [i.2 for i in [generate(f,[1,1])]]
Compiling function f with type List Integer -> List
Integer
```

```
[1, 1, 2, 3, 5, 8, 13, ...] (4)
```

Type: Stream Integer

You can create the stream of odd non-negative integers by either filtering them from the integers, or by evaluating an expression for each integer.

```
[i for i in ints | odd? i]
[1, 3, 5, 7, 9, 11, 13, ...] (5)
```

Type: Stream NonNegativeInteger

```
odds := [2*i+1 for i in ints]
[1, 3, 5, 7, 9, 11, 13, ...] (6)
```

Type: Stream NonNegativeInteger

You can accumulate the initial segments of a stream using the `scan` operation.

```
scan(0,+,odds)
[1, 4, 9, 16, 25, 36, 49, ...] (7)
```

Type: Stream NonNegativeInteger

The corresponding elements of two or more streams can be combined in this way.

```
[i*j for i in ints for j in odds]
[0, 3, 10, 21, 36, 55, 78, ...] (8)
```

Type: Stream NonNegativeInteger

Many operations similar to those applicable to lists are available for streams.

```
map(*,ints,odds)
[0, 3, 10, 21, 36, 55, 78, ...]
```

(9)

Type: Stream NonNegativeInteger

```
first ints
0
```

(10)

Type: NonNegativeInteger

```
rest ints
[1, 2, 3, 4, 5, 6, 7, ...]
```

(11)

Type: Stream NonNegativeInteger

```
fib 20
6765
```

(12)

Type: PositiveInteger

The packages `StreamFunctions1`, `StreamFunctions2` and `StreamFunctions3` export some useful stream manipulation operations. For more information, see Section 5.5 on page 171, Section 8.9 on page 295, ‘`ContinuedFraction`’ on page 385, and ‘`List`’ on page 489. Issue the system command `)show Stream` to display the full list of operations defined by `Stream`.

## 9.72 String

The type `String` provides character strings. Character strings provide all the operations for a one-dimensional array of characters, plus additional operations for manipulating text. For more information on related topics, see ‘`Character`’ on page 374 and ‘`CharacterClass`’ on page 376. You can also issue the system command `)show String` to display the full list of operations defined by `String`.

String values can be created using double quotes.

```
hello := "Hello, I'm AXIOM!"  
"Hello, I'm AXIOM!"
```

(1)

Type: String

Note, however, that double quotes and underscores must be preceded by an extra underscore.

```
said := "Jane said, _"Look!_"  
"Jane said, " Look ! "
```

(2)

Type: String

```
saw := "She saw exactly one underscore: _."  
"She saw exactly one underscore: _."
```

(3)

Type: String

It is also possible to use `new` to create a string of any size filled with a given character. Since there are many `new` functions it is necessary to indicate the desired type.

```
gasp: String := new(32, char "x")  
"xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx"
```

(4)

Type: String

The length of a string is given by `#`.

```
#gasp  
32
```

(5)

Type: PositivelInteger

Indexing operations allow characters to be extracted or replaced in strings. For any string `s`, indices lie in the range `1..#s`.

```
hello.2  
e
```

(6)

Type: Character

Indexing is really just the application of a string to a subscript, so any application syntax works.

```
hello 2  
e
```

(7)

Type: Character

```
hello(2)  
e
```

(8)

Type: Character

If it is important not to modify a given string, it should be copied before any updating operations are used.

```
hullo := copy hello  
"Hello, I'm AXIOM!"
```

(9)

Type: String

	<pre> hullo.2 := char "u"; [hello, hullo] ["Hello, I'm AXIOM!", "Hullo, I'm AXIOM!"] </pre>	(10)	Type: List String
Operations are provided to split and join strings. The <b>concat</b> operation allows several strings to be joined together.	<pre> saw := concat ["alpha","---","omega"] "alpha---omega" </pre>	(11)	Type: String
There is a version of <b>concat</b> that works with two strings.	<pre> concat("hello ", "goodbye") "hello goodbye" </pre>	(12)	Type: String
Juxtaposition can also be used to concatenate strings.	<pre> "This " "is " "several " "strings " "concatenated." "This is several strings concatenated." </pre>	(13)	Type: String
Substrings are obtained by giving an index range.	<pre> hello(1..5) "Hello"  hello(8..) "I'm AXIOM!" </pre>	(14) (15)	Type: String
A string can be split into several substrings by giving a separation character or character class.	<pre> split(hello, char " ") ["Hello,", "I'm", "AXIOM!"]  other := complement alphanumeric();  split(saw, other) ["alpha", "omega"] </pre>	(16) (17) (18)	Type: List String Type: CharacterClass
Unwanted characters can be trimmed from the beginning or end of a string using the operations <b>trim</b> , <b>leftTrim</b> and <b>rightTrim</b> .	<pre> trim ("## ++ relax ++ ##", char "#") " ++ relax ++ " </pre>	(19)	Type: String



Each of these functions takes a string and a second argument to specify the characters to be discarded.

```
trim ("## ++ relax ++ ##", other)
"relax"
(20)
Type: String
```

The second argument can be given either as a single character or as a character class.

```
leftTrim ("## ++ relax ++ ##", other)
"relax ++ ##"
(21)
Type: String
```

```
rightTrim("## ++ relax ++ ##", other)
"## ++ relax"
(22)
Type: String
```

Strings can be changed to upper case or lower case using the operations **upperCase**, **upperCase!**, **lowerCase** and **lowerCase!**.

```
upperCase hello
"HELLO, I'M AXIOM!"
(23)
Type: String
```

The versions with the exclamation mark change the original string, while the others produce a copy.

```
lowerCase hello
"hello, i'm axiom!"
(24)
Type: String
```

Some basic string matching is provided. The function **prefix?** tests whether one string is an initial prefix of another.

```
prefix? ("He", "Hello")
true
(25)
Type: Boolean
```

```
prefix? ("Her", "Hello")
false
(26)
Type: Boolean
```

A similar function, **suffix?**, tests for suffixes.

```
suffix? ("", "Hello")
true
(27)
Type: Boolean
```

```
suffix? ("LO", "Hello")
false
(28)
Type: Boolean
```

The function **substring?** tests for a substring given a starting position.

```
substring? ("ll", "Hello", 3)
true
(29)
Type: Boolean
```

```
substring?("ll", "Hello", 4)
false
```

(30)

Type: Boolean

A number of **position** functions locate things in strings. If the first argument to **position** is a string, then **position(s,t,i)** finds the location of **s** as a substring of **t** starting the search at position **i**.

```
n := position("nd", "underground", 1)
2
```

(31)

Type: PositiveInteger

```
n := position("nd", "underground", n+1)
10
```

(32)

Type: PositiveInteger

If **s** is not found, then 0 is returned (**minIndex(s)-1** in **IndexedString**).

```
n := position("nd", "underground", n+1)
0
```

(33)

Type: NonNegativeInteger

To search for a specific character or a member of a character class, a different first argument is used.

```
position(char "d", "underground", 1)
3
```

(34)

Type: PositiveInteger

```
position(hexDigit(), "underground", 1)
3
```

(35)

Type: PositiveInteger

## 9.73 StringTable

---

This domain provides a table type in which the keys are known to be strings so special techniques can be used. Other than performance, the type `StringTable(S)` should behave exactly the same way as `Table(String,S)`. See ‘Table’ on page 585 for general information about tables. Issue the system command `)show StringTable` to display the full list of operations defined by `StringTable`.

This creates a new table whose keys are strings.

```
t: StringTable(Integer) := table()
table() (1)
```

Type: StringTable Integer

The value associated with each string key is the number of characters in the string.

```
for s in split("My name is Ian Watt.",char " ")
  repeat
    t.s := #s
```

Type: Void

```
for key in keys t repeat output [key, t.key]
["Ian",3]
["My",2]
["Watt.",5]
["name",4]
["is",2]
```

Type: Void

## 9.74 Symbol

Symbols are one of the basic types manipulated by AXIOM. The Symbol domain provides ways to create symbols of many varieties. Issue the system command `)show Symbol` to display the full list of operations defined by Symbol.

The simplest way to create a symbol is to “single quote” an identifier.

```
X: Symbol := 'x
x
(1)
Type: Symbol
```

This gives the symbol even if `x` has been assigned a value. If `x` has not been assigned a value, then it is possible to omit the quote.

```
XX: Symbol := x
x
(2)
Type: Symbol
```

Declarations must be used when working with symbols, because otherwise the interpreter tries to place values in a more specialized type Variable.

```
A := 'a
a
(3)
Type: Variable a
```

```
B := b
b
(4)
Type: Variable b
```

The normal way of entering polynomials uses this fact.

```
x**2 + 1
x2 + 1
(5)
Type: Polynomial Integer
```

Another convenient way to create symbols is to convert a string. This is useful when the name is to be constructed by a program.

```
"Hello"::Symbol
Hello
(6)
Type: Symbol
```

Sometimes it is necessary to generate new unique symbols, for example, to name constants of integration. The expression `new()` generates a symbol starting with `%`.

```
new()$Symbol
%A
(7)
Type: Symbol
```

Successive calls to `new` produce different symbols.

```
new()$Symbol
%B
(8)
Type: Symbol
```

The expression `new("s")` produces a symbol starting with `%s`.

```
new("xyz")$Symbol
%xyz0
(9)
Type: Symbol
```

A symbol can be adorned in various ways. The most basic thing is applying a symbol to a list of subscripts.

`X[i,j]`  
 $x_{i,j}$   
 (10)  
 Type: Symbol

Somewhat less pretty is to attach subscripts, superscripts or arguments.

`U := subscript(u, [1,2,1,2])`  
 $u_{1,2,1,2}$   
 (11)  
 Type: Symbol

`V := superscript(v, [n])`  
 $v^n$   
 (12)  
 Type: Symbol

`P := argscript(p, [t])`  
 $p(t)$   
 (13)  
 Type: Symbol

It is possible to test whether a symbol has scripts using the **scripted?** test.

`scripted? U`  
`true`  
 (14)  
 Type: Boolean

`scripted? X`  
`false`  
 (15)  
 Type: Boolean

If a symbol is not scripted, then it may be converted to a string.

`string X`  
`"x"`  
 (16)  
 Type: String

The basic parts can always be extracted using the **name** and **scripts** operations.

`name U`  
 $u$   
 (17)  
 Type: Symbol

`scripts U`  
 $[sub = [1, 2, 1, 2], sup = [], presup = [], presub = [], args = []]$   
 (18)  
 Type: Record(sub: List OutputForm, sup: List OutputForm, presup: List OutputForm, presub: List OutputForm, args: List OutputForm)

`name X`  
 $x$   
 (19)  
 Type: Symbol

The most general form is obtained using the **script** operation. This operation takes an argument which is a list containing, in this order, lists of subscripts, superscripts, presuperscripts, presubscripts and arguments to a symbol.

If trailing lists of scripts are omitted, they are assumed to be empty.

scripts X

$$[sub = [], sup = [], presup = [], presub = [], args = []] \quad (20)$$

Type: Record(sub: List OutputForm, sup: List OutputForm, presup: List OutputForm, presub: List OutputForm, args: List OutputForm)

$$M := \text{script}(\text{Mammoth}, [[i, j], [k, l], [0, 1], [2], [u, v, w]])$$

$${}^{0,1}_{2}\text{Mammoth}^{k,l}_{i,j}(u, v, w) \quad (21)$$

Type: Symbol

scripts M

$$[sub = [i, j], sup = [k, l], presup = [0, 1], presub = [2], args = [u, v, w]] \quad (22)$$

Type: Record(sub: List OutputForm, sup: List OutputForm, presup: List OutputForm, presub: List OutputForm, args: List OutputForm)

$$N := \text{script}(\text{Nut}, [[i, j], [k, l], [0, 1]])$$

$${}^{0,1}\text{Nut}^{k,l}_{i,j} \quad (23)$$

Type: Symbol

scripts N

$$[sub = [i, j], sup = [k, l], presup = [0, 1], presub = [], args = []] \quad (24)$$

Type: Record(sub: List OutputForm, sup: List OutputForm, presup: List OutputForm, presub: List OutputForm, args: List OutputForm)

## 9.75 Table

---

The Table constructor provides a general structure for associative storage. This type provides hash tables in which data objects can be saved according to keys of any type. For a given table, specific types must be chosen for the keys and entries.

In this example the keys to the table are polynomials with integer coefficients. The entries in the table are strings.

To save an entry in the table, the **setelt** operation is used. This can be called directly, giving the table a key and an entry.

Alternatively, you can use assignment syntax.

Entries are retrieved from the table by calling the **elt** operation.

This operation is called when a table is “applied” to a key using this or the following syntax.

Parentheses are used only for grouping. They are needed if the key is an infix expression.

Note that the **elt** operation is used only when the key is known to be in the table—otherwise an error is generated.

```
t: Table(Polynomial Integer, String) := table()
table()
Type: Table(Polynomial Integer, String)
```

```
setelt(t, x**2 - 1, "Easy to factor")
"Easy to factor"
Type: String
```

```
t(x**3 + 1) := "Harder to factor"
"Harder to factor"
Type: String
```

```
t(x) := "The easiest to factor"
"The easiest to factor"
Type: String
```

```
elt(t, x)
"The easiest to factor"
Type: String
```

```
t.x
"The easiest to factor"
Type: String
```

```
t x
"The easiest to factor"
Type: String
```

```
t.(x**2 - 1)
"Easy to factor"
Type: String
```

```
t (x**3 + 1)
"Harder to factor"
Type: String
```

You can get a list of all the keys to a table using the <b>keys</b> operation.	<pre>keys t [<math>x</math>, <math>x^3 + 1</math>, <math>x^2 - 1</math>]</pre> <div>(10)</div> <div>Type: List Polynomial Integer</div>
If you wish to test whether a key is in a table, the <b>search</b> operation is used. This operation returns either an entry or "failed".	<pre>search(x, t) "The easiest to factor"</pre> <div>(11)</div> <div>Type: Union(String, ...)</div> <pre>search(x**2, t) "failed"</pre> <div>(12)</div> <div>Type: Union("failed", ...)</div>
The return type is a union so the success of the search can be tested using <b>case</b> .	<pre>search(x**2, t) case "failed" true</pre> <div>(13)</div> <div>Type: Boolean</div>
The <b>remove!</b> operation is used to delete values from a table.	<pre>remove!(x**2-1, t) "Easy to factor"</pre> <div>(14)</div> <div>Type: Union(String, ...)</div>
If an entry exists under the key, then it is returned. Otherwise <b>remove!</b> returns "failed".	<pre>remove!(x-1, t) "failed"</pre> <div>(15)</div> <div>Type: Union("failed", ...)</div>
The number of key-entry pairs can be found using the <b>#</b> operation.	<pre>#t 2</pre> <div>(16)</div> <div>Type: PositiveInteger</div>
Just as <b>keys</b> returns a list of keys to the table, a list of all the entries can be obtained using the <b>members</b> operation.	<pre>members t ["The easiest to factor", "Harder to factor"]</pre> <div>(17)</div> <div>Type: List String</div>
A number of useful operations take functions and map them on to the table to compute the result. Here we count the entries which have "Hard" as a prefix.	<pre>count(s: String +-&gt; prefix?("Hard", s), t) 1</pre> <div>(18)</div> <div>Type: PositiveInteger</div>

Other table types are provided to support various needs.

- **AssociationList** gives a list with a table view. This allows new entries to be appended onto the front of the list to cover up old entries. This is useful when table entries need to be stacked or when frequent list traversals are required. See 'AssociationList' on page 352 for more



information.

- `EqTable` gives tables in which keys are considered equal only when they are in fact the same instance of a structure. See ‘`EqTable`’ on page 406 for more information.
- `StringTable` should be used when the keys are known to be strings. See ‘`StringTable`’ on page 581 for more information.
- `SparseTable` provides tables with default entries, so lookup never fails. The `GeneralSparseTable` constructor can be used to make any table type behave this way. See ‘`SparseTable`’ on page 568 for more information.
- `KeyedAccessFile` allows values to be saved in a file, accessed as a table. See ‘`KeyedAccessFile`’ on page 460 for more information.

Issue the system command `)show Table` to display the full list of operations defined by `Table`.

## 9.76 TextFile

---

The domain TextFile allows AXIOM to read and write character data and exchange text with other programs. This type behaves in AXIOM much like a File of strings, with additional operations to cause new lines. We give an example of how to produce an upper case copy of a file.

This is the file from which we read the text.

```
f1: TextFile := open("/etc/group", "input")
"/etc/group"
```

(1)

Type: TextFile

This is the file to which we read the text.

```
f2: TextFile := open("/tmp/MOTD", "output")
"/tmp/MOTD"
```

(2)

Type: TextFile

Entire lines are handled using the **readLine!** and **writeLine!** operations.

```
l := readLine! f1
"system*:0:root"
```

(3)

Type: String

```
writeLine!(f2, upperCase l)
"SYSTEM*:0:ROOT"
```

(4)

Type: String

Use the **endOfFile?** operation to check if you have reached the end of the file.

```
while not endOfFile? f1 repeat
  s := readLine! f1
  writeLine!(f2, upperCase s)
```

Type: Void

The file **f1** is exhausted and should be closed.

```
close! f1
"/etc/group"
```

(6)

Type: TextFile

It is sometimes useful to write lines a bit at a time. The **write!** operation allows this.

```
write!(f2, "-The-")
"-The-"
```

(7)

Type: String

```
write!(f2, "-End-")
"-End-"
```

(8)

Type: String

This ends the line. This is done in a machine-dependent manner.

```
writeLine! f2
""
```

(9)

Type: String

```
close! f2
```

```
"/tmp/MOTD"
```

(10)

Type: TextFile

Finally, clean up.

```
)system rm /tmp/MOTD
```

For more information on related topics, see ‘File’ on page 420, ‘KeyedAccessFile’ on page 460, and ‘Library’ on page 474. Issue the system command `)show TextFile` to display the full list of operations defined by TextFile.

## 9.77 TwoDimensional- Array

---

The TwoDimensionalArray domain is used for storing data in a two-dimensional data structure indexed by row and by column. Such an array is a homogeneous data structure in that all the entries of the array must belong to the same AXIOM domain (although see Section 2.6 on page 112). Each array has a fixed number of rows and columns specified by the user and arrays are not extensible. In AXIOM, the indexing of two-dimensional arrays is one-based. This means that both the “first” row of an array and the “first” column of an array are given the index 1. Thus, the entry in the upper left corner of an array is in position (1,1).

The operation **new** creates an array with a specified number of rows and columns and fills the components of that array with a specified entry. The arguments of this operation specify the number of rows, the number of columns, and the entry.

This creates a five-by-four array of integers, all of whose entries are zero.

```
arr : ARRAY2 INT := new(5,4,0)
```

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (1)$$

Type: TwoDimensionalArray Integer

The entries of this array can be set to other integers using the operation **setelt**.

Issue this to set the element in the upper left corner of this array to 17.

```
setelt(arr,1,1,17)
```

$$17 \quad (2)$$

Type: PositiveInteger

Now the first element of the array is 17.

```
arr
```

$$\begin{bmatrix} 17 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (3)$$

Type: TwoDimensionalArray Integer

Likewise, elements of an array are extracted using the operation **elt**.

```
elt(arr,1,1)
```

$$17 \quad (4)$$

Type: PositiveInteger

Another way to use these two operations is as follows. This sets the element in position (3,2) of the array to 15.

```
arr(3,2) := 15
```

$$15 \quad (5)$$

Type: PositiveInteger

This extracts the element in position (3,2) of the array.

```
arr(3,2)
```

15

(6)

Type: PositiveInteger

The operations **elt** and **setelt** come equipped with an error check which verifies that the indices are in the proper ranges. For example, the above array has five rows and four columns, so if you ask for the entry in position (6,2) with **arr(6,2)** AXIOM displays an error message. If there is no need for an error check, you can call the operations **qelt** and **qsetelt!** which provide the same functionality but without the error check. Typically, these operations are called in well-tested programs.

The operations **row** and **column** extract rows and columns, respectively, and return objects of OneDimensionalArray with the same underlying element type.

```
row(arr,1)
```

[17, 0, 0, 0]

(7)

Type: OneDimensionalArray Integer

```
column(arr,1)
```

[17, 0, 0, 0, 0]

(8)

Type: OneDimensionalArray Integer

You can determine the dimensions of an array by calling the operations **nrows** and **ncols**, which return the number of rows and columns, respectively.

```
nrows(arr)
```

5

(9)

Type: PositiveInteger

```
ncols(arr)
```

4

(10)

Type: PositiveInteger

To apply an operation to every element of an array, use **map**. This creates a new array. This expression negates every element.

```
map(-,arr)
```

$$\begin{bmatrix} -17 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -15 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(11)

Type: TwoDimensionalArray Integer

This creates an array where all the elements are doubled.

```
map((x +-> x + x),arr)
```

$$\begin{bmatrix} 34 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 30 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (12)$$

Type: TwoDimensionalArray Integer

To change the array destructively, use **map!** instead of **map**. If you need to make a copy of any array, use **copy**.

```
arrc := copy(arr)
```

$$\begin{bmatrix} 17 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 15 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (13)$$

Type: TwoDimensionalArray Integer

```
map!(-,arrc)
```

$$\begin{bmatrix} -17 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -15 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (14)$$

Type: TwoDimensionalArray Integer

```
arrc
```

$$\begin{bmatrix} -17 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -15 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (15)$$

Type: TwoDimensionalArray Integer

```
arr
```

$$\begin{bmatrix} 17 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 15 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (16)$$

Type: TwoDimensionalArray Integer

Use **member?** to see if a given element is in an array.

```
member?(17,arr)
```

$$\text{true} \quad (17)$$

Type: Boolean

```
member?(10317,arr)
false
```

(18)

Type: Boolean

To see how many times an  
element appears in an array, use  
**count**.

```
count(17,arr)
1
```

(19)

Type: PositivelInteger

```
count(0,arr)
18
```

(20)

Type: PositivelInteger

For more information about the operations available for `TwoDimensionalArray`, issue `)show TwoDimensionalArray`. For information on related topics, see ‘Matrix’ on page 500 and ‘OneDimensionalArray’ on page 514.

## 9.78 Univariate- Polynomial

The domain constructor `UnivariatePolynomial` (abbreviated `UP`) creates domains of univariate polynomials in a specified variable. For example, the domain `UP(a1,POLY FRAC INT)` provides polynomials in the single variable `a1` whose coefficients are general polynomials with rational number coefficients.

### Restriction:

AXIOM does not allow you to create types where `UnivariatePolynomial` is contained in the coefficient type of `Polynomial`. Therefore, `UP(x,POLY INT)` is legal but `POLY UP(x,INT)` is not.

`UP(x,INT)` is the domain of polynomials in the single variable `x` with integer coefficients.

`(p,q) : UP(x,INT)`

Type: Void

`p := (3*x-1)**2 * (2*x + 8)`

$18x^3 + 60x^2 - 46x + 8$  (2)

Type: UnivariatePolynomial(x, Integer)

`q := (1 - 6*x + 9*x**2)**2`

$81x^4 - 108x^3 + 54x^2 - 12x + 1$  (3)

Type: UnivariatePolynomial(x, Integer)

The usual arithmetic operations are available for univariate polynomials.

`p**2 + p*q`

$1458x^7 + 3240x^6 - 7074x^5 + 10584x^4 - 9282x^3 + 4120x^2 - 878x + 72$  (4)

Type: UnivariatePolynomial(x, Integer)

The operation **leadingCoefficient** extracts the coefficient of the term of highest degree.

`leadingCoefficient p`

18 (5)

Type: PositiveInteger

The operation **degree** returns the degree of the polynomial. Since the polynomial has only one variable, the variable is not supplied to operations like **degree**.

`degree p`

3 (6)

Type: PositiveInteger

The reductum of the polynomial, the polynomial obtained by subtracting the term of highest order, is returned by **reductum**.

`reductum p`

$60x^2 - 46x + 8$  (7)

Type: UnivariatePolynomial(x, Integer)



The operation <b>gcd</b> computes the greatest common divisor of two polynomials.	$\text{gcd}(p, q)$ $9x^2 - 6x + 1$	(8)	Type: UnivariatePolynomial(x, Integer)
The operation <b>lcm</b> computes the least common multiple.	$\text{lcm}(p, q)$ $162x^5 + 432x^4 - 756x^3 + 408x^2 - 94x + 8$	(9)	Type: UnivariatePolynomial(x, Integer)
The operation <b>resultant</b> computes the resultant of two univariate polynomials. In the case of <b>p</b> and <b>q</b> , the resultant is 0 because they share a common root.	$\text{resultant}(p, q)$ 0	(10)	Type: NonNegativeInteger
To compute the derivative of a univariate polynomial with respect to its variable, use <b>D</b> .	$D\ p$ $54x^2 + 120x - 46$	(11)	Type: UnivariatePolynomial(x, Integer)
Univariate polynomials can also be used as if they were functions. To evaluate a univariate polynomial at some point, apply the polynomial to the point.	$p(2)$ 300	(12)	Type: PositiveInteger
The same syntax is used for composing two univariate polynomials, i.e. substituting one polynomial for the variable in another. This substitutes <b>q</b> for the variable in <b>p</b> .	$p(q)$ $9565938x^{12} - 38263752x^{11} + 70150212x^{10} - 77944680x^9 + 58852170x^8 - 32227632x^7 + 13349448x^6 - 4280688x^5 + 1058184x^4 - 192672x^3 + 23328x^2 - 1536x + 40$	(13)	Type: UnivariatePolynomial(x, Integer)
This substitutes <b>p</b> for the variable in <b>q</b> .	$q(p)$ $8503056x^{12} + 113374080x^{11} + 479950272x^{10} + 404997408x^9 - 1369516896x^8 - 626146848x^7 + 2939858712x^6 - 2780728704x^5 + 1364312160x^4 - 396838872x^3 + 69205896x^2 - 6716184x + 279841$	(14)	Type: UnivariatePolynomial(x, Integer)
To obtain a list of coefficients of the polynomial, use <b>coefficients</b> .	$l := \text{coefficients } p$ $[18, 60, -46, 8]$	(15)	Type: List Integer
From this you can use <b>gcd</b> and <b>reduce</b> to compute the content of the polynomial.	$\text{reduce}(\text{gcd}, l)$ 2	(16)	Type: PositiveInteger

Alternatively (and more easily), you can just call **content**.

```
content p
2
```

(17)

Type: PositiveInteger

Note that the operation **coefficients** omits the zero coefficients from the list. Sometimes it is useful to convert a univariate polynomial to a vector whose  $i$ <sup>th</sup> position contains the degree  $i-1$  coefficient of the polynomial.

```
ux := (x**4+2*x+3)::UP(x,INT)
```

$$x^4 + 2x + 3$$

(18)

Type: UnivariatePolynomial(x, Integer)

To get a complete vector of coefficients, use the operation **vectorise**, which takes a univariate polynomial and an integer denoting the length of the desired vector.

```
vectorise(ux,5)
[3, 2, 0, 0, 1]
```

(19)

Type: Vector Integer

It is common to want to do something to every term of a polynomial, creating a new polynomial in the process.

This is a function for iterating across the terms of a polynomial, squaring each term.

```
squareTerms(p) ==
  reduce(+,[t**2 for t in monomials p])
```

Type: Void

Recall what **p** looked like.

```
p
```

$$18x^3 + 60x^2 - 46x + 8$$

(21)

Type: UnivariatePolynomial(x, Integer)

We can demonstrate **squareTerms** on **p**.

```
squareTerms p
Compiling function squareTerms with type
  UnivariatePolynomial(x,Integer) ->
  UnivariatePolynomial(x,Integer)
```

$$324x^6 + 3600x^4 + 2116x^2 + 64$$

(22)

Type: UnivariatePolynomial(x, Integer)

When the coefficients of the univariate polynomial belong to a field,<sup>7</sup> it is possible to compute quotients and remainders.

```
(r,s) : UP(a1,FRAC INT)
```

Type: Void

---

<sup>7</sup>For example, when the coefficients are rational numbers, as opposed to integers. The important property of a field is that non-zero elements can be divided and produce another element. The quotient of the integers 2 and 3 is not another integer.

$$r := a1^{**2} - 2/3$$

$$a1^2 - \frac{2}{3} \quad (24)$$

Type: UnivariatePolynomial(a1, Fraction Integer)

$$s := a1 + 4$$

$$a1 + 4 \quad (25)$$

Type: UnivariatePolynomial(a1, Fraction Integer)

When the coefficients are rational numbers or rational expressions, the operation **quo** computes the quotient of two polynomials.

$$r \text{ quo } s$$

$$a1 - 4 \quad (26)$$

Type: UnivariatePolynomial(a1, Fraction Integer)

The operation **rem** computes the remainder.

$$r \text{ rem } s$$

$$\frac{46}{3} \quad (27)$$

Type: UnivariatePolynomial(a1, Fraction Integer)

The operation **divide** can be used to return a record of both components.

$$d := \text{divide}(r, s)$$

$$\left[ \text{quotient} = a1 - 4, \text{remainder} = \frac{46}{3} \right] \quad (28)$$

Type: Record(quotient: UnivariatePolynomial(a1, Fraction Integer), remainder: UnivariatePolynomial(a1, Fraction Integer))

Now we check the arithmetic!

$$r - (d.\text{quotient} * s + d.\text{remainder})$$

$$0 \quad (29)$$

Type: UnivariatePolynomial(a1, Fraction Integer)

It is also possible to integrate univariate polynomials when the coefficients belong to a field.

$$\text{integrate } r$$

$$\frac{1}{3} a1^3 - \frac{2}{3} a1 \quad (30)$$

Type: UnivariatePolynomial(a1, Fraction Integer)

$$\text{integrate } s$$

$$\frac{1}{2} a1^2 + 4 a1 \quad (31)$$

Type: UnivariatePolynomial(a1, Fraction Integer)

One application of univariate polynomials is to see expressions in terms of a specific variable.

We start with a polynomial in **a1** whose coefficients are quotients of polynomials in **b1** and **b2**.

$$t : \text{UP}(a1, \text{FRAC POLY INT})$$

Type: Void

Since in this case we are not talking about using multivariate polynomials in only two variables, we use Polynomial. We also use Fraction because we want fractions.

We push all the variables into a single quotient of polynomials.

$$t := a1^{**2} - a1/b2 + (b1^{**2}-b1)/(b2+3)$$

$$a1^2 - \frac{1}{b2} a1 + \frac{b1^2 - b1}{b2 + 3} \quad (33)$$

Type: UnivariatePolynomial(a1, Fraction Polynomial Integer)

$$u : \text{FRAC POLY INT} := t$$

$$\frac{a1^2 b2^2 + (b1^2 - b1 + 3 a1^2 - a1) b2 - 3 a1}{b2^2 + 3 b2} \quad (34)$$

Type: Fraction Polynomial Integer

Alternatively, we can view this as a polynomial in the variable *b1*. This is a *mode-directed* conversion: you indicate as much of the structure as you care about and let AXIOM decide on the full type and how to do the transformation.

$$u :: \text{UP}(b1, ?)$$

$$\frac{1}{b2 + 3} b1^2 - \frac{1}{b2 + 3} b1 + \frac{a1^2 b2 - a1}{b2} \quad (35)$$

Type: UnivariatePolynomial(b1, Fraction Polynomial Integer)

See Section 8.2 on page 274 for a discussion of the factorization facilities in AXIOM for univariate polynomials. For more information on related topics, see Section 1.9 on page 73, Section 2.7 on page 113, ‘Polynomial’ on page 529, ‘MultivariatePolynomial’ on page 508, and ‘Distributed-MultivariatePolynomial’ on page 402. Issue the system command `)show UnivariatePolynomial` to display the full list of operations defined by UnivariatePolynomial.

## 9.79 Universal- Segment

---

The UniversalSegment domain generalizes Segment by allowing segments without a “hi” end point.

```
pints := 1..
1..
```

(1)

Type: UniversalSegment PositiveInteger

```
nevens := (0..) by -2
0.. by -2
```

(2)

Type: UniversalSegment NonNegativeInteger

Values of type Segment are automatically converted to type UniversalSegment when appropriate.

```
useg: UniversalSegment(Integer) := 3..10
3..10
```

(3)

Type: UniversalSegment Integer

The operation **hasHi** is used to test whether a segment has a hi end point.

```
hasHi pints
false
```

(4)

Type: Boolean

```
hasHi nevens
false
```

(5)

Type: Boolean

```
hasHi useg
true
```

(6)

Type: Boolean

All operations available on type Segment apply to UniversalSegment, with the proviso that expansions produce streams rather than lists. This is to accommodate infinite expansions.

```
expand pints
[1, 2, 3, 4, 5, 6, 7, ...]
```

(7)

Type: Stream Integer

```
expand nevens
[0, -2, -4, -6, -8, -10, -12, ...]
```

(8)

Type: Stream Integer

```
expand [1, 3, 10..15, 100..]
[1, 3, 10, 11, 12, 13, 14, ...]
```

(9)

Type: Stream Integer

For more information on related topics, see ‘Segment’ on page 559, ‘SegmentBinding’ on page 561, ‘List’ on page 489, and ‘Stream’ on page 575. Issue the system command `)show UniversalSegment` to display the full

list of operations defined by UniversalSegment.

## 9.80 Vector

---

The Vector domain is used for storing data in a one-dimensional indexed data structure. A vector is a homogeneous data structure in that all the components of the vector must belong to the same AXIOM domain. Each vector has a fixed length specified by the user; vectors are not extensible. This domain is similar to the OneDimensionalArray domain, except that when the components of a Vector belong to a Ring, arithmetic operations are provided. For more examples of operations that are defined for both Vector and OneDimensionalArray, see ‘OneDimensionalArray’ on page 514.

As with the OneDimensionalArray domain, a Vector can be created by calling the operation **new**, its components can be accessed by calling the operations **elt** and **qelt**, and its components can be reset by calling the operations **setelt** and **qsetelt**!

This creates a vector of integers of length 5 all of whose components are 12.

```
u : VECTOR INT := new(5,12)
[12, 12, 12, 12, 12]
```

(1)

Type: Vector Integer

This is how you create a vector from a list of its components.

```
v : VECTOR INT := vector([1,2,3,4,5])
[1, 2, 3, 4, 5]
```

(2)

Type: Vector Integer

Indexing for vectors begins at 1. The last element has index equal to the length of the vector, which is computed by “#”.

```
#(v)
5
```

(3)

Type: PositiveInteger

This is the standard way to use **elt** to extract an element. Functionally, it is the same as if you had typed **elt(v,2)**.

```
v.2
2
```

(4)

Type: PositiveInteger

This is the standard way to use **setelt** to change an element. It is the same as if you had typed **setelt(v,3,99)**.

```
v.3 := 99
99
```

(5)

Type: PositiveInteger

Now look at **v** to see the change. You can use **qelt** and **qsetelt**! (instead of **elt** and **setelt**, respectively) but *only* when you know that the index is within the valid range.

```
v
[1, 2, 99, 4, 5]
```

(6)

Type: Vector Integer

When the components belong to a Ring, AXIOM provides arithmetic operations for Vector. These include left and right scalar multiplication.

```
5 * v
[5, 10, 495, 20, 25]
```

(7)

Type: Vector Integer

```
v * 7
[7, 14, 693, 28, 35]
```

(8)

Type: Vector Integer

```
w : VECTOR INT := vector([2,3,4,5,6])
[2, 3, 4, 5, 6]
```

(9)

Type: Vector Integer

Addition and subtraction are also available.

```
v + w
[3, 5, 103, 9, 11]
```

(10)

Type: Vector Integer

Of course, when adding or subtracting, the two vectors must have the same length or an error message is displayed.

```
v - w
[-1, -1, 95, -1, -1]
```

(11)

Type: Vector Integer

For more information about other aggregate domains, see the following: ‘List’ on page 489, ‘Matrix’ on page 500, ‘OneDimensionalArray’ on page 514, ‘Set’ on page 563, ‘Table’ on page 585, and ‘TwoDimensionalArray’ on page 590. Issue the system command `)show Vector` to display the full list of operations defined by Vector.



## 9.81 Void

---

When an expression is not in a value context, it is given type Void. For example, in the expression

```
r := (a; b; if c then d else e; f)
```

values are used only from the subexpressions **c** and **f**: all others are thrown away. The subexpressions **a**, **b**, **d** and **e** are evaluated for side-effects only and have type Void. There is a unique value of type Void.

You will most often see results of type Void when you declare a variable.

```
a : Integer
```

Type: Void

Usually no output is displayed for Void results. You can force the display of a rather ugly object by issuing `)set message void on`.

```
)set message void on
```

```
b : Fraction Integer
```

```
"()"
```

Type: Void

```
)set message void off
```

All values can be converted to type Void.

```
3::Void
```

Type: Void

Once a value has been converted to Void, it cannot be recovered.

```
% :: PositiveInteger
```

```
Cannot convert from type Void to PositiveInteger for  
value  
"()"
```

## 9.82

### WuWenTsun-TriangularSet

---

The WuWenTsunTriangularSet domain constructor implements the characteristic set method of Wu Wen Tsun. This algorithm decomposes an algebraic variety into a union of regular zeros set of finitely many triangular sets. The constructor takes four arguments. The first one, **R**, is the coefficient ring of the polynomials; it must belong to the category Integral-Domain. The second one, **E**, is the exponent monoid of the polynomials; it must belong to the category OrderedAbelianMonoidSup. The third one, **V**, is the ordered set of variables; it must belong to the category OrderedSet. The last one is the polynomial ring; it must belong to the category RecursivePolynomialCategory(R,E,V). The abbreviation for WuWenTsunTriangularSet is WUTSET.

Let us illustrate the facilities by an example.

Define the coefficient ring.	<code>R := Integer</code>	
	Integer	(1)
		Type: Domain
Define the list of variables,	<code>ls : List Symbol := [x,y,z,t]</code>	
	<code>[x, y, z, t]</code>	(2)
		Type: List Symbol
and make it an ordered set;	<code>V := OVAR(ls)</code>	
	OrderedVariableList [ x , y , z , t ]	(3)
		Type: Domain
then define the exponent monoid.	<code>E := IndexedExponents V</code>	
	IndexedExponents OrderedVariableList [ x , y , z , t ]	(4)
		Type: Domain
Define the polynomial ring.	<code>P := NSMP(R, V)</code>	
	NewSparseMultivariatePolynomial ( Integer , OrderedVariableList [ x , y , z , t ] )	(5)
		Type: Domain
Let the variables be polynomial.	<code>x: P := 'x</code>	
	<code>x</code>	(6)
	Type: NewSparseMultivariatePolynomial(Integer, OrderedVariableList [x, y, z, t])	
	<code>y: P := 'y</code>	
	<code>y</code>	(7)
	Type: NewSparseMultivariatePolynomial(Integer, OrderedVariableList [x, y, z, t])	

z: P := 'z  
z

Type: NewSparseMultivariatePolynomial(Integer, OrderedVariableList [x, y, z, t])

t: P := 't  
t

Type: NewSparseMultivariatePolynomial(Integer, OrderedVariableList [x, y, z, t])

Now call the  
WuWenTsunTriangularSet domain  
constructor.

T := WUTSET(R, E, V, P)  
WuWenTsunTriangularSet ( Integer , IndexedExponents OrderedVariableList  
[ x , y , z , t ] , OrderedVariableList [ x , y , z , t ] ,  
NewSparseMultivariatePolynomial ( Integer , OrderedVariableList [ x , y , z  
, t ]))

Type: Domain

Define a polynomial system.

p1 := x \*\* 31 - x \*\* 6 - x - y  
 $x^{31} - x^6 - x - y$

Type: NewSparseMultivariatePolynomial(Integer, OrderedVariableList [x, y, z, t])

p2 := x \*\* 8 - z  
 $x^8 - z$

Type: NewSparseMultivariatePolynomial(Integer, OrderedVariableList [x, y, z, t])

p3 := x \*\* 10 - t  
 $x^{10} - t$

Type: NewSparseMultivariatePolynomial(Integer, OrderedVariableList [x, y, z, t])

lp := [p1, p2, p3]  
 $[x^{31} - x^6 - x - y, x^8 - z, x^{10} - t]$

Type: List NewSparseMultivariatePolynomial(Integer, OrderedVariableList [x, y, z,  
t])

Compute a characteristic set of  
the system.

characteristicSet(lp)\$T  
$$\left\{ \begin{array}{l} z^5 - t^4, \\ t^4 z^2 y^2 + 2 t^3 z^4 y + (-t^7 + 2 t^4 - t) z^6 + t^6 z, \\ (t^3 - 1) z^3 x - z^3 y - t^3 \end{array} \right\}$$

Type: Union(WuWenTsunTriangularSet(Integer, IndexedExponents  
OrderedVariableList [x, y, z, t], OrderedVariableList [x, y, z, t],  
NewSparseMultivariatePolynomial(Integer, OrderedVariableList [x, y, z, t])),  
...)

Solve the system.

`zeroSetSplit(lp)$T`

$$\left[ \begin{array}{l} \{t, z, y, x\}, \\ \{t^3 - 1, z^5 - t^4, z^3 y + t^3, z x^2 - t\}, \\ \{z^5 - t^4, t^4 z^2 y^2 + 2 t^3 z^4 y + (-t^7 + 2 t^4 - t) z^6 + t^6 z, \\ (t^3 - 1) z^3 x - z^3 y - t^3\} \end{array} \right] \quad (16)$$

Type: List WuWenTsunTriangularSet(Integer, IndexedExponents  
OrderedVariableList [x, y, z, t], OrderedVariableList [x, y, z, t],  
NewSparseMultivariatePolynomial(Integer, OrderedVariableList [x, y, z, t]))

The RegularTriangularSet and SquareFreeRegularTriangularSet domain constructors, and the LazardSetSolvingPackage and ZeroDimensionalSolvePackage package constructors also provide operations to compute triangular decompositions of algebraic varieties. These four constructors use a special kind of characteristic sets, called regular triangular sets. These special characteristic sets have better properties than the general ones. Regular triangular sets and their related concepts are presented in the paper "On the Theories of Triangular sets" By P. Aubry, D. Lazard and M. Moreno Maza (to appear in the Journal of Symbolic Computation). The decomposition algorithm (due to the third author) available in the four above constructors provide generally better timings than the characteristic set method. In fact, the WUTSET constructor remains interesting for the purpose of manipulating characteristic sets whereas the other constructors are more convenient for solving polynomial systems.

Note that the way of understanding triangular decompositions is detailed in the documentation of the RegularTriangularSet constructor.

## 9.83 ZeroDimensional- SolvePackage

---

The ZeroDimensionalSolvePackage package constructor provides operations for computing symbolically the complex or real roots of zero-dimensional algebraic systems.

The package provides **no** multiplicity information (i.e. some returned roots may be double or higher) but only distinct roots are returned.

Complex roots are given by means of univariate representations of irreducible regular chains. These univariate representations are computed by the InternalRationalUnivariateRepresentationPackage package constructor. Real roots are given by means of tuples of coordinates lying in the Real-Closure of the coefficient ring.

The ZeroDimensionalSolvePackage constructor takes three arguments. The first one **R** is the coefficient ring; it must belong to the categories OrderedRing, EuclideanDomain, CharacteristicZero and RealConstant. This means essentially that **R** is Integer or Fraction(Integer). The second argument **ls** is the list of variables involved in the systems to solve. The third one must be **concat(ls,s)** where **s** is an additional symbol used for the univariate representations. The abbreviation for ZeroDimensionalSolvePackage is ZDSOLVE.

Both computations of complex roots and real roots rely on triangular decompositions by means of the RegularTriangularSet domain constructor. No Groebner bases are computed.

We illustrate now how to use the constructor ZDSOLVE by two examples: the *Arnborg-Lazard* system and the *L-3* system (Aubry-Moreno Maza).

Define the coefficient ring.

```
R := Integer
Integer
(1)
Type: Domain
```

Define the lists of variables:

```
ls : List Symbol := [x,y,z,t]
[x, y, z, t]
(2)
Type: List Symbol
```

and:

```
ls2 : List Symbol := [x,y,z,t,new()$Symbol]
[x, y, z, t, %A]
(3)
Type: List Symbol
```

Call the package:

```
pack := ZDSOLVE(R,ls,ls2)
ZeroDimensionalSolvePackage(Integer,[x,y,z,t],[x,y,z,t,
(4)
Type: Domain
```

Define a polynomial system  
(Arnborg-Lazard)

$$p1 := \frac{x^{**2} * y * z + x * y^{**2} * z + x * y * z^{**2} + x * y * z + x * y + x * z + y * z}{y * z}$$

$$x \ y \ z^2 + \left( x \ y^2 + \left( x^2 + x + 1 \right) y + x \right) z + x \ y \quad (5)$$

Type: Polynomial Integer

$$p2 := \frac{x^{**2} * y^{**2} * z + x * y^{**2} * z^{**2} + x^{**2} * y * z + x * y * z + y * z + x}{x + z}$$

$$x \ y^2 \ z^2 + \left( x^2 \ y^2 + \left( x^2 + x + 1 \right) y + 1 \right) z + x \quad (6)$$

Type: Polynomial Integer

$$p3 := \frac{x^{**2} * y^{**2} * z^{**2} + x^{**2} * y^{**2} * z + x * y^{**2} * z + x * y * z + x * z + z + 1}{x * z + z + 1}$$

$$x^2 \ y^2 \ z^2 + \left( \left( x^2 + x \right) y^2 + x \ y + x + 1 \right) z + 1 \quad (7)$$

Type: Polynomial Integer

$$lp := [p1, p2, p3]$$

$$\begin{bmatrix} x \ y \ z^2 + \left( x \ y^2 + \left( x^2 + x + 1 \right) y + x \right) z + x \ y, \\ x \ y^2 \ z^2 + \left( x^2 \ y^2 + \left( x^2 + x + 1 \right) y + 1 \right) z + x, \\ x^2 \ y^2 \ z^2 + \left( \left( x^2 + x \right) y^2 + x \ y + x + 1 \right) z + 1 \end{bmatrix} \quad (8)$$

Type: List Polynomial Integer

Note that these polynomials do not involve the variable **t**; we will use it in the second example.

First compute a decomposition into regular chains (i.e. regular triangular sets).

`triangSolve(lp)$pack`

$$\left[ \begin{array}{l} z^{20} - 6 z^{19} - 41 z^{18} + 71 z^{17} + 106 z^{16} + 92 z^{15} + 197 z^{14} + \\ 145 z^{13} + 257 z^{12} + 278 z^{11} + 201 z^{10} + 278 z^9 + 257 z^8 + \\ 145 z^7 + 197 z^6 + 92 z^5 + 106 z^4 + 71 z^3 - 41 z^2 - 6 z + 1, \\ (14745844 z^{19} + 50357474 z^{18} - 130948857 z^{17} \\ - 185261586 z^{16} - 180077775 z^{15} - 338007307 z^{14} \\ - 275379623 z^{13} - 453190404 z^{12} - 474597456 z^{11} \\ - 366147695 z^{10} - 481433567 z^9 - 430613166 z^8 \\ - 261878358 z^7 - 326073537 z^6 - 163008796 z^5 \\ - 177213227 z^4 - 104356755 z^3 + 65241699 z^2 \\ + 9237732 z - 1567348) y + 1917314 z^{19} + 6508991 z^{18} \\ - 16973165 z^{17} - 24000259 z^{16} - 23349192 z^{15} - 43786426 z^{14} \\ - 35696474 z^{13} - 58724172 z^{12} - 61480792 z^{11} - 47452440 z^{10} \\ - 62378085 z^9 - 55776527 z^8 - 33940618 z^7 - 42233406 z^6 \\ - 21122875 z^5 - 22958177 z^4 - 13504569 z^3 + 8448317 z^2 + \\ 1195888 z - 202934, \\ ((z^3 - 2 z) y^2 + (-z^3 - z^2 - 2 z - 1) y - z^2 - z + 1) x + \\ z^2 - 1 \end{array} \right] \quad (9)$$

Type: List RegularChain(Integer, [x, y, z, t])

We can see easily from this decomposition (consisting of a single regular chain) that the input system has 20 complex roots.

Then we compute a univariate representation of this regular chain.

`univariateSolve(lp)$pack`

$$\left[ \begin{array}{l} \text{complexRoots} = \left( \begin{array}{l} ?^{12} - 12 ?^{11} + 24 ?^{10} + 4 ?^9 - 9 ?^8 + \\ 27 ?^7 - 21 ?^6 + 27 ?^5 - 9 ?^4 + 4 ?^3 + \\ 24 ?^2 - 12 ? + 1 \end{array} \right), \\ \text{coordinates} = \left[ \begin{array}{l} 63 x + 62 \%A^{11} - 721 \%A^{10} + 1220 \%A^9 + \\ 705 \%A^8 - 285 \%A^7 + 1512 \%A^6 - \\ 735 \%A^5 + 1401 \%A^4 - 21 \%A^3 + \\ 215 \%A^2 + 1577 \%A - 142, \\ 63 y - 75 \%A^{11} + 890 \%A^{10} - 1682 \%A^9 - \\ 516 \%A^8 + 588 \%A^7 - 1953 \%A^6 + \\ 1323 \%A^5 - 1815 \%A^4 + 426 \%A^3 - \\ 243 \%A^2 - 1801 \%A + 679, \\ z - \%A \end{array} \right] \end{array} \right], \quad (10)$$

$$\left[ \begin{array}{l} \text{complexRoots} = ?^6 + ?^5 + ?^4 + ?^3 + ?^2 + ? + 1, \\ \text{coordinates} = [x - \%A^5, y - \%A^3, z - \%A] \end{array} \right],$$

$$\left[ \begin{array}{l} \text{complexRoots} = ?^2 + 5 ? + 1, \\ \text{coordinates} = [x - 1, y - 1, z - \%A] \end{array} \right]$$

Type: List Record(complexRoots: SparseUnivariatePolynomial Integer, coordinates: List Polynomial Integer)

We see that the zeros of our regular chain are split into three components. This is due to the use of univariate polynomial factorization.

Each of these components consist of two parts. The first one is an irreducible univariate polynomial  $\mathbf{p}(?)$  which defines a simple algebraic extension of the field of fractions of  $\mathbf{R}$ . The second one consists of multivariate polynomials  $\mathbf{pol1}(\mathbf{x}, \%A)$ ,  $\mathbf{pol2}(\mathbf{y}, \%A)$  and  $\mathbf{pol3}(\mathbf{z}, \%A)$ . Each of these polynomials involve two variables: one is an indeterminate  $\mathbf{x}$ ,  $\mathbf{y}$  or  $\mathbf{z}$  of the input system  $\mathbf{lp}$  and the other is  $\%A$  which represents any root of  $\mathbf{p}(?)$ . Recall that this  $\%A$  is the last element of the third parameter of ZDSOLVE. Thus any complex root  $?$  of  $\mathbf{p}(?)$  leads to a solution of the input system  $\mathbf{lp}$  by replacing  $\%A$  by this  $?$  in  $\mathbf{pol1}(\mathbf{x}, \%A)$ ,  $\mathbf{pol2}(\mathbf{y}, \%A)$  and  $\mathbf{pol3}(\mathbf{z}, \%A)$ . Note that the polynomials  $\mathbf{pol1}(\mathbf{x}, \%A)$ ,  $\mathbf{pol2}(\mathbf{y}, \%A)$  and  $\mathbf{pol3}(\mathbf{z}, \%A)$  have degree one w.r.t.  $\mathbf{x}$ ,  $\mathbf{y}$  or  $\mathbf{z}$  respectively. This is always the case for all univariate representations. Hence the operation **univariateSolve** replaces a system of multivariate polynomials by a list of univariate polynomials, what justifies its name.



We now compute the solutions with real coordinates:

```
lr := realSolve(lp)$pack
```

$$\left[ \left[ \%R1, \frac{1184459}{1645371} \%R1^{19} - \frac{2335702}{548457} \%R1^{18} - \frac{5460230}{182819} \%R1^{17} + \frac{79900378}{1645371} \%R1^{16} - \dots \right] \right]$$

Type: List List RealClosure Fraction Integer

The number of real solutions for the input system is:

```
# lr
```

8

(12)

Type: PositiveInteger

Each of these real solutions is given by a list of elements in RealClosure(R). In these 8 lists, the first element is a value of **z**, the second of **y** and the last of **x**. This is logical since by setting the list of variables of the package to **[x,y,z,t]** we mean that the elimination ordering on the variables is **t ; z ; y ; x**. Note that each system treated by the ZDSOLVE package constructor needs only to be zero-dimensional w.r.t. the variables involved in the system it-self and not necessarily w.r.t. all the variables used to define the package.

We can approximate these real numbers as follows. This computation takes between 30 sec. and 5 min, depending on your machine.

```
[[approximate(r,1/1000000) for r in point] for point in lr]
```

$$\left[ \left[ -\frac{10048059}{2097152}, \frac{45030573169853879435243979138389664145967319762117682193358}{45030572830252454885165118069858266350831006937573204652805}, \dots \right] \right]$$

Type: List List Fraction Integer

We can also concentrate on the solutions with real (strictly) positive coordinates:

```
lpr := positiveSolve(lp)$pack
```

[]

(14)

Type: List List RealClosure Fraction Integer

Thus we have checked that the input system has no solution with strictly positive coordinates.

Let us define another polynomial system (*L-3*).

```
f0 := x**3 + y + z + t- 1
```

$$z + y + x^3 + t - 1$$

(15)

Type: Polynomial Integer

```
f1 := x + y**3 + z + t -1
```

$$z + y^3 + x + t - 1$$

(16)

Type: Polynomial Integer

```
f2 := x + y + z**3 + t-1
```

$$z^3 + y + x + t - 1$$

(17)

Type: Polynomial Integer

$$f3 := x + y + z + t^{**3} - 1$$

$$z + y + x + t^3 - 1 \quad (18)$$

Type: Polynomial Integer

$$lf := [f0, f1, f2, f3]$$

$$\begin{bmatrix} z + y + x^3 + t - 1, z + y^3 + x + t - 1, \\ z^3 + y + x + t - 1, z + y + x + t^3 - 1 \end{bmatrix} \quad (19)$$

Type: List Polynomial Integer

First compute a decomposition into regular chains (i.e. regular triangular sets).

$$lts := \text{triangSolve}(lf)\$pack$$

$$\left[ \left\{ t^2 + t + 1, z^3 - z - t^3 + t, \left( 3z + 3t^3 - 3 \right) y^2 + \left( 3z^2 + \left( 6t^3 - 6 \right) z + 3t^6 - 6t^3 \right) y + \left( 3z^3 + \left( 6t^3 - 6 \right) z + 3t^6 - 6t^3 \right) \right\} \right]$$

Type: List RegularChain(Integer, [x, y, z, t])

Then we compute a univariate representation.

$$\text{univariateSolve}(lf)\$pack$$

$$\left[ [complexRoots = ?, coordinates = [x - 1, y - 1, z + 1, t - \%A]], [complexRoots = ?^4 + 5?^3 + 16?^2 + 30? + 57, coordinates = [151x + 15\%A^3 + 151y + 15\%A^3 + 151z + 15\%A^3 + 151t + 15\%A^3]] \right]$$

Type: List Record(complexRoots: SparseUnivariatePolynomial Integer, coordinates: List Polynomial Integer)

Note that this computation is made from the input system **lf**.

However it is possible to reuse a pre-computed regular chain as follows:

$$ts := lts.1$$

$$\left\{ t^2 + t + 1, z^3 - z - t^3 + t, \left( 3z + 3t^3 - 3 \right) y^2 + \left( 3z^2 + \left( 6t^3 - 6 \right) z + 3t^6 - 6t^3 \right) y + \left( 3z^3 + \left( 6t^3 - 6 \right) z + 3t^6 - 6t^3 \right) \right\}$$

Type: RegularChain(Integer, [x, y, z, t])

$$\text{univariateSolve}(ts)\$pack$$

$$\left[ [complexRoots = ?^4 + 5?^3 + 16?^2 + 30? + 57, coordinates = [151x + 15\%A^3 + 151y + 15\%A^3 + 151z + 15\%A^3 + 151t + 15\%A^3]] \right]$$

Type: List Record(complexRoots: SparseUnivariatePolynomial Integer, coordinates: List Polynomial Integer)

$$\text{realSolve}(ts)\$pack$$

$$[] \quad (24)$$

Type: List List RealClosure Fraction Integer

We compute now the full set of points with real coordinates:

$$lr2 := \text{realSolve}(lf)\$pack$$

$$\left[ [0, -1, 1, 1], [0, 0, 1, 0], [1, 0, 0, 0], [0, 0, 0, 1], [0, 1, 0, 0], [1, 0, \%R37, -\%R37] \right]$$

Type: List List RealClosure Fraction Integer

The number of real solutions for the input system is:

$$\#lr2$$

$$27 \quad (26)$$

Type: PositivelInteger

We concentrate now on the solutions with real (strictly) positive coordinates:

$$\text{lpr2} := \text{positiveSolve(lf)}\$pack$$

$$\left[ \left[ \%R40, -\frac{1}{3} \%R40^3 + \frac{1}{3}, -\frac{1}{3} \%R40^3 + \frac{1}{3}, -\frac{1}{3} \%R40^3 + \frac{1}{3} \right] \right] \quad (27)$$

Type: List List RealClosure Fraction Integer

Finally, we approximate the coordinates of this point with 20 exact digits:

$$[\text{approximate}(r, 1/10^{**21})::\text{Float for } r \text{ in } \text{lpr2}.1]$$

$$[0.32218535462608559291, 0.32218535462608559291, 0.32218535462608559291, 0.32218535462608559291]$$

Type: List Float



---

## PART IV

---

# Advanced Programming in AXIOM



---

# Interactive Programming

Programming in the interpreter is easy. So is the use of AXIOM's graphics facility. Both are rather flexible and allow you to use them for many interesting applications. However, both require learning some basic ideas and skills.

All graphics examples in the AXIOM Images section are either produced directly by interactive commands or by interpreter programs. Four of these programs are introduced here. By the end of this chapter you will know enough about graphics and programming in the interpreter to not only understand all these examples, but to tackle interesting and difficult problems on your own. Appendix F lists all the remaining commands and programs used to create these images.

## 10.1 Drawing Ribbons Interactively

We begin our discussion of interactive graphics with the creation of a useful facility: plotting ribbons of two-graphs in three-space. Suppose you want to draw the two-dimensional graphs of  $n$  functions  $f_i(x)$ ,  $1 \leq i \leq n$ , all over some fixed range of  $x$ . One approach is to create a two-dimensional graph for each one, then superpose one on top of the other. What you will more than likely get is a jumbled mess. Even if you make each function a different color, the result is likely to be confusing.

A better approach is to display each of the  $f_i(x)$  in three dimensions as a “ribbon” of some appropriate width along the  $y$ -direction, laying down each ribbon next to the previous one. A ribbon is simply a function of  $x$  and  $y$  depending only on  $x$ .

We illustrate this for  $f_i(x)$  defined as simple powers of  $x$  for  $x$  ranging between  $-1$  and  $1$ .

Draw the ribbon for  $z = x^2$ .

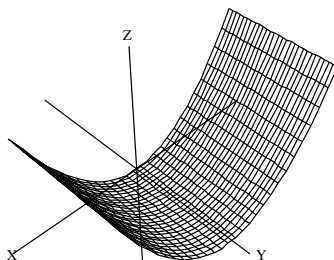
```
draw(x**2,x=-1..1,y=0..1)
```

```
Compiling function %B with type (DoubleFloat,  
    DoubleFloat) -> DoubleFloat  
Transmitting data...
```

```
ThreeDimensionalViewport: "x*x"
```

(1)

Type: ThreeDimensionalViewport



Now that was easy! What you get is a “wire-mesh” rendition of the ribbon. That’s fine for now. Notice that the mesh-size is small in both the  $x$  and the  $y$  directions. AXIOM normally computes points in both these directions. This is unnecessary. One step is all we need in the  $y$ -direction. To have AXIOM economize on  $y$ -points, we re-draw the ribbon with option `var2Steps == 1`.



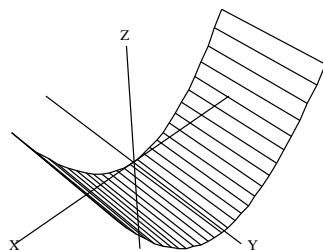
Re-draw the ribbon, but with option `var2Steps == 1` so that only 1 step is computed in the  $y$  direction.

```
vp := draw(x**2,x=-1..1,y=0..1,var2Steps==1)
Compiling function %D with type (DoubleFloat,
    DoubleFloat) -> DoubleFloat
Transmitting data...
```

ThreeDimensionalViewport: "`x*x`"

(2)

Type: ThreeDimensionalViewport



The operation has created a viewport, that is, a graphics window on your screen. We assigned the viewport to `vp` and now we manipulate its contents.

Graphs are objects, like numbers and algebraic expressions. You may want to do some experimenting with graphs. For example, say

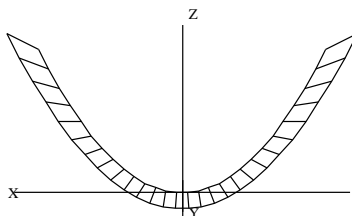
```
showRegion(vp, "on")
```

to put a bounding box around the ribbon. Try it! Issue `rotate(vp, -45, 90)` to rotate the figure  $-45$  longitudinal degrees and  $90$  latitudinal degrees.

Here is a different rotation. This turns the graph so you can view it along the  $y$ -axis.

```
rotate(vp, 0, -90)
```

Type: Void



There are many other things you can do. In fact, most everything you

can do interactively using the three-dimensional control panel (such as translating, zooming, resizing, coloring, perspective and lighting selections) can also be done directly by operations (see Chapter 7 for more details).

When you are done experimenting, say `reset(vp)` to restore the picture to its original position and settings.

Let's add another ribbon to our picture—one for  $x^3$ . Since  $y$  ranges from 0 to 1 for the first ribbon, now let  $y$  range from 1 to 2. This puts the second ribbon next to the first one.

How do you add a second ribbon to the viewport? One method is to extract the “space” component from the viewport using the operation **subspace**. You can think of the space component as the object inside the window (here, the ribbon). Let's call it **sp**. To add the second ribbon, you draw the second ribbon using the option `space == sp`.

Extract the space component of `vp`.

```
sp := subspace(vp)
3 – Space with 1 component
```

(4)

Type: ThreeSpace DoubleFloat

Add the ribbon for  $x^3$  alongside that for  $x^2$ .

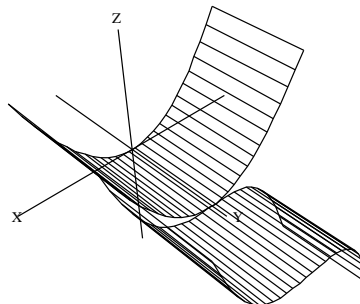
```
vp := draw(x**3,x=-1..1,y=1..2,var2Steps==1, space==sp)
```

```
Compiling function %F with type (DoubleFloat,
DoubleFloat) -> DoubleFloat
Transmitting data...
```

```
ThreeDimensionalViewport: "x**3"
```

(5)

Type: ThreeDimensionalViewport



Unless you moved the original viewport, the new viewport covers the old one. You might want to check that the old object is still there by moving the top window.

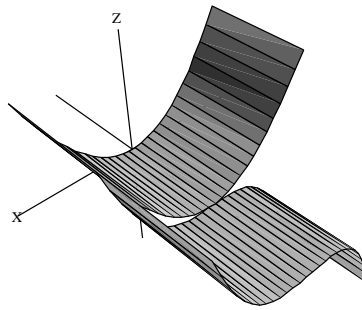
Let's show quadrilateral polygon outlines on the ribbons and then enclose

Show quadrilateral polygon outlines.

the ribbons in a box.

```
drawStyle(vp,"shade");outlineRender(vp,"on")
```

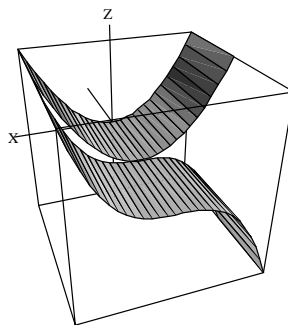
Type: Void



Enclose the ribbons in a box.

```
rotate(vp,20,-60); showRegion(vp,"on")
```

Type: Void



This process has become tedious! If we had to add two or three more ribbons, we would have to repeat the above steps several more times. It is time to write an interpreter program to help us take care of the details.

## 10.2 A Ribbon Program

The above approach creates a new viewport for each additional ribbon. A better approach is to build one object composed of all ribbons before creating a viewport. To do this, use **makeObject** rather than **draw**. The operations have similar formats, but **draw** returns a viewport and **makeObject** returns a space object.

We now create a function **drawRibbons** of two arguments: **flist**, a list of formulas for the ribbons you want to draw, and **xrange**, the range over which you want them drawn. Using this function, you can just say

```
drawRibbons([x**2, x**3], x=-1..1)
```

to do all of the work required in the last section. Here is the **drawRibbons** program. Invoke your favorite editor and create a file called **ribbon.input** containing the following program.

Create empty space <b>sp</b> .	<code>drawRibbons(flist, xrange) ==</code>	1
The initial ribbon position.	<code>sp := createThreeSpace()</code>	2
For each function <b>f</b> ,	<code>y0 := 0</code>	3
create and add a ribbon	<code>for f in flist repeat</code>	4
for <b>f</b> to the space <b>sp</b> .	<code>    makeObject(f, xrange, y=y0..y0+1,</code>	5
The next ribbon position.	<code>        space==sp, var2Steps == 1)</code>	6
Create viewport.	<code>y0 := y0 + 1</code>	7
Select shading style.	<code>vp := makeViewport3D(sp, "Ribbons")</code>	8
Show polygon outlines.	<code>drawStyle(vp, "shade")</code>	9
Enclose in a box.	<code>outlineRender(vp, "on")</code>	10
The number of ribbons	<code>showRegion(vp, "on")</code>	11
Zoom in x- and z-directions.	<code>n := # flist</code>	12
Change the angle of view.	<code>zoom(vp, n, 1, n)</code>	13
Return the viewport.	<code>rotate(vp, 0, 75)</code>	14
	<code>vp</code>	15

Figure 10.1: The first **drawRibbons** function.

Here are some remarks on the syntax used in the **drawRibbons** function (consult Chapter 6 for more details). Unlike most other programming languages which use semicolons, parentheses, or *begin-end* brackets to delineate the structure of programs, the structure of an AXIOM program is determined by indentation. The first line of the function definition always begins in column 1. All other lines of the function are indented with respect to the first line and form a *pile* (see Section 5.2 on page 153).

The definition of **drawRibbons** consists of a pile of expressions to be executed one after another. Each expression of the pile is indented at the same level. Lines 4-7 designate one single expression: since lines 5-7 are indented with respect to the others, these lines are treated as a continuation of line 4. Also since lines 5 and 7 have the same indentation level, these lines designate a pile within the outer pile.

The last line of a pile usually gives the value returned by the pile. Here it is also the value returned by the function. AXIOM knows this is the last line of the function because it is the last line of the file. In other cases, a new expression beginning in column one signals the end of a function.

The line `drawStyle(vp,"shade")` is given after the viewport has been created to select the draw style. We have also used the **zoom** option. Without the zoom, the viewport region would be scaled equally in all three coordinate directions.

Let's try the function **drawRibbons**. First you must read the file to give AXIOM the function definition.

Read the input file.

```
)read ribbon
--Copyright The Numerical Algorithms Group Limited
1994.
--the first attempt
drawRibbons(flist,xrange) ==
  sp := createThreeSpace()
  y0 := 0
  for f in flist repeat
    makeObject(f,xrange,y=y0..y0+1,
      space==sp, var2Steps ==1)
  y0 := y0+1
  vp:=makeViewport3D(sp,"Ribbons")
  drawStyle(vp,"shade")
  outlineRender(vp,"on")
  showRegion(vp,"on")
  n := # flist
  zoom(vp,n,1,n)
  rotate(vp,0,75)
  vp
```

Type: Void

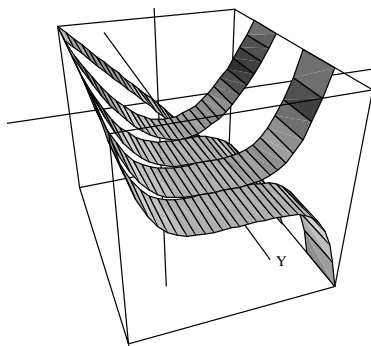
Draw ribbons for  $x, x^2, \dots, x^5$   
for  $-1 \leq x \leq 1$

```
drawRibbons([x**i for i in 1..5],x=-1..1)
Compiling function drawRibbons with type (List
Polynomial Integer,SegmentBinding Integer) ->
ThreeDimensionalViewport
Compiling function %H with type (DoubleFloat,
DoubleFloat) -> DoubleFloat
Compiling function %J with type (DoubleFloat,
DoubleFloat) -> DoubleFloat
Compiling function %L with type (DoubleFloat,
DoubleFloat) -> DoubleFloat
Compiling function %N with type (DoubleFloat,
DoubleFloat) -> DoubleFloat
Compiling function %P with type (DoubleFloat,
DoubleFloat) -> DoubleFloat
Transmitting data...
```

ThreeDimensionalViewport: "Ribbons"

(2)

Type: ThreeDimensionalViewport



## 10.3 Coloring and Positioning Ribbons

---

Before leaving the ribbon example, we make two improvements. Normally, the color given to each point in the space is a function of its height within a bounding box. The points at the bottom of the box are red, those at the top are purple.

To change the normal coloring, you can give an option `colorFunction == function`. When AXIOM goes about displaying the data, it determines the range of colors used for all points within the box. AXIOM then distributes these numbers uniformly over the number of hues. Here we use the simple color function  $(x, y) \mapsto i$  for the  $i^{\text{th}}$  ribbon.

Also, we add an argument `yrange` so you can give the range of  $y$  occupied by the ribbons. For example, if the `yrange` is given as `y=0..1` and there are 5 ribbons to be displayed, each ribbon would have width 0.2 and would appear in the range  $0 \leq y \leq 1$ .

Refer to lines 4-9. Line 4 assigns to `yVar` the variable part of the `yrange` (after all, it need not be  $y$ ). Suppose that `yrange` is given as `t = a..b` where `a` and `b` have numerical values. Then line 5 assigns the value of `a` to the variable `y0`. Line 6 computes the width of the ribbon by dividing the difference of `a` and `b` by the number, `num`, of ribbons. The result is assigned to the variable `width`. Note that in the for-loop in line 7, we are iterating in parallel; it is not a nested loop.

Create empty space `sp`.

The number of ribbons.

The ribbon variable.

The first ribbon coordinate.

The width of a ribbon.

For each function `f`,

create and add ribbon to  
`sp` of a different color.

The next ribbon coordinate.

Create viewport.

Select shading style.

Show polygon outlines.

Enclose in a box.

Return the viewport.

```
drawRibbons(flist, xrange, yrange) ==
  sp := createThreeSpace()
  num := # flist
  yVar := variable yrange
  y0:Float := lo segment yrange
  width:Float := (hi segment yrange - y0)/num
  for f in flist for color in 1..num repeat
    makeObject(f, xrange, yVar = y0..y0+width,
      var2Steps == 1, colorFunction == (x,y) +-> color,
      space == sp)
  y0 := y0 + width
  vp := makeViewport3D(sp, "Ribbons")
  drawStyle(vp, "shade")
  outlineRender(vp, "on")
  showRegion(vp, "on")
  vp
```

Figure 10.2: The final `drawRibbons` function.

## 10.4 Points, Lines, and Curves

---

What you have seen so far is a high-level program using the graphics facility. We now turn to the more basic notions of points, lines, and curves in three-dimensional graphs. These facilities use small floats (objects of type `DoubleFloat`) for data. Let us first give names to the small float values 0 and 1.

The small float 0.

```
zero := 0.0@DFLOAT
0.0
(1)
```

Type: DoubleFloat

The small float 1.

```
one := 1.0@DFLOAT
1.0
(2)
```

Type: DoubleFloat

The “@” sign means “of the type.” Thus `zero` is 0.0 of the type `DoubleFloat`. You can also say `0.0::DFLOAT`.

Points can have four small float components:  $x, y, z$  coordinates and an optional color. A “curve” is simply a list of points connected by straight line segments.

Create the point `origin` with color `zero`, that is, the lowest color on the color map.

```
origin := point [zero,zero,zero,zero]
[0.0, 0.0, 0.0, 0.0]
(3)
```

Type: Point DoubleFloat

Create the point `unit` with color `zero`.

```
unit := point [one,one,one,zero]
[1.0, 1.0, 1.0, 0.0]
(4)
```

Type: Point DoubleFloat

Create the curve (well, here, a line) from `origin` to `unit`.

```
line := [origin, unit]
[[0.0, 0.0, 0.0, 0.0], [1.0, 1.0, 1.0, 0.0]]
(5)
```

Type: List Point DoubleFloat

We make this line segment into an arrow by adding an arrowhead. The arrowhead extends to, say, `p3` on the left, and to, say, `p4` on the right. To describe an arrow, you tell AXIOM to draw the two curves `[p1, p2, p3]` and `[p2, p4]`. We also decide through experimentation on values for `arrowScale`, the ratio of the size of the arrowhead to the stem of the arrow, and `arrowAngle`, the angle between the arrowhead and the arrow.

Invoke your favorite editor and create an input file called `arrows.input`. This input file first defines the values of `arrowAngle` and `arrowScale`, then defines the function `makeArrow(p1,p2)` to draw an arrow from point  $p_1$  to  $p_2$ .



The angle of the arrowhead.	<code>arrowAngle := %pi-%pi/10.0@DFLOAT</code>	1
The size of the arrowhead relative to the stem.	<code>arrowScale := 0.2@DFLOAT</code>	2
		3
The arrow.	<code>makeArrow(p1, p2) ==</code>	4
The length of the arrowhead.	<code>delta := p2 - p1</code>	5
The angle from the x-axis	<code>len := arrowScale * length delta</code>	6
The x-coord of left endpoint.	<code>theta := atan(delta.1, delta.2)</code>	7
The y-coord of left endpoint.	<code>c1 := len*cos(theta + arrowAngle)</code>	8
The x-coord of right endpoint.	<code>s1 := len*sin(theta + arrowAngle)</code>	9
The y-coord of right endpoint.	<code>c2 := len*cos(theta - arrowAngle)</code>	10
The z-coord of both endpoints.	<code>s2 := len*sin(theta - arrowAngle)</code>	11
The left endpoint of head.	<code>z := p2.3*(1 - arrowScale)</code>	12
The right endpoint of head.	<code>p3 := point [p2.1 + c1, p2.2 + s1, z, p2.4]</code>	13
The arrow as a list of curves.	<code>p4 := point [p2.1 + c2, p2.2 + s2, z, p2.4]</code>	14
	<code>[[p1, p2, p3], [p2, p4]]</code>	15

Read the file and then create an arrow from the point `origin` to the point `unit`.

Read the input file defining  
**makeArrow.**

```
)read arrows
--Copyright The Numerical Algorithms Group Limited
1991.
```

```
arrowAngle:=%pi-%pi/10.0@SF
```

```
2.8274333882308138
```

(6)  
Type: DoubleFloat

```
arrowScale:=0.2@SF
```

```
0.20000000000000001
```

(7)  
Type: DoubleFloat

```
makeArrow(p1,p2) ==
delta      :=p2 -p1
len        := arrowScale * length delta
theta      := atan(delta.1, delta.2)
c1:= len*cos(theta+arrowAngle)
s1:= len*sin(theta+arrowAngle)
c2:= len*cos(theta-arrowAngle)
s2:= len*sin(theta-arrowAngle)
z:= p2.3*(1-arrowScale)
p3:=point[p2.1+c1,p2.2+s1,z,p2.4]
p4:=point[p2.1+c2,p2.2+s2,z,p2.4]
[[p1,p2,p3],[p2,p4]]
```

Type: Void

Construct the arrow (a list of two curves).

```
arrow := makeArrow(origin,unit)
```

Compiling function makeArrow with type (Point DoubleFloat,Point DoubleFloat) -> List List Point DoubleFloat

$$\left[ \begin{array}{c} \left[ \begin{array}{c} [0.0, 0.0, 0.0, 0.0], [1.0, 1.0, 1.0, 0.0], \\ [0.69134628604607973, 0.842733077659504, \\ 0.80000000000000004, 0.0] \end{array} \right], \\ \left[ \begin{array}{c} [1.0, 1.0, 1.0, 0.0], \\ [0.842733077659504, 0.69134628604607973, \\ 0.80000000000000004, 0.0] \end{array} \right] \end{array} \right]$$

(9)

Type: List List Point DoubleFloat

Create an empty object **sp** of type **ThreeSpace**.

```
sp := createThreeSpace()
```

3 – Space *with* 0 components (10)

Type: ThreeSpace DoubleFloat

Add each curve of the arrow to the space **sp**.

```
for a in arrow repeat sp := curve(sp,a)
```

Type: Void

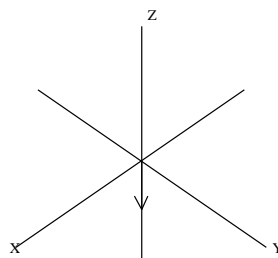
Create a three-dimensional viewport containing that space.

```
vp := makeViewport3D(sp,"Arrow")
```

Transmitting data...

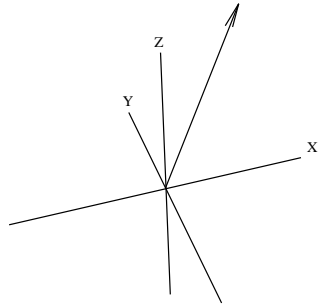
ThreeDimensionalViewport: "Arrow" (12)

Type: ThreeDimensionalViewport



Here is a better viewing angle.     `rotate(vp,200,-60)`

Type: Void



## 10.5 A Bouquet of Arrows

---

Let's draw a "bouquet" of arrows. Each arrow is identical. The arrow-heads are uniformly placed on a circle parallel to the  $xy$ -plane. Thus the position of each arrow differs only by the angle  $\theta$ ,  $0 \leq \theta < 2\pi$ , between the arrow and the  $x$ -axis on the  $xy$ -plane.

Our bouquet is rather special: each arrow has a different color (which won't be evident here, unfortunately). This is arranged by letting the color of each successive arrow be denoted by  $\theta$ . In this way, the color of arrows ranges from red to green to violet. Here is a program to draw a bouquet of  $n$  arrows.

The initial angle.	<code>drawBouquet(n,title) ==</code>	1
Create empty space <code>sp</code> .	<code>angle := 0.0@DFLOAT</code>	2
For each index $i$ , create:	<code>sp := createThreeSpace()</code>	3
—the point at base of arrow;	<code>for i in 0..n-1 repeat</code>	4
—the point at tip of arrow;	<code>start:=point[0.0@DFLOAT,0.0@DFLOAT,0.0@DFLOAT,angle]</code>	5
—the $i$ th arrow.	<code>end :=point[cos angle, sin angle, 1.0@DFLOAT, angle]</code>	6
For each arrow component,	<code>arrow := makeArrow(start,end)</code>	7
add the component to <code>sp</code> .	<code>for a in makeArrow(start,end) repeat</code>	8
The next angle.	<code>curve(sp,a)</code>	9
Create the viewport from <code>sp</code> .	<code>angle := angle + 2*%pi/n</code>	10
	<code>makeViewport3D(sp,title)</code>	11
Read the input file.	<code>)read bouquet</code>	
	<code>--Copyright The Numerical Algorithms Group Limited</code>	
	<code>1994.</code>	
relative size of the arrow head compared to the length of the arrow	<code>arrowScale := 0.2@DFLOAT</code>	
	<code>0.20000000000000001</code>	(1)
		Type: DoubleFloat
angle of the arrow head	<code>arrowAngle := %pi-%pi/10.0@DFLOAT</code>	
	<code>2.8274333882308138</code>	(2)
		Type: DoubleFloat

Add an arrow head to a line segment, which starts at 'p1', ends at 'p2', has length 'len', and angle 'arg'. We pass 'len' and 'arg' as arguments since they were already computed by the calling program

```
makeArrow(p1, p2) ==
  delta := p2 - p1
  len := arrowScale * length delta
  theta := atan(delta.1, delta.2)
  c1 := len * cos(theta + arrowAngle)
  s1 := len * sin(theta + arrowAngle)
  c2 := len * cos(theta - arrowAngle)
  s2 := len * sin(theta - arrowAngle)
  z := p2.3*(1 - arrowScale)
  p3 := point [p2.1 + c1, p2.2 + s1, z, p2.4]
  p4 := point [p2.1 + c2, p2.2 + s2, z, p2.4]
  [[p1, p2, p3], [p2, p4]]
```

Type: Void

```
drawBouquet(n,title) ==
  angle := 0.0@DFLOAT
  sp := create3Space()$ThreeSpace(DFLOAT)
  for i in 0..n-1 repeat
    start := point
    [0.0@DFLOAT,0.0@DFLOAT,0.0@DFLOAT,angle]
    end := point [cos angle, sin angle, 1.0@DFLOAT, an-
    gle]
    arrow := makeArrow(start, end)
    for a in arrow repeat curve(sp,a)
    angle := angle + 2*%pi/n
  makeViewport3D(sp,title)$VIEW3D
```

A bouquet of a dozen arrows.

```
drawBouquet(12,"A Dozen Arrows")
```

Type: Void

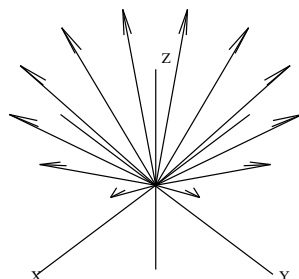
```
Compiling function makeArrow with type (Point
  DoubleFloat,Point DoubleFloat) -> List List Point
  DoubleFloat
```

```
+++ |*2;makeArrow;1;initial| redefined
Compiling function drawBouquet with type (
  PositiveInteger,String) ->
  ThreeDimensionalViewport
Transmitting data...
```

```
ThreeDimensionalViewport: "A Dozen Arrows"
```

(5)

Type: ThreeDimensionalViewport



## 10.6 Drawing Complex Vector Fields

We now put our arrows to good use drawing complex vector fields. These vector fields give a representation of complex-valued functions of complex variables. Consider a Cartesian coordinate grid of points  $(x, y)$  in the plane, and some complex-valued function  $f$  defined on this grid. At every point on this grid, compute the value of  $f(x + iy)$  and call it  $z$ . Since  $z$  has both a real and imaginary value for a given  $(x, y)$  grid point, there are four dimensions to plot. What do we do? We represent the values of  $z$  by arrows planted at each grid point. Each arrow represents the value of  $z$  in polar coordinates  $(r, \theta)$ . The length of the arrow is proportional to  $r$ . Its direction is given by  $\theta$ .

The code for drawing vector fields is in the file **vectors.input**. We discuss its contents from top to bottom.

Before showing you the code, we have two small matters to take care of. First, what if the function has large spikes, say, ones that go off to infinity? We define a variable `clipValue` for this purpose. When `r` exceeds the value of `clipValue`, then the value of `clipValue` is used instead of that for `r`. For convenience, we define a function `clipFun(x)` which uses `clipValue` to “clip” the value of `x`.

Maximum value allowed.

---

```
clipValue : DFLOAT := 6                                1
clipFun(x) == min(max(x, -clipValue), clipValue)        2
```

---

Notice that we identify `clipValue` as a small float but do not declare the type of the function **clipFun**. As it turns out, **clipFun** is called with a small float value. This declaration ensures that **clipFun** never does a conversion when it is called.

The second matter concerns the possible “poles” of a function, the actual points where the spikes have infinite values. AXIOM uses normal DoubleFloat arithmetic which does not directly handle infinite values. If your function has poles, you must adjust your step size to avoid landing directly on them (AXIOM calls **error** when asked to divide a value by 0, for example).

We set the variables `realSteps` and `imagSteps` to hold the number of steps taken in the real and imaginary directions, respectively. Most examples will have ranges centered around the origin. To avoid a pole at the origin, the number of points is taken to be odd.

Number of real steps.

Number of imaginary steps.

---

```
realSteps: INT := 25                                    3
imagSteps: INT := 25                                    4
)read arrows                                            5
```

---

Now define the function **drawComplexVectorField** to draw the arrows.

It is good practice to declare the type of the main function in the file. This one declaration is usually sufficient to ensure that other lower-level functions are compiled with the correct types.

---

C := Complex DoubleFloat	6
S := Segment DoubleFloat	7
drawComplexVectorField: (C -> C, S, S) -> VIEW3D	8

---

The first argument is a function mapping complex small floats into complex small floats. The second and third arguments give the range of real and imaginary values as segments like `a..b`. The result is a three-dimensional viewport. Here is the full function definition:

	<code>drawComplexVectorField(f, realRange, imagRange) ==</code>	9
The real step size.	<code>delReal := (hi(realRange)-lo(realRange))/realSteps</code>	10
The imaginary step size.	<code>delImag := (hi(imagRange)-lo(imagRange))/imagSteps</code>	11
Create empty space <code>sp</code> .	<code>sp := createThreeSpace()</code>	12
The initial real value.	<code>real := lo(realRange)</code>	13
Begin real iteration.	<code>for i in 1..realSteps+1 repeat</code>	14
The initial imaginary value.	<code>imag := lo(imagRange)</code>	15
Begin imaginary iteration.	<code>for j in 1..imagSteps+1 repeat</code>	16
The value of <code>f</code> at the point.	<code>z := f complex(real,imag)</code>	17
The direction of the arrow.	<code>arg := argument z</code>	18
The length of the arrow.	<code>len := clipFun sqrt norm z</code>	19
The base point of the arrow.	<code>p1 := point [real, imag, 0.0@DFLOAT, arg]</code>	20
The scaled length of the arrow.	<code>scaleLen := delReal * len</code>	21
The tip point of the arrow.	<code>p2 := point [p1.1 + scaleLen*cos(arg),</code>	22
	<code>p1.2 + scaleLen*sin(arg), 0.0@DFLOAT, arg]</code>	23
Create the arrow.	<code>arrow := makeArrow(p1, p2)</code>	24
Add arrow to the space <code>sp</code> .	<code>for a in arrow repeat curve(sp, a)</code>	25
The next imaginary value.	<code>imag := imag + delImag</code>	26
The next real value.	<code>real := real + delReal</code>	27
Draw it!	<code>makeViewport3D(sp, "Complex Vector Field")</code>	28

---

As a first example, let us draw `f(z) == sin(z)`. There is no need to create a user function: just pass the `sin` from Complex DoubleFloat.

Read the file.

```
)read vectors
--Copyright The Numerical Algorithms Group Limited
1991.

)r arrows
--Copyright The Numerical Algorithms Group Limited
1991.

arrowAngle:=%pi-%pi/10.0@SF
```

2.8274333882308138 (1)  
Type: DoubleFloat

```

arrowScale:=0.2@SF
0.20000000000000001

```

(2)

Type: DoubleFloat

```

makeArrow(p1,p2) ==
  delta      :=p2 -p1
  len        := arrowScale * length delta
  theta      := atan(delta.1, delta.2)
  c1:= len*cos(theta+arrowAngle)
  s1:= len*sin(theta+arrowAngle)
  c2:= len*cos(theta-arrowAngle)
  s2:= len*sin(theta-arrowAngle)
  z:= p2.3*(1-arrowScale)
  p3:=point[p2.1+c1,p2.2+s1,z,p2.4]
  p4:=point[p2.1+c2,p2.2+s2,z,p2.4]
  [[p1,p2,p3],[p2,p4]]

```

Type: Void

```

clipValue :SF := 6
6.0

```

(4)

Type: DoubleFloat

```

clipFun(x) == min(max(x,-clipValue),clipValue)

```

Type: Void

```

realSteps :INT := 25
25

```

(6)

Type: Integer

```

imagSteps :INT := 25
25

```

(7)

Type: Integer

```

C := Complex SF
Complex DoubleFloat

```

(8)

Type: Domain

```

S := Segment SF
Segment DoubleFloat

```

(9)

Type: Domain



```
drawComplexVectorField : (C -> C, S, S) -> VIEW3D
```

Type: Void

```
drawComplexVectorField(f, realRange, imagRange) ==
  delReal := (hi realRange - lo realRange)/realSteps
  delImag := (hi imagRange - lo imagRange)/imagSteps
  sp := create3Space()$ThreeSpace SF
  real := lo realRange
  for i in 1..realSteps + 1 repeat
    imag := lo imagRange
    for j in 1..imagSteps + 1 repeat
      z := f complex(real, imag)
      arg := argument z
      len := clipFun sqrt norm z
      p1 := point[real, imag, 0.0@SF, arg]
      scaleLen := delReal * len
      p2 := point[p1.1 + scaleLen * cos(arg),
                  p1.2 + scaleLen * sin(arg), 0.0@SF,
arg]
      arrow := makeArrow(p1, p2)
      for a in arrow repeat curve(sp, a)
      imag := imag + delImag
      real := real + delReal
      makeViewport3D(sp, "Complex Vector Field")$VIEW3D
```

Type: Void

```
drawComplex : (C->C, S, S) -> VIEW3D
```

Type: Void

```

drawComplex(f, realRange, imagRange) ==
  deltaReal :SF := (hi realRange - lo
realRange)/realSteps
  deltaImag :SF := (hi imagRange - lo
imagRange)/imagSteps
  llp:List List Point SF := []
  real :SF := lo realRange
  for i in 1..realSteps + 1 repeat
    imag :SF := lo imagRange
    lp := []$(List Point SF)
    for j in 1..imagSteps + 1 repeat
      z :COMPLEX SF := f(complex(real, imag))
      pt :Point SF := point[real, imag, clipFun sqrt norm
z, argument z]
      lp := cons(pt, lp)
      imag := imag + deltaImag
      real := real + deltaReal
    llp := cons(reverse! lp, llp)
    lp := reverse! lp
  makeViewport3D(mesh(llp), "Complex Function")$VIEW3D

```

Type: Void

Draw the complex vector field of  
sin(x).

```

drawComplexVectorField(sin,-2..2,-2..2)
Compiling function clipFun with type DoubleFloat ->
DoubleFloat
Compiling function makeArrow with type (Point
DoubleFloat,Point DoubleFloat) -> List List Point
DoubleFloat

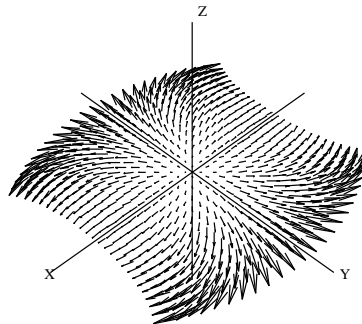
+++ |*2;makeArrow;1;initial| redefined
Compiling function drawComplexVectorField with type (
(Complex DoubleFloat -> Complex DoubleFloat),
Segment DoubleFloat,Segment DoubleFloat) ->
ThreeDimensionalViewport
Transmitting data...

```

ThreeDimensionalViewport: "Complex Vector Field"

(14)

Type: ThreeDimensionalViewport



## 10.7 Drawing Complex Functions

Here is another way to graph a complex function of complex arguments. For each complex value  $z$ , compute  $f(z)$ , again expressing the value in polar coordinates  $(r, \theta)$ . We draw the complex valued function, again considering the  $(x, y)$ -plane as the complex plane, using  $r$  as the height (or  $z$ -coordinate) and  $\theta$  as the color. This is a standard plot—we learned how to do this in Chapter 7—but here we write a new program to illustrate the creation of polygon meshes, or grids.

Call this function **drawComplex**. It displays the points using the “mesh” of points. The function definition is in three parts.

The first part.  
The real step size.  
The imaginary step size.  
Initial list of list of points `llp`.

---

```
drawComplex: (C -> C, S, S) -> VIEW3D      1
drawComplex(f, realRange, imagRange) ==    2
  delReal := (hi(realRange)-lo(realRange))/realSteps  3
  delImag := (hi(imagRange)-lo(imagRange))/imagSteps  4
  llp:List List Point DFLOAT := []          5
```

---

Variables `delReal` and `delImag` give the step sizes along the real and imaginary directions as computed by the values of the global variables `realSteps` and `imagSteps`. The mesh is represented by a list of lists of points `llp`, initially empty. Now `[]` alone is ambiguous, so to set this initial value you have to tell AXIOM what type of empty list it is. Next comes the loop which builds `llp`.

The initial real value.  
Begin real iteration.  
The initial imaginary value.  
The initial list of points `lp`.  
Begin imaginary iteration.  
The value of  $f$  at the point.  
Create a point.  
  
Add the point to `lp`.  
The next imaginary value.  
The next real value.  
Add `lp` to `llp`.

---

```
  real := lo(realRange)                      6
  for i in 1..realSteps+1 repeat              7
    imag := lo(imagRange)                    8
    lp := []$(List Point DFLOAT)             9
    for j in 1..imagSteps+1 repeat           10
      z := f complex(real,imag)              11
      pt := point [real,imag, clipFun sqrt norm z, 12
                  argument z]                13
      lp := cons(pt,lp)                      14
      imag := imag + delImag                 15
      real := real + delReal                 16
      llp := cons(lp, llp)                   17
```

---

The code consists of both an inner and outer loop. Each pass through the inner loop adds one list `lp` of points to the list of lists of points `llp`. The elements of `lp` are collected in reverse order.

Create a mesh and display.

---

```
  makeViewport3D(mesh(llp), "Complex Function") 18
```

---

The operation **mesh** then creates an object of type `ThreeSpace(DoubleFloat)` from the list of lists of points. This is then passed to **makeViewport3D** to display the image.

Now add this function directly to your **vectors.input** file and re-read the

file using `)read vectors`. We try **drawComplex** using a user-defined function `f`.

Read the file.

```
)read vectors
--Copyright The Numerical Algorithms Group Limited
1991.

)r arrows
--Copyright The Numerical Algorithms Group Limited
1991.
```

```
arrowAngle:=%pi-%pi/10.0@SF
```

```
2.8274333882308138 (1)
```

Type: DoubleFloat

```
arrowScale:=0.2@SF
```

```
0.20000000000000001 (2)
```

Type: DoubleFloat

```
makeArrow(p1,p2) ==
  delta      :=p2 -p1
  len        := arrowScale * length delta
  theta := atan(delta.1, delta.2)
  c1:= len*cos(theta+arrowAngle)
  s1:= len*sin(theta+arrowAngle)
  c2:= len*cos(theta-arrowAngle)
  s2:= len*sin(theta-arrowAngle)
  z:= p2.3*(1-arrowScale)
  p3:=point[p2.1+c1,p2.2+s1,z,p2.4]
  p4:=point[p2.1+c2,p2.2+s2,z,p2.4]
  [[p1,p2,p3],[p2,p4]]
```

Type: Void

```
clipValue :SF := 6
```

```
6.0 (4)
```

Type: DoubleFloat

```
clipFun(x) == min(max(x,-clipValue),clipValue)
```

Type: Void

```
realSteps :INT := 25
```

```
25 (6)
```

Type: Integer

```

imagSteps :INT := 25
25
Type: Integer

C := Complex SF
Complex DoubleFloat
Type: Domain

S := Segment SF
Segment DoubleFloat
Type: Domain

drawComplexVectorField : (C -> C, S, S) -> VIEW3D
Type: Void

drawComplexVectorField(f, realRange, imagRange) ==
  delReal := (hi realRange - lo realRange)/realSteps
  delImag := (hi imagRange - lo imagRange)/imagSteps
  sp := create3Space()$ThreeSpace SF
  real := lo realRange
  for i in 1..realSteps + 1 repeat
    imag := lo imagRange
    for j in 1..imagSteps + 1 repeat
      z := f complex(real, imag)
      arg := argument z
      len := clipFun sqrt norm z
      p1 := point[real, imag, 0.0@SF, arg]
      scaleLen := delReal * len
      p2 := point[p1.1 + scaleLen * cos(arg),
                  p1.2 + scaleLen * sin(arg), 0.0@SF,
arg]
      arrow := makeArrow(p1, p2)
      for a in arrow repeat curve(sp, a)
      imag := imag + delImag
      real := real + delReal
      makeViewport3D(sp, "Complex Vector Field")$VIEW3D
Type: Void

drawComplex : (C->C, S, S) -> VIEW3D
Type: Void

```

```

drawComplex(f, realRange, imagRange) ==
  deltaReal :SF := (hi realRange - lo
realRange)/realSteps
  deltaImag :SF := (hi imagRange - lo
imagRange)/imagSteps
  llp:List List Point SF := []
  real :SF := lo realRange
  for i in 1..realSteps + 1 repeat
    imag :SF := lo imagRange
    lp := []$(List Point SF)
    for j in 1..imagSteps + 1 repeat
      z :COMPLEX SF := f(complex(real, imag))
      pt :Point SF := point[real, imag, clipFun sqrt norm
z, argument z]
      lp := cons(pt, lp)
      imag := imag + deltaImag
      real := real + deltaReal
    llp := cons(reverse! lp, llp)
    llp := reverse! llp
  makeViewport3D(mesh(llp), "Complex Function")$VIEW3D

```

This one has a pole at  $z = 0$ .

$f(z) == \exp(1/z)$

Type: Void

Draw it with an odd number of steps to avoid the pole.

```

drawComplex(f, -2..2, -2..2)
Compiling function f with type Complex DoubleFloat
-> Complex DoubleFloat
Compiling function clipFun with type DoubleFloat ->
DoubleFloat

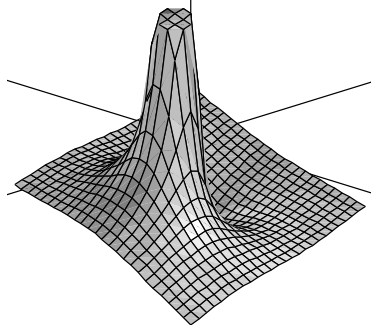
+++ |*1;clipFun;1;initial| redefined
Compiling function drawComplex with type ((Complex
DoubleFloat -> Complex DoubleFloat),Segment
DoubleFloat,Segment DoubleFloat) ->
ThreeDimensionalViewport
Transmitting data...

```

ThreeDimensionalViewport: "Complex Function"

(15)

Type: ThreeDimensionalViewport



## 10.8 Functions Producing Functions

---

In Section 6.14 on page 207, you learned how to use the operation **function** to create a function from symbolic formulas. Here we introduce a similar operation which not only creates functions, but functions from functions.

The facility we need is provided by the package `MakeUnaryCompiledFunction(E,S,T)`. This package produces a unary (one-argument) compiled function from some symbolic data generated by a previous computation.<sup>1</sup> The `E` tells where the symbolic data comes from; the `S` and `T` give AXIOM the source and target type of the function, respectively. The compiled function produced has type  $S \rightarrow T$ . To produce a compiled function with definition `p(x) == expr`, call `compiledFunction(expr, x)` from this package. The function you get has no name. You must assign the function to the variable `p` to give it that name.

Do some computation.

$$(x+1/3)^{**5}$$

$$x^5 + \frac{5}{3}x^4 + \frac{10}{9}x^3 + \frac{10}{27}x^2 + \frac{5}{81}x + \frac{1}{243} \quad (1)$$

Type: Polynomial Fraction Integer

Convert this to an anonymous function of `x`. Assign it to the variable `p` to give the function a name.

```
p := compiledFunction(%,x)$MakeUnaryCompiledFunction(POLY
  FRAC INT,DFLOAT,DFLOAT)
Compiling function %Q with type DoubleFloat ->
  DoubleFloat

theMap (...)
```

(2)

Type: (DoubleFloat → DoubleFloat)

Apply the function.

```
p(sin(1.3))
3.668751115057229
```

(3)

Type: DoubleFloat

For a more sophisticated application, read on.

---

<sup>1</sup>MakeBinaryCompiledFunction is available for binary functions.

## 10.9 Automatic Newton Iteration Formulas

We resume our continuing saga of arrows and complex functions. Suppose we want to investigate the behavior of Newton's iteration function in the complex plane. Given a function  $f$ , we want to find the complex values  $z$  such that  $f(z) = 0$ .

The first step is to produce a Newton iteration formula for a given  $f$ :  $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ . We represent this formula by a function  $g$  that performs the computation on the right-hand side, that is,  $x_{n+1} = g(x_n)$ .

The type Expression Integer (abbreviated `EXPR INT`) is used to represent general symbolic expressions in AXIOM. To make our facility as general as possible, we assume  $f$  has this type. Given  $f$ , we want to produce a Newton iteration function  $g$  which, given a complex point  $x_n$ , delivers the next Newton iteration point  $x_{n+1}$ .

This time we write an input file called **newton.input**. We need to import `MakeUnaryCompiledFunction` (discussed in the last section), call it with appropriate types, and then define the function `newtonStep` which references it. Here is the function `newtonStep`:

The complex numbers.  
Package for making functions.

Newton's iteration function.  
Function for  $f$ .  
Function for  $f'$ .  
Return the iterator function.

Turn an expression  $f$  into a  
function.

Create an  $n$ th derivative  
function.

Returns the variable in  $f$ .  
The list of variables.  
The number of variables.

Return a dummy variable.

---

```

C := Complex DoubleFloat
complexFunPack:=MakeUnaryCompiledFunction(EXPR INT,C,C)
1
2
3
newtonStep(f) ==
4
    fun := complexNumericFunction f
5
    deriv := complexDerivativeFunction(f,1)
6
    (x:C):C +->
7
        x - fun(x)/deriv(x)
8
9
complexNumericFunction f ==
10
    v := theVariableIn f
11
    compiledFunction(f, v)$complexFunPack
12
13
complexDerivativeFunction(f,n) ==
14
    v := theVariableIn f
15
    df := D(f,v,n)
16
    compiledFunction(df, v)$complexFunPack
17
18
theVariableIn f ==
19
    v1 := variables f
20
    nv := # v1
21
    nv > 1 => error "Expression is not univariate."
22
    nv = 0 => 'x
23
    first v1
24

```

---

Do you see what is going on here? A formula  $f$  is passed into the function **newtonStep**. First, the function turns  $f$  into a compiled program mapping complex numbers into complex numbers. Next, it does the same thing for the derivative of  $f$ . Finally, it returns a function which computes a single step of Newton's iteration.



The function **complexNumericFunction** extracts the variable from the expression **f** and then turns **f** into a function which maps complex numbers into complex numbers. The function **complexDerivativeFunction** does the same thing for the derivative of **f**. The function **theVariableIn** extracts the variable from the expression **f**, calling the function **error** if **f** has more than one variable. It returns the dummy variable **x** if **f** has no variables.

Let's now apply **newtonStep** to the formula for computing cube roots of two.

Read the input file with the definitions.

```
)read newton
--Copyright The Numerical Algorithms Group Limited
1994.
```

Newton's Iteration function **newtonStep(f)** returns a newton's iteration function for the expression **f**.

```
newtonStep(f) ==
  fun := complexNumericFunction f
  deriv := complexDerivativeFunction(f,1)
  (b:Complex DoubleFloat):Complex DoubleFloat +->
  b - fun(b)/deriv(b)
```

Type: Void

create complex numeric functions from an expression

```
complexFunPack := MakeUnaryCompiledFunction(EXPR INT,
  Complex DoubleFloat, Complex DoubleFloat)
MakeUnaryCompiledFunction ( Expression Integer , Complex DoubleFloat
, Complex DoubleFloat ) (2)
```

Type: Domain

create a complex numeric function from an expression

```
complexNumericFunction x ==
  v := theVariable x
  compiledFunction(x, v)$complexFunPack
```

Type: Void

create a complex numeric derivative function from an expression

```
complexDerivativeFunction(x,n) ==
  v := theVariable x
  df := differentiate(x,v,n)
  compiledFunction(df, v)$complexFunPack
```

Type: Void

return the unique variable in x,  
or an error if it is multivariate

```
theVariable x ==
  v1 := variables x
  nv := # v1
  nv > 1 => error "Expression is not univariate."
  nv = 0 => 'x
  first v1
```

Type: Void

```
)read vectors
--Copyright The Numerical Algorithms Group Limited
1991.
```

```
)r arrows
--Copyright The Numerical Algorithms Group Limited
1991.
```

```
arrowAngle:=%pi-%pi/10.0@SF
```

```
2.8274333882308138
```

(6)

Type: DoubleFloat

```
arrowScale:=0.2@SF
```

```
0.20000000000000001
```

(7)

Type: DoubleFloat

```
makeArrow(p1,p2) ==
  delta      :=p2 -p1
  len        := arrowScale * length delta
  theta      := atan(delta.1, delta.2)
  c1:= len*cos(theta+arrowAngle)
  s1:= len*sin(theta+arrowAngle)
  c2:= len*cos(theta-arrowAngle)
  s2:= len*sin(theta-arrowAngle)
  z:= p2.3*(1-arrowScale)
  p3:=point[p2.1+c1,p2.2+s1,z,p2.4]
  p4:=point[p2.1+c2,p2.2+s2,z,p2.4]
  [[p1,p2,p3],[p2,p4]]
```

Type: Void

```
clipValue :SF := 6
```

```
6.0
```

(9)

Type: DoubleFloat

```
clipFun(x) == min(max(x,-clipValue),clipValue)
```

Type: Void

```

realSteps :INT := 25
25
(11)
Type: Integer

imagSteps :INT := 25
25
(12)
Type: Integer

C := Complex SF
Complex DoubleFloat
(13)
Type: Domain

S := Segment SF
Segment DoubleFloat
(14)
Type: Domain

drawComplexVectorField : (C -> C, S, S) -> VIEW3D

Type: Void

drawComplexVectorField(f, realRange, imagRange) ==
    delReal := (hi realRange - lo realRange)/realSteps
    delImag := (hi imagRange - lo imagRange)/imagSteps
    sp := create3Space()$ThreeSpace SF
    real := lo realRange
    for i in 1..realSteps + 1 repeat
    imag := lo imagRange
    for j in 1..imagSteps + 1 repeat
        z := f complex(real, imag)
        arg := argument z
        len := clipFun sqrt norm z
        p1 := point[real, imag, 0.0@SF, arg]
        scaleLen := delReal * len
        p2 := point[p1.1 + scaleLen * cos(arg),
                    p1.2 + scaleLen * sin(arg), 0.0@SF,
arg]
        arrow := makeArrow(p1, p2)
        for a in arrow repeat curve(sp, a)
        imag := imag + delImag
    real := real + delReal
    makeViewport3D(sp, "Complex Vector Field")$VIEW3D

Type: Void

drawComplex : (C -> C, S, S) -> VIEW3D

Type: Void

```

```

drawComplex(f, realRange, imagRange) ==
  deltaReal :SF := (hi realRange - lo
realRange)/realSteps
  deltaImag :SF := (hi imagRange - lo
imagRange)/imagSteps
  llp:List List Point SF := []
  real :SF := lo realRange
  for i in 1..realSteps + 1 repeat
  imag :SF := lo imagRange
  lp := []$(List Point SF)
  for j in 1..imagSteps + 1 repeat
    z :COMPLEX SF := f(complex(real, imag))
    pt :Point SF := point[real, imag, clipFun sqrt norm
z, argument z]
    lp := cons(pt, lp)
    imag := imag + deltaImag
    real := real + deltaReal
  llp := cons(reverse! lp, llp)
  llp := reverse! llp
  makeViewport3D(mesh(llp), "Complex Function")$VIEW3D

```

Type: Void

The cube root of two.

```
f := x**3 - 2
```

$$x^3 - 2$$

(19)

Type: Polynomial Integer

Get Newton's iteration formula.

```
g := newtonStep f
```

```

Compiling function theVariable with type Polynomial
Integer -> Symbol
Compiling function complexNumericFunction with type
Polynomial Integer -> (Complex DoubleFloat ->
Complex DoubleFloat)
Compiling function complexDerivativeFunction with
type (Polynomial Integer,PositiveInteger) -> (
Complex DoubleFloat -> Complex DoubleFloat)
Compiling function newtonStep with type Polynomial
Integer -> (Complex DoubleFloat -> Complex
DoubleFloat)
Compiling function %R with type Complex DoubleFloat
-> Complex DoubleFloat
Compiling function %S with type Complex DoubleFloat
-> Complex DoubleFloat

```

```
theMap (...)
```

(20)

Type: (Complex DoubleFloat → Complex DoubleFloat)

Let **a** denote the result of  
applying Newton's iteration  
once to the complex number  $1 +$   
 $\%i$ .

```
a := g(1.0 + %i)
```

```
0.66666666666666674 + 0.3333333333333337 i
```

(21)

Type: Complex DoubleFloat

Now apply it repeatedly. How fast does it converge?

```
[(a := g(a)) for i in 1..]
[1.1644444444444444 - 0.7377777777777775 i,
 0.92614004697164776 - 0.17463006425584393 i,
 1.3164444838140228 + 0.15690694583015852 i,
 1.2462991025761463 + 0.015454763610132094 i,
 1.2598725296532081 - 0.00033827162059311272 i,
 1.259920960928212 + 2.6023534653422681e - 08 i,
 1.259921049894879 - 3.6751942591616685e - 15 i,
 1.2599210498948732 - 3.3132158019282496e - 29 i,
 1.2599210498948732 - 5.6051938572992683e - 45 i,
 1.2599210498948732, ...]
```

Type: Stream Complex DoubleFloat

Check the accuracy of the last iterate.

```
a ** 3
2.0
```

Type: Complex DoubleFloat

In ‘MappingPackage1’ on page 496, we show how functions can be manipulated as objects in AXIOM. A useful operation to consider here is “\*”, which means composition. For example `g*g` causes the Newton iteration formula to be applied twice. Correspondingly, `g**n` means to apply the iteration formula `n` times.

Apply `g` twice to the point `1 + %i`.

```
(g*g) (1.0 + %i)
1.1644444444444444 - 0.7377777777777775 i
```

Type: Complex DoubleFloat

Apply `g` 11 times.

```
(g**11) (1.0 + %i)
1.2599210498948732
```

Type: Complex DoubleFloat

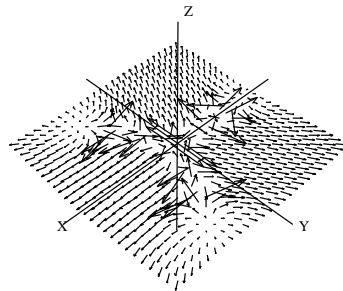
Look now at the vector field and surface generated after two steps of Newton’s formula for the cube root of two. The poles in these pictures represent bad starting values, and the flat areas are the regions of convergence to the three roots.

The vector field.

```
drawComplexVectorField(g**3,-3..3,-3..3)
Compiling function clipFun with type DoubleFloat ->
  DoubleFloat
+++ |*1;clipFun;1;initial| redefined
Compiling function makeArrow with type (Point
  DoubleFloat,Point DoubleFloat) -> List List Point
  DoubleFloat
+++ |*2;makeArrow;1;initial| redefined
Compiling function drawComplexVectorField with type (
  (Complex DoubleFloat -> Complex DoubleFloat),
  Segment DoubleFloat,Segment DoubleFloat) ->
  ThreeDimensionalViewport
+++ |*3;drawComplexVectorField;1;initial| redefined
Transmitting data...
```

ThreeDimensionalViewport: "Complex Vector Field" (26)

Type: ThreeDimensionalViewport

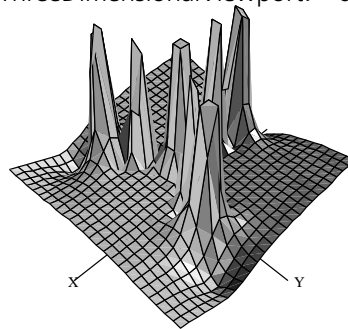


The surface.

```
drawComplex(g**3,-3..3,-3..3)
Compiling function drawComplex with type ((Complex
  DoubleFloat -> Complex DoubleFloat),Segment
  DoubleFloat,Segment DoubleFloat) ->
  ThreeDimensionalViewport
+++ |*3;drawComplex;1;initial| redefined
Transmitting data...
```

ThreeDimensionalViewport: "Complex Function" (27)

Type: ThreeDimensionalViewport



---

# Packages

Packages provide the bulk of AXIOM's algorithmic library, from numeric packages for computing special functions to symbolic facilities for differential equations, symbolic integration, and limits.

In Chapter 10, we developed several useful functions for drawing vector fields and complex functions. We now show you how you can add these functions to the AXIOM library to make them available for general use.

The way we created the functions in Chapter 10 is typical of how you, as an advanced AXIOM user, may interact with AXIOM. You have an application. You go to your editor and create an input file defining some functions for the application. Then you run the file and try the functions. Once you get them all to work, you will often want to extend them, add new features, perhaps write additional functions.

Eventually, when you have a useful set of functions for your application, you may want to add them to your local AXIOM library. To do this, you embed these function definitions in a package and add that package to the library.

To introduce new packages, categories, and domains into the system, you need to use the AXIOM compiler to convert the constructors into executable machine code. An existing compiler in AXIOM is available on an "as-is" basis. A new, faster compiler will be available in version 2.0 of AXIOM.

## 11.1 Names, Abbreviations, and File Structure

---

Each package has a name and an abbreviation. For a package of the complex draw functions from Chapter 10, we choose the name `DrawComplex` and abbreviation `DRAWCX`.<sup>1</sup> To be sure that you have not chosen a name or abbreviation already used by the system, issue the system command `)show` for both the name and the abbreviation.

Once you have named the package and its abbreviation, you can choose any new filename you like with extension “`.spad`” to hold the definition of your package. We choose the name **`drawpak.spad`**. If your application involves more than one package, you can put them all in the same file. AXIOM assumes no relationship between the name of a library file, and the name or abbreviation of a package.

Near the top of the “`.spad`” file, list all the abbreviations for the packages using `)abbrev`, each command beginning in column one. Macros giving names to AXIOM expressions can also be placed near the top of the file. The macros are only usable from their point of definition until the end of the file.

Consider the definition of `DrawComplex` in Figure 11.1. After the macro definition

```
S      ==> Segment DoubleFloat
```

the name `S` can be used in the file as a shorthand for `Segment DoubleFloat`.<sup>2</sup> The abbreviation command for the package

```
)abbrev package DRAWCX DrawComplex
```

is given after the macros (although it could precede them).

## 11.2 Syntax

---

The definition of a package has the syntax:

$$PackageForm : Exports == Implementation$$

The syntax for defining a package constructor is the same as that for defining any function in AXIOM. In practice, the definition extends over many lines so that this syntax is not practical. Also, the type of a package is expressed by the operator `with` followed by an explicit list of operations. A preferable way to write the definition of a package is with a `where` expression:

---

<sup>1</sup>An abbreviation can be any string of between two and seven capital letters and digits, beginning with a letter. See Section 2.2.5 on page 101 for more information.

<sup>2</sup>The interpreter also allows `macro` for macro definitions.



The definition of a package usually has the form:

```
PackageForm : Exports == Implementation where
  optional type declarations
  Exports == with
    list of exported operations
  Implementation == add
    list of function definitions for exported operations
```

The DrawComplex package takes no parameters and exports five operations, each a separate item of a *pile*. Each operation is described as a *declaration*: a name, followed by a colon (“:”), followed by the type of the operation. All operations have types expressed as *mappings* with the syntax

*source* -> *target*

## 11.3 Abstract Datatypes

---

A constructor as defined in AXIOM is called an *abstract datatype* in the computer science literature. Abstract datatypes separate “specification” (what operations are provided) from “implementation” (how the operations are implemented). The **Exports** (specification) part of a constructor is said to be “public” (it provides the user interface to the package) whereas the **Implementation** part is “private” (information here is effectively hidden—programs cannot take advantage of it).

The **Exports** part specifies what operations the package provides to users. As an author of a package, you must ensure that the **Implementation** part provides a function for each operation in the **Exports** part.<sup>3</sup>

An important difference between interactive programming and the use of packages is in the handling of global variables such as **realSteps** and **imagSteps**. In interactive programming, you simply change the values of variables by *assignment*. With packages, such variables are local to the package—their values can only be set using functions exported by the package. In our example package, we provide two functions **setRealSteps** and **setImagSteps** for this purpose.

Another local variable is **clipValue** which can be changed using the exported operation **setClipValue**. This value is referenced by the internal function **clipFun** that decides whether to use the computed value of the

---

<sup>3</sup>The DrawComplex package enhances the facility described in Chapter 10.7 by allowing a complex function to have arrows emanating from the surface to indicate the direction of the complex argument.

function at a point or, if the magnitude of that value is too large, the value assigned to `clipValue` (with the appropriate sign).

## 11.4 Capsules

---

The part to the right of `add` in the **Implementation** part of the definition is called a *capsule*. The purpose of a capsule is:

- to define a function for each exported operation, and
- to define a *local environment* for these functions to run.

What is a local environment? First, what is an environment? Think of the capsule as an input file that AXIOM reads from top to bottom. Think of the input file as having a `)clear all` at the top so that initially no variables or functions are defined. When this file is read, variables such as `realSteps` and `arrowSize` in `DrawComplex` are set to initial values. Also, all the functions defined in the capsule are compiled. These include those that are exported (like `drawComplex`), and those that are not (like `makeArrow`). At the end, you get a set of name-value pairs: variable names (like `realSteps` and `arrowSize`) are paired with assigned values, while operation names (like `drawComplex` and `makeArrow`) are paired with function values.

This set of name-value pairs is called an *environment*. Actually, we call this environment the “initial environment” of a package: it is the environment that exists immediately after the package is first built. Afterwards, functions of this capsule can access or reset a variable in the environment. The environment is called *local* since any changes to the value of a variable in this environment can be seen *only* by these functions.

Only the functions from the package can change the variables in the local environment. When two functions are called successively from a package, any changes caused by the first function called are seen by the second.

Since the environment is local to the package, its names don’t get mixed up with others in the system or your workspace. If you happen to have a variable called `realSteps` in your workspace, it does not affect what the `DrawComplex` functions do in any way.

The functions in a package are compiled into machine code. Unlike function definitions in input files that may be compiled repeatedly as you use them with varying argument types, functions in packages have a unique type (generally parameterized by the argument parameters of a package) and a unique compilation residing on disk.

The capsule itself is turned into a compiled function. This so-called *capsule function* is what builds the initial environment spoken of above. If the package has arguments (see below), then each call to the package con-

structor with a distinct pair of arguments builds a distinct package, each with its own local environment.

## 11.5 Input Files vs. Packages

---

A good question at this point would be “Is writing a package more difficult than writing an input file?”

The programs in input files are designed for flexibility and ease-of-use. AXIOM can usually work out all of your types as it reads your program and does the computations you request. Let’s say that you define a one-argument function without giving its type. When you first apply the function to a value, this value is understood by AXIOM as identifying the type for the argument parameter. Most of the time AXIOM goes through the body of your function and figures out the target type that you have in mind. AXIOM sometimes fails to get it right. Then—and only then—do you need a declaration to tell AXIOM what type you want.

Input files are usually written to be read by AXIOM—and by you. Without suitable documentation and declarations, your input files are likely incomprehensible to a colleague—and to you some months later!

Packages are designed for legibility, as well as run-time efficiency. There are few new concepts you need to learn to write packages. Rather, you just have to be explicit about types and type conversions. The types of all functions are pre-declared so that AXIOM—and the reader— knows precisely what types of arguments can be passed to and from the functions (certainly you don’t want a colleague to guess or to have to work this out from context!). The types of local variables are also declared. Type conversions are explicit, never automatic.<sup>4</sup>

In summary, packages are more tedious to write than input files. When writing input files, you can casually go ahead, giving some facts now, leaving others for later. Writing packages requires forethought, care and discipline.

## 11.6 Compiling Packages

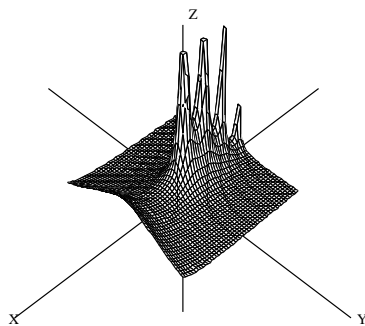
---

Once you have defined the package `DrawComplex`, you need to compile and test it. To compile the package, issue the system command `)compile drawpak`. AXIOM reads the file `drawpak.spad` and compiles its contents into machine binary. If all goes well, the file `DRAWCX.NRLIB` is created in your local directory for the package. To test the package, you must load the package before trying an operation.

---

<sup>4</sup>There is one exception to this rule: conversions from a subdomain to a domain are automatic. After all, the objects both have the domain as a common type.

Compile the package.	<code>)compile drawpak</code>	
Expose the package.	<code>)expose DRAWWCX</code>	
	<code>DrawComplex</code> is now explicitly exposed in frame	
Use an odd step size to avoid a pole at the origin.	<code>setRealSteps 51</code>	(1)
	<code>51</code>	Type: PositivelInteger
	<code>setImagSteps 51</code>	(2)
	<code>51</code>	Type: PositivelInteger
Define <b>f</b> to be the Gamma function.	<code>f(z) == Gamma(z)</code>	
		Type: Void
Clip values of function with magnitude larger than 7.	<code>setClipValue 7</code>	(4)
	<code>7.0</code>	Type: DoubleFloat
Draw the <b>Gamma</b> function.	<code>drawComplex(f, -%pi..%pi, -%pi..%pi, false)</code>	
	Compiling function f with type Complex DoubleFloat -> Complex DoubleFloat Transmitting data...	
	ThreeDimensionalViewport: "Complex Function"	(5)
		Type: ThreeDimensionalViewport



## 11.7 Parameters

---

The power of packages becomes evident when packages have parameters. Usually these parameters are domains and the exported operations have types involving these parameters.

In Chapter 2, you learned that categories denote classes of domains. Although we cover this notion in detail in the next chapter, we now give you a sneak preview of its usefulness.

In Section 6.15 on page 210, we defined functions `bubbleSort(m)` and `insertionSort(m)` to sort a list of integers. If you look at the code for these functions, you see that they may be used to sort *any* structure `m` with the right properties. Also, the functions can be used to sort lists of *any* elements—not just integers. Let us now recall the code for `bubbleSort`.

```
bubbleSort(m) ==  
  n := #m  
  for i in 1..(n-1) repeat  
    for j in n..(i+1) by -1 repeat  
      if m.j < m.(j-1) then swap!(m,j,j-1)  
  m
```

What properties of “lists of integers” are assumed by the sorting algorithm? In the first line, the operation `#` computes the maximum index of the list. The first obvious property is that `m` must have a finite number of elements. In AXIOM, this is done by your telling AXIOM that `m` has the “attribute” `finiteAggregate`. An *attribute* is a property that a domain either has or does not have. As we show later in Section 12.9 on page 670, programs can query domains as to the presence or absence of an attribute.

The operation `swap!` swaps elements of `m`. Using Browse, you find that `swap!` requires its elements to come from a domain of category `IndexedAggregate` with attribute `shallowlyMutable`. This attribute means that you can change the internal components of `m` without changing its external structure. Shallowly-mutable data structures include lists, streams, one- and two-dimensional arrays, vectors, and matrices.

The category `IndexedAggregate` designates the class of aggregates whose elements can be accessed by the notation `m.s` for suitable selectors `s`. The category `IndexedAggregate` takes two arguments: `Index`, a domain of selectors for the aggregate, and `Entry`, a domain of entries for the aggregate. Since the sort functions access elements by integers, we must choose `Index = Integer`. The most general class of domains for which `bubbleSort` and `insertionSort` are defined are those of category `IndexedAggregate(Integer,Entry)` with the two attributes `shallowlyMutable` and `finiteAggregate`.

Using Browse, you can also discover that AXIOM has many kinds of

domains with attribute `shallowlyMutable`. Those of class `IndexedAggregate(Integer,Entry)` include `Bits`, `FlexibleArray`, `OneDimensionalArray`, `List`, `String`, and `Vector`, and also `HashTable` and `EqTable` with integer keys. Although you may never want to sort all such structures, we nonetheless demonstrate AXIOM's ability to do so.

Another requirement is that `Entry` has an operation “<”. One way to get this operation is to assume that `Entry` has category `OrderedSet`. By definition, will then export a “<” operation. A more general approach is to allow any comparison function `f` to be used for sorting. This function will be passed as an argument to the sorting functions.

Our sorting package then takes two arguments: a domain `S` of objects of *any* type, and a domain `A`, an aggregate of type `IndexedAggregate(Integer,S)` with the above two attributes. Here is its definition using what are close to the original definitions of `bubbleSort` and `insertionSort` for sorting lists of integers. The symbol “!” is added to the ends of the operation names. This uniform naming convention is used for AXIOM operation names that destructively change one or more of their arguments.

---

<code>SortPackage(S,A) : Exports == Implementation where</code>	1
<code>  S: Object</code>	2
<code>  A: IndexedAggregate(Integer,S)</code>	3
<code>    with (finiteAggregate; shallowlyMutable)</code>	4
	5
<code>  Exports == with</code>	6
<code>    bubbleSort!: (A,(S,S) -&gt; Boolean) -&gt; A</code>	7
<code>    insertionSort!: (A, (S,S) -&gt; Boolean) -&gt; A</code>	8
	9
<code>  Implementation == add</code>	10
<code>    bubbleSort!(m,f) ==</code>	11
<code>      n := #m</code>	12
<code>      for i in 1..(n-1) repeat</code>	13
<code>        for j in n..(i+1) by -1 repeat</code>	14
<code>          if f(m.j,m.(j-1)) then swap!(m,j,j-1)</code>	15
<code>      m</code>	16
<code>    insertionSort!(m,f) ==</code>	17
<code>      for i in 2..#m repeat</code>	18
<code>        j := i</code>	19
<code>        while j &gt; 1 and f(m.j,m.(j-1)) repeat</code>	20
<code>          swap!(m,j,j-1)</code>	21
<code>        j := (j - 1) pretend PositiveInteger</code>	22
<code>      m</code>	23

---

## 11.8 Conditionals

When packages have parameters, you can say that an operation is or is not exported depending on the values of those parameters. When the domain of objects *S* has an “<” operation, we can supply one-argument versions of `bubbleSort` and `insertionSort` which use this operation for sorting. The presence of the operation “<” is guaranteed when *S* is an ordered set.

---

```
Exports == with
  bubbleSort!: (A,(S,S) -> Boolean) -> A
  insertionSort!: (A, (S,S) -> Boolean) -> A

  if S has OrderedSet then
    bubbleSort!: A -> A
    insertionSort!: A -> A
```

---

In addition to exporting the one-argument sort operations conditionally, we must provide conditional definitions for the operations in the **Implementation** part. This is easy: just have the one-argument functions call the corresponding two-argument functions with the operation “<” from *S*.

---

```
Implementation == add
  ...
  if S has OrderedSet then
    bubbleSort!(m) == bubbleSort!(m,<$S)
    insertionSort!(m) == insertionSort!(m,<$S)
```

---

In Section 6.15 on page 210, we give an alternative definition of **bubbleSort** using **first** and **rest** that is more efficient for a list (for which access to any element requires traversing the list from its first node). To implement a more efficient algorithm for lists, we need the operation **setelt** which allows us to destructively change the **first** and **rest** of a list. Using Browse, you find that these operations come from category `UnaryRecursiveAggregate`. Several aggregate types are unary recursive aggregates including those of `List` and `AssociationList`. We provide two different implementations for **bubbleSort!** and **insertionSort!**: one for list-like structures, another for array-like structures.

---

```
Implementation == add
  ...
  if A has UnaryRecursiveAggregate(S) then
    bubbleSort!(m,fn) ==
      empty? m => m
      l := m
      while not empty? (r := l.rest) repeat
        r := bubbleSort! r
        x := l.first
        if fn(r.first,x) then
          l.first := r.first
```

---

r.first := x	12
l.rest := r	13
l := l.rest	14
m	15
insertionSort!(m,fn) ==	16
...	17

---

The ordering of definitions is important. The standard definitions come first and then the predicate

A has UnaryRecursiveAggregate(S)

is evaluated. If **true**, the special definitions cover up the standard ones.

Another equivalent way to write the capsule is to use an **if-then-else** expression:

if A has UnaryRecursiveAggregate(S) then	1
...	2
else	3
...	4

---

## 11.9 Testing

---

Once you have written the package, embed it in a file, for example, **sortpak.spad**. Be sure to include an **)abbrev** command at the top of the file:

**)abbrev package SORTPAK SortPackage**

Now compile the file (using **)compile sortpak.spad**).

Expose the constructor. You are then ready to begin testing.

**)expose SORTPAK**

SortPackage is now explicitly exposed in frame

Define a list.

l := [1,7,4,2,11,-7,3,2]	
[1, 7, 4, 2, 11, -7, 3, 2]	(1)
	Type: List Integer

Since the integers are an ordered set, a one-argument operation will do.

bubbleSort!(l)	
[-7, 1, 2, 2, 3, 4, 7, 11]	(2)
	Type: List Integer

Re-sort it using “greater than.”

bubbleSort!(l,(x,y) +-> x > y)	
[11, 7, 4, 3, 2, 2, 1, -7]	(3)
	Type: List Integer



Now sort it again using “<” on integers.	<pre>bubbleSort!(1, &lt;\$Integer) [-7, 1, 2, 2, 3, 4, 7, 11]</pre>	(4)	Type: List Integer
A string is an aggregate of characters so we can sort them as well.	<pre>bubbleSort! "Mathematical Sciences" " MSaaaccceeehiilmnstt"</pre>	(5)	Type: String
Is “<” defined on booleans?	<pre>false &lt; true true</pre>	(6)	Type: Boolean
Good! Create a bit string representing ten consecutive boolean values <b>true</b> .	<pre>u : Bits := new(10,true) "1111111111"</pre>	(7)	Type: Bits
Set bits 3 through 5 to <b>false</b> , then display the result.	<pre>u(3..5) := false; u "1100011111"</pre>	(8)	Type: Bits
Now sort these booleans.	<pre>bubbleSort! u "0001111111"</pre>	(9)	Type: Bits
Create an “eq-table” (see ‘EqTable’ on page 406), a table having integers as keys and strings as values.	<pre>t : EqTable(Integer,String) := table() table()</pre>	(10)	Type: EqTable(Integer, String)
Give the table a first entry.	<pre>t.1 := "robert" "robert"</pre>	(11)	Type: String
And a second.	<pre>t.2 := "richard" "richard"</pre>	(12)	Type: String
What does the table look like?	<pre>t table(2 = "richard", 1 = "robert")</pre>	(13)	Type: EqTable(Integer, String)
Now sort it.	<pre>bubbleSort! t table(2 = "robert", 1 = "richard")</pre>	(14)	Type: EqTable(Integer, String)

## 11.10 How Packages Work

---

See the modemap for  
**bubbleSort!**.

Recall that packages as abstract datatypes are compiled independently and put into the library. The curious reader may ask: “How is the interpreter able to find an operation such as **bubbleSort!**? Also, how is a single compiled function such as **bubbleSort!** able to sort data of different types?”

After the interpreter loads the package `SortPackage`, the four operations from the package become known to the interpreter. Each of these operations is expressed as a *modemap* in which the type of the operation is written in terms of symbolic domains.

```
)display op bubbleSort!
```

There are 2 exposed functions called **bubbleSort!** :

```
[1] D1 -> D1 from SortPackage(D2,D1)
    if D2 has ORDSET and D2 has OBJECT and D1 has
    IndexedAggregate(Integer, D2) with
        finiteAggregate
        shallowlyMutable

[2] (D1,((D3,D3) -> Boolean)) -> D1 from SortPackage(D3,D1)
    if D3 has OBJECT and D1 has
    IndexedAggregate(Integer,D3) with
        finiteAggregate
        shallowlyMutable
```

What happens if you ask for **bubbleSort!([1,-5,3])**? There is a unique modemap for an operation named **bubbleSort!** with one argument. Since **[1,-5,3]** is a list of integers, the symbolic domain **D1** is defined as **List(Integer)**. For some operation to apply, it must satisfy the predicate for some **D2**. What **D2**? The third expression of the **and** requires **D1 has IndexedAggregate(Integer, D2) with two attributes**. So the interpreter searches for an **IndexedAggregate** among the ancestors of **List(Integer)** (see Section 12.4 on page 667). It finds one: **IndexedAggregate(Integer, Integer)**. The interpreter tries defining **D2** as **Integer**. After substituting for **D1** and **D2**, the predicate evaluates to **true**. An applicable operation has been found!

Now **AXIOM** builds the package **SortPackage(List(Integer), Integer)**. According to its definition, this package exports the required operation: **bubbleSort!: List Integer → List Integer**. The interpreter then asks the package for a function implementing this operation. The package gets all the functions it needs (for example, **rest** and **swap!**) from the appropriate domains and then it returns a **bubbleSort!** to the interpreter together with the local environment for **bubbleSort!**. The interpreter applies the function to the argument **[1,-5,3]**. The **bubbleSort!** function is executed in its local environment and produces the result.

All constructors used in a file must be spelled out in full unless abbreviated by macros like these at the top of a file.	C ==> Complex DoubleFloat S ==> Segment DoubleFloat INT ==> Integer DFLOAT ==> DoubleFloat VIEW3D ==> ThreeDimensionalViewport CURVE ==> List List Point DFLOAT	1 2 3 4 5 6 7
Identify kinds and abbreviations Type definition begins here.	)abbrev package DRAWCX DrawComplex DrawComplex(): Exports == Implementation where	8 9 10
Export part begins. Exported Operations	Exports == with drawComplex: (C -> C,S,S,Boolean) -> VIEW3D drawComplexVectorField: (C -> C,S,S) -> VIEW3D setRealSteps: INT -> INT setImagSteps: INT -> INT setClipValue: DFLOAT-> DFLOAT	11 12 13 14 15 16 17
Implementation part begins.	Implementation == add	18
Local variable 1.	arrowScale : DFLOAT := (0.2)::DFLOAT --relative size	19
Local variable 2.	arrowAngle : DFLOAT := pi()-pi()/(20::DFLOAT)	20
Local variable 3.	realSteps : INT := 11 --# real steps	21
Local variable 4.	imagSteps : INT := 11 --# imaginary steps	22
Local variable 5.	clipValue : DFLOAT := 10::DFLOAT --maximum vector length	23 24
Exported function definition 1.	setRealSteps(n) == realSteps := n	25
Exported function definition 2.	setImagSteps(n) == imagSteps := n	26
Exported function definition 3.	setClipValue(c) == clipValue := c	27 28
	clipFun: DFLOAT -> DFLOAT --Clip large magnitudes.	29
Local function definition 1.	clipFun(x) == min(max(x, -clipValue), clipValue)	30 31
	makeArrow: (Point DFLOAT,Point DFLOAT,DFLOAT,DFLOAT) -> CURVE	32
Local function definition 2.	makeArrow(p1, p2, len, arg) == ...	33 34
Exported function definition 4.	drawComplex(f, realRange, imagRange, arrows?) == ...	35

Figure 11.1: The DrawComplex package.



---

# Categories

This chapter unravels the mysteries of categories—what they are, how they are related to domains and packages, how they are defined in AXIOM, and how you can extend the system to include new categories of your own.

We assume that you have read the introductory material on domains and categories in Section 2.1.1 on page 93. There you learned that the notion of packages covered in the previous chapter are special cases of domains. While this is in fact the case, it is useful here to regard domains as distinct from packages.

Think of a domain as a datatype, a collection of objects (the objects of the domain). From your “sneak preview” in the previous chapter, you might conclude that categories are simply named clusters of operations exported by domains. As it turns out, categories have a much deeper meaning. Categories are fundamental to the design of AXIOM. They control the interactions between domains and algorithmic packages, and, in fact, between all the components of AXIOM.

Categories form hierarchies as shown on the inside cover pages of this book. The inside front-cover pages illustrate the basic algebraic hierarchy of the AXIOM programming language. The inside back-cover pages show the hierarchy for data structures.

Think of the category structures of AXIOM as a foundation for a city on which superstructures (domains) are built. The algebraic hierarchy, for example, serves as a foundation for constructive mathematical algorithms embedded in the domains of AXIOM. Once in place, domains can be constructed, either independently or from one another.

Superstructures are built for quality—domains are compiled into machine

code for run-time efficiency. You can extend the foundation in directions beyond the space directly beneath the superstructures, then extend selected superstructures to cover the space. Because of the compilation strategy, changing components of the foundation generally means that the existing superstructures (domains) built on the changed parts of the foundation (categories) have to be rebuilt—that is, recompiled.

Before delving into some of the interesting facts about categories, let’s see how you define them in AXIOM.

## 12.1 Definitions

A category is defined by a function with exactly the same format as any other function in AXIOM.

The definition of a category has the syntax:

*CategoryForm* : *Category* == *Extensions* [ *with Exports* ]

The brackets [ ] here indicate optionality.

The first example of a category definition is SetCategory, the most basic of the algebraic categories in AXIOM.

SetCategory() : Category ==	1
Join(Type,CoercibleTo OutputForm) with	2
"=" : (\$, \$) -> Boolean	3

The definition starts off with the name of the category (SetCategory); this is always in column one in the source file. All parts of a category definition are then indented with respect to this first line.

In Chapter 2, we talked about Ring as denoting the class of all domains that are rings, in short, the class of all rings. While this is the usual naming convention in AXIOM, it is also common to use the word “Category” at the end of a category name for clarity. The interpretation of the name SetCategory is, then, “the category of all domains that are (mathematical) sets.”

The name SetCategory is followed in the definition by its formal parameters enclosed in parentheses “()”. Here there are no parameters. As required, the type of the result of this category function is the distinguished name Category.

Then comes the “==”. As usual, what appears to the right of the “==” is a definition, here, a category definition. A category definition always has

two parts separated by the reserved word `with`.

The first part tells what categories the category extends. Here, the category extends two categories: `Type`, the category of all domains, and `CoercibleTo(OutputForm)`. The operation `Join` is a system-defined operation that forms a single category from two or more other categories.

Every category other than `Type` is an extension of some other category. If, for example, `SetCategory` extended only the category `Type`, the definition here would read “`Type with ...`”. In fact, the `Type` is optional in this line; “`with ...`” suffices.

## 12.2 Exports

---

To the right of the `with` is a list of all the *exports* of the category. Each exported operation has a name and a type expressed by a *declaration* of the form “*name: type*”.

Categories can export symbols, as well as 0 and 1 which denote domain constants.<sup>1</sup> In the current implementation, all other exports are operations with types expressed as *mappings* with the syntax

*source* `->` *target*

The category `SetCategory` has a single export: the operation “`=`” whose type is given by the mapping `($, $) -> Boolean`. The “`$`” in a mapping type always means “the domain.” Thus the operation “`=`” takes two arguments from the domain and returns a value of type `Boolean`.

The source part of the mapping here is given by a *tuple* consisting of two or more types separated by commas and enclosed in parentheses. If an operation takes only one argument, you can drop the parentheses around the source type. If the mapping has no arguments, the source part of the mapping is either left blank or written as “`()`”. Here are examples of formats of various operations with some contrived names.

```
someIntegerConstant :      $
aZeroArgumentOperation:  () -> Integer
aOneArgumentOperation:   Integer -> $
aTwoArgumentOperation:   (Integer,$) -> Void
aThreeArgumentOperation: ($,Integer,$) -> Fraction($)
```

## 12.3 Documentation

---

The definition of `SetCategory` above is missing an important component: its library documentation. Here is its definition, complete with documentation.

---

<sup>1</sup>The numbers 0 and 1 are operation names in AXIOM.

---

++ Description:	1
++ \axiomType{SetCategory} is the basic category	2
++ for describing a collection of elements with	3
++ \axiomOp{=} (equality) and a \axiomFun{coerce}	4
++ to \axiomType{OutputForm}.	5
	6
SetCategory(): Category ==	7
Join(Type, CoercibleTo OutputForm) with	8
"=": (\$, \$) -> Boolean	9
++ \axiom{x = y} tests if \axiom{x} and	10
++ \axiom{y} are equal.	11

---

Documentary comments are an important part of constructor definitions. Documentation is given both for the category itself and for each export. A description for the category precedes the code. Each line of the description begins in column one with “++”. The description starts with the word **Description:**.<sup>2</sup> All lines of the description following the initial line are indented by the same amount.

Surround the name of any constructor (with or without parameters) with an . Similarly, surround an operator name with ‘‘ ’’, an AXIOM operation with , and a variable or AXIOM expression with . Library documentation is given in a T<sub>E</sub>X-like language so that it can be used both for hard-copy and for Browse. These different wrappings cause operations and types to have mouse-active buttons in Browse. For hard-copy output, wrapped expressions appear in a different font. The above documentation appears in hard-copy as:

SetCategory is the basic category for describing a collection of elements with “=” (equality) and a **coerce** to OutputForm.

and

x = y tests if x and y are equal.

For our purposes in this chapter, we omit the documentation from further category descriptions.

---

<sup>2</sup>Other information such as the author’s name, date of creation, and so on, can go in this area as well but are currently ignored by AXIOM.



## 12.4 Hierarchies

A second example of a category is `SemiGroup`, defined by:

<code>SemiGroup(): Category == SetCategory with</code>	1
<code>    "": (\$,\$) -&gt; \$</code>	2
<code>    "***": (\$, PositiveInteger) -&gt; \$</code>	3

This definition is as simple as that for `SetCategory`, except that there are two exported operations. Multiple exported operations are written as a *pile*, that is, they all begin in the same column. Here you see that the category mentions another type, `PositiveInteger`, in a signature. Any domain can be used in a signature.

Since categories extend one another, they form hierarchies. Each category other than `Type` has one or more parents given by the one or more categories mentioned before the `with` part of the definition. `SemiGroup` extends `SetCategory` and `SetCategory` extends both `Type` and `CoercibleTo (OutputForm)`. Since `CoercibleTo (OutputForm)` also extends `Type`, the mention of `Type` in the definition is unnecessary but included for emphasis.

## 12.5 Membership

We say a category designates a class of domains. What class of domains? That is, how does AXIOM know what domains belong to what categories? The simple answer to this basic question is key to the design of AXIOM:

**Domains belong to categories by assertion.**

When a domain is defined, it is asserted to belong to one or more categories. Suppose, for example, that an author of domain `String` wishes to use the binary operator “`*`” to denote concatenation. Thus “`hello` ” `*` “`there`” would produce the string “`hello there`”<sup>3</sup>. The author of `String` could then assert that `String` is a member of `SemiGroup`. According to our definition of `SemiGroup`, strings would then also have the operation “`**`” defined automatically. Then “`--`” `**` 4 would produce a string of eight dashes “`-----`”. Since `String` is a member of `SemiGroup`, it also is a member of `SetCategory` and thus has an operation “`=`” for testing that two strings are equal.

Now turn to the algebraic category hierarchy inside the front cover of this book. Any domain that is a member of a category extending `SemiGroup` is a member of `SemiGroup` (that is, it *is* a semigroup). In particular, any

<sup>3</sup>Actually, concatenation of strings in AXIOM is done by juxtaposition or by using the operation `concat`. The expression “`hello` ” “`there`” produces the string “`hello there`”.

domain asserted to be a Ring is a semigroup since Ring extends Monoid, that, in turn, extends SemiGroup. The definition of Integer in AXIOM asserts that Integer is a member of category IntegerNumberSystem, that, in turn, asserts that it is a member of EuclideanDomain. Now EuclideanDomain extends PrincipalIdealDomain and so on. If you trace up the hierarchy, you see that EuclideanDomain extends Ring, and, therefore, SemiGroup. Thus Integer is a semigroup and also exports the operations “\*” and “\*\*”.

## 12.6 Defaults

We actually omitted the last part of the definition of SemiGroup in Section 12.4 on page 667. Here now is its complete AXIOM definition.

---

SemiGroup(): Category == SetCategory with	1
"*": (\$, \$) -> \$	2
"***": (\$, PositiveInteger) -> \$	3
add	4
import RepeatedSquaring(\$)	5
x: \$ ** n: PositiveInteger == expt(x,n)	6

---

The add part at the end is used to give “default definitions” for exported operations. Once you have a multiplication operation “\*”, you can define exponentiation for positive integer exponents using repeated multiplication:

$$x^n = \underbrace{xxx \cdots x}_{n \text{ times}}$$

This definition for “\*\*” is called a *default* definition. In general, a category can give default definitions for any operation it exports. Since SemiGroup and all its category descendants in the hierarchy export “\*\*”, any descendant category may redefine “\*\*” as well.

A domain of category SemiGroup (such as Integer) may or may not choose to define its own “\*\*” operation. If it does not, a default definition that is closest (in a “tree-distance” sense of the hierarchy) to the domain is chosen.

The part of the category definition following an “add” operation is a *capsule*, as discussed in the previous chapter. The line

```
import RepeatedSquaring($)
```

references the package RepeatedSquaring(\$), that is, the package RepeatedSquaring that takes “this domain” as its parameter. For example, if the semigroup Polynomial(Integer) does not define its own exponentiation operation, the definition used may come from the package RepeatedSquaring(Polynomial(Integer)). The next line gives the definition in terms of **expt** from that package.

The default definitions are collected to form a “default package” for the category. The name of the package is the same as the category but with an ampersand (“&”) added at the end. A default package always takes an additional argument relative to the category. Here is the definition of the default package `SemiGroup&` as automatically generated by AXIOM from the above definition of `SemiGroup`.

---

<code>SemiGroup&amp;(\$): Exports == Implementation where</code>	1
<code>  \$: SemiGroup</code>	2
<code>  Exports == with</code>	3
<code>    "***": (\$, PositiveInteger) -&gt; \$</code>	4
<code>  Implementation == add</code>	5
<code>    import RepeatedSquaring(\$)</code>	6
<code>    x:\$ ** n:PositiveInteger == expt(x,n)</code>	7

---

## 12.7 Axioms

---

In the previous section you saw the complete AXIOM program defining `SemiGroup`. According to this definition, semigroups (that is, are sets with the operations “\*” and “\*\*”).

You might ask: “Aside from the notion of default packages, isn’t a category just a *macro*, that is, a shorthand equivalent to the two operations “\*” and “\*\*” with their types?” If a category were a macro, every time you saw the word `SemiGroup`, you would rewrite it by its list of exported operations. Furthermore, every time you saw the exported operations of `SemiGroup` among the exports of a constructor, you could conclude that the constructor exported `SemiGroup`.

A category is *not* a macro and here is why. The definition for `SemiGroup` has documentation that states:

Category `SemiGroup` denotes the class of all multiplicative semi-groups, that is, a set with an associative operation “\*”.

Axioms:

`associative("*" : ($,$)->$) -- (x*y)*z = x*(y*z)`

According to the author’s remarks, the mere exporting of an operation named “\*” and “\*\*” is not enough to qualify the domain as a `SemiGroup`. In fact, a domain can be a semigroup only if it explicitly exports a “\*\*” and a “\*” satisfying the associativity axiom.

In general, a category name implies a set of axioms, even mathematical theorems. There are numerous axioms from `Ring`, for example, that are well-understood from the literature. No attempt is made to list them all.

Nonetheless, all such mathematical facts are implicit by the use of the name Ring.

## 12.8 Correctness

---

While such statements are only comments, AXIOM can enforce their intention simply by shifting the burden of responsibility onto the author of a domain. A domain belongs to category **Ring** only if the author asserts that the domain belongs to Ring or to a category that extends Ring.

This principle of assertion is important for large user-extendable systems. AXIOM has a large library of operations offering facilities in many areas. Names such as **norm** and **product**, for example, have diverse meanings in diverse contexts. An inescapable hindrance to users would be to force those who wish to extend AXIOM to always invent new names for operations. AXIOM allows you to reuse names, and then use context to disambiguate one from another.

Here is another example of why this is important. Some languages, such as **APL**, denote the Boolean constants **true** and **false** by the integers 1 and 0. You may want to let infix operators “+” and “\*” serve as the logical operators **or** and **and**, respectively. But note this: Boolean is not a ring. The *inverse axiom* for Ring states:

Every element  $x$  has an additive inverse  $y$  such that  $x + y = 0$ .

Boolean is not a ring since **true** has no inverse—there is no inverse element  $a$  such that  $1 + a = 0$  (in terms of booleans,  $(\text{true or } a) = \text{false}$ ). Nonetheless, AXIOM *could* easily and correctly implement Boolean this way. Boolean simply would not assert that it is of category Ring. Thus the “+” for Boolean values is not confused with the one for Ring. Since the Polynomial constructor requires its argument to be a ring, AXIOM would then refuse to build the domain Polynomial(Boolean). Also, AXIOM would refuse to wrongfully apply algorithms to Boolean elements that presume that the ring axioms for “+” hold.

## 12.9 Attributes

---

Most axioms are not computationally useful. Those that are can be explicitly expressed by what AXIOM calls an *attribute*. The attribute `commutative("*")`, for example, is used to assert that a domain has commutative multiplication. Its definition is given by its documentation:

A domain  $R$  has `commutative("*")` if it has an operation “\*”:  $(R,R) \rightarrow R$  such that  $x * y = y * x$ .

Just as you can test whether a domain has the category Ring, you can test that a domain has a given attribute.

Do polynomials over the integers have commutative multiplication?

```
Polynomial Integer has commutative("")
true
```

(1)

Type: Boolean

Do matrices over the integers have commutative multiplication?

```
Matrix Integer has commutative("")
false
```

(2)

Type: Boolean

Attributes are used to conditionally export and define operations for a domain (see Section 13.3 on page 677). Attributes can also be asserted in a category definition.

After mentioning category Ring many times in this book, it is high time that we show you its definition:

---

```
Ring(): Category == 1
  Join(Rng, Monoid, LeftModule($: Rng)) with 2
    characteristic: -> NonNegativeInteger 3
    coerce: Integer -> $ 4
    unitsKnown 5
  add 6
    n: Integer 7
    coerce(n) == n * 1$$ 8
```

---

There are only two new things here. First, look at the “\$\$” on the last line. This is not a typographic error! The first “\$” says that the 1 is to come from some domain. The second “\$” says that the domain is “this domain.” If “\$” is Fraction(Integer), this line reads `coerce(n) == n * 1$Fraction(Integer)`.

The second new thing is the presence of attribute “unitsKnown”. AXIOM can always distinguish an attribute from an operation. An operation has a name and a type. An attribute has no type. The attribute `unitsKnown` asserts a rather subtle mathematical fact that is normally taken for granted when working with rings.<sup>4</sup> Because programs can test for this attribute, AXIOM can correctly handle rather more complicated mathematical structures (ones that are similar to rings but do not have this attribute).

---

<sup>4</sup>With this axiom, the units of a domain are the set of elements  $x$  that each have a multiplicative inverse  $y$  in the domain. Thus 1 and -1 are units in domain Integer. Also, for Fraction Integer, the domain of rational numbers, all non-zero elements are units.

## 12.10 Parameters

Like domain constructors, category constructors can also have parameters. For example, category `MatrixCategory` is a parameterized category for defining matrices over a ring `R` so that the matrix domains can have different representations and indexing schemes. Its definition has the form:

---

<code>MatrixCategory(R,Row,Col): Category ==</code>	1
<code>TwoDimensionalArrayCategory(R,Row,Col) with ...</code>	2

---

The category extends `TwoDimensionalArrayCategory` with the same arguments. You cannot find `TwoDimensionalArrayCategory` in the algebraic hierarchy listing. Rather, it is a member of the data structure hierarchy, given inside the back cover of this book. In particular, `TwoDimensionalArrayCategory` is an extension of `HomogeneousAggregate` since its elements are all one type.

The domain `Matrix(R)`, the class of matrices with coefficients from domain `R`, asserts that it is a member of category `MatrixCategory(R, Vector(R), Vector(R))`. The parameters of a category must also have types. The first parameter to `MatrixCategory` `R` is required to be a ring. The second and third are required to be domains of category `FiniteLinearAggregate(R)`.<sup>5</sup> In practice, examples of categories having parameters other than domains are rare.

Adding the declarations for parameters to the definition for `MatrixCategory`, we have:

---

<code>R: Ring</code>	1
<code>(Row, Col): FiniteLinearAggregate(R)</code>	2
	3
<code>MatrixCategory(R, Row, Col): Category ==</code>	4
<code>TwoDimensionalArrayCategory(R, Row, Col) with ...</code>	5

---

## 12.11 Conditionals

As categories have parameters, the actual operations exported by a category can depend on these parameters. As an example, the operation **determinant** from category `MatrixCategory` is only exported when the underlying domain `R` has commutative multiplication:

```
if R has commutative("*") then
  determinant: $ -> R
```

Conditionals can also define conditional extensions of a category. Here is a portion of the definition of `QuotientFieldCategory`:

---

<code>QuotientFieldCategory(R) : Category == ... with ...</code>	1
--	---

---

<sup>5</sup>This is another extension of `HomogeneousAggregate` that you can see in the data structure hierarchy.

if R has OrderedSet then OrderedSet	2
if R has IntegerNumberSystem then	3
ceiling: \$ -> R	4
...	5

---

Think of category `QuotientFieldCategory(R)` as denoting the domain `Fraction(R)`, the class of all fractions of the form  $a/b$  for elements of  $\mathbf{R}$ . The first conditional means in English: “If the elements of  $\mathbf{R}$  are totally ordered ( $\mathbf{R}$  is an `OrderedSet`), then so are the fractions  $a/b$ ”.

The second conditional is used to conditionally export an operation **ceiling** which returns the smallest integer greater than or equal to its argument. Clearly, “ceiling” makes sense for integers but not for polynomials and other algebraic structures. Because of this conditional, the domain `Fraction(Integer)` exports an operation **ceiling**: `Fraction Integer  $\rightarrow$  Integer`, but `Fraction Polynomial Integer` does not.

Conditionals can also appear in the default definitions for the operations of a category. For example, a default definition for **ceiling** within the part following the “**add**” reads:

```
if R has IntegerNumberSystem then
  ceiling x == ...
```

Here the predicate used is identical to the predicate in the **Exports** part. This need not be the case. See Section 11.8 on page 657 for a more complicated example.

## 12.12 Anonymous Categories

---

The part of a category to the right of a **with** is also regarded as a category—an “anonymous category.” Thus you have already seen a category definition in Chapter 11. The **Exports** part of the package `DrawComplex` (Section 11.3 on page 651) is an anonymous category. This is not necessary. We could, instead, give this category a name:

<code>DrawComplexCategory(): Category == with</code>	1
<code>drawComplex: (C -&gt; C,S,S,Boolean) -&gt; VIEW3D</code>	2
<code>drawComplexVectorField: (C -&gt; C,S,S) -&gt; VIEW3D</code>	3
<code>setRealSteps: INT -&gt; INT</code>	4
<code>setImagSteps: INT -&gt; INT</code>	5
<code>setClipValue: DFLOAT-&gt; DFLOAT</code>	6

---

and then define `DrawComplex` by:

<code>DrawComplex(): DrawComplexCategory == Implementation</code>	1
<code>where</code>	2
<code>...</code>	3

---

There is no reason, however, to give this list of exports a name since no other domain or package exports it. In fact, it is rare for a package to export a named category. As you will see in the next chapter, however, it is very common for the definition of domains to mention one or more category before the `with`.



---

# Domains

We finally come to the *domain constructor*. A few subtle differences between packages and domains turn up some interesting issues. We first discuss these differences then describe the resulting issues by illustrating a program for the QuadraticForm constructor. After a short example of an algebraic constructor, CliffordAlgebra, we show how you use domain constructors to build a database query facility.

### 13.1 Domains vs. Packages

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Packages are special cases of domains. What is the difference between a package and a domain that is not a package? By definition, there is only one difference: a domain that is not a package has the symbol “\$” appearing somewhere among the types of its exported operations. The “\$” denotes “this domain.” If the “\$” appears before the “->” in the type of a signature, it means the operation takes an element from the domain as an argument. If it appears after the “->”, then the operation returns an element of the domain.

If no exported operations mention “\$”, then evidently there is nothing of interest to do with the objects of the domain. You might then say that a package is a “boring” domain! But, as you saw in Chapter 11, packages are a very useful notion indeed. The exported operations of a package depend solely on the parameters to the package constructor and other explicit domains.

To summarize, domain constructors are versatile structures that serve two distinct practical purposes: Those like Polynomial and List describe classes of computational objects; others, like SortPackage, describe packages of useful operations. As in the last chapter, we focus here on the first kind.

## 13.2 Definitions

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The syntax for defining a domain constructor is the same as for any function in AXIOM:

$$\text{DomainForm} : \text{Exports} == \text{Implementation}$$

As this definition usually extends over many lines, a **where** expression is generally used instead.

A recommended format for the definition of a domain is:

```
DomainForm : Exports == Implementation where
  optional type declarations
Exports == [Category Assertions] with
  list of exported operations
Implementation == [Add Domain] add
  [Rep := Representation]
  list of function definitions for exported operations
```

Note: The brackets [ ] here denote optionality.

A complete domain constructor definition for QuadraticForm is shown in Figure 13.1. Interestingly, this little domain illustrates all the new concepts you need to learn.

A domain constructor can take any number and type of parameters. QuadraticForm takes a positive integer **n** and a field **K** as arguments. Like a package, a domain has a set of explicit exports and an implementation described by a capsule. Domain constructors are documented in the same way as package constructors.

Domain QuadraticForm(**n**, **K**), for a given positive integer **n** and domain **K**, explicitly exports three operations:

- **quadraticForm(A)** creates a quadratic form from a matrix **A**.
- **matrix(q)** returns the matrix **A** used to create the quadratic form **q**.
- **q.v** computes the scalar  $v^T A v$  for a given vector **v**.

Compared with the corresponding syntax given for the definition of a package, you see that a domain constructor has three optional parts to its definition: *Category Assertions*, *Add Domain*, and *Representation*.

	)abbrev domain QFORM QuadraticForm	1
		2
	++ Description:	3
	++ This domain provides modest support for	4
	++ quadratic forms.	5
	QuadraticForm(n, K): Exports == Implementation where	6
	n: PositiveInteger	7
	K: Field	8
		9
The exports.	Exports == AbelianGroup with	10
The export <b>quadraticForm</b> .	quadraticForm: SquareMatrix(n,K) -> \$	11
	++ \axiom{quadraticForm(m)} creates a quadratic	12
	++ quadratic form from a symmetric,	13
	++ square matrix \axiom{m}.	14
The export <b>matrix</b> .	matrix: \$ -> SquareMatrix(n,K)	15
	++ \axiom{matrix(qf)} creates a square matrix	16
	++ from the quadratic form \axiom{qf}.	17
The export <b>elt</b> .	elt: (\$, DirectProduct(n,K)) -> K	18
	++ \axiom{qf(v)} evaluates the quadratic form	19
	++ \axiom{qf} on the vector \axiom{v},	20
	++ producing a scalar.	21
		22
The definitions of the exports	Implementation == SquareMatrix(n,K) add	23
The “representation.”	Rep := SquareMatrix(n,K)	24
The definition of	quadraticForm m ==	25
<b>quadraticForm</b> .	not symmetric? m => error	26
	"quadraticForm requires a symmetric matrix"	27
	m :: \$	28
The definition of <b>matrix</b> .	matrix q == q :: Rep	29
The definition of <b>elt</b> .	elt(q,v) == dot(v, (matrix q * v))	30

Figure 13.1: The QuadraticForm domain.

### 13.3 Category Assertions

The *Category Assertions* part of your domain constructor definition lists those categories of which all domains created by the constructor are unconditionally members. The word “unconditionally” means that membership in a category does not depend on the values of the parameters to the domain constructor. This part thus defines the link between the domains and the category hierarchies given on the inside covers of this book. As described in Section 12.8 on page 670, it is this link that makes it possible for you to pass objects of the domains as arguments to other operations in AXIOM.

Every QuadraticForm domain is declared to be unconditionally a member of category AbelianGroup. An abelian group is a collection of elements closed under addition. Every object  $x$  of an abelian group has an additive inverse  $y$  such that  $x + y = 0$ . The exports of an abelian group include 0, “+”, “−”, and scalar multiplication by an integer. After asserting that

QuadraticForm domains are abelian groups, it is possible to pass quadratic forms to algorithms that only assume arguments to have these abelian group properties.

In Section 12.11 on page 672, you saw that `Fraction(R)`, a member of `QuotientFieldCategory(R)`, is a member of `OrderedSet` if `R` is a member of `OrderedSet`. Likewise, from the **Exports** part of the definition of `ModMonic(R, S)`,

```
UnivariatePolynomialCategory(R) with
  if R has Finite then Finite
  ...
```

you see that `ModMonic(R, S)` is a member of `Finite` if `R` is.

The **Exports** part of a domain definition is the same kind of expression that can appear to the right of an “==” in a category definition. If a domain constructor is unconditionally a member of two or more categories, a **Join** form is used. The **Exports** part of the definition of `FlexibleArray(S)` reads, for example:

```
Join(ExtensibleLinearAggregate(S),
     OneDimensionalArrayAggregate(S)) with...
```

## 13.4 A Demo

---

Before looking at the *Implementation* part of `QuadraticForm`, let’s try some examples.

Build a domain `QF`.

```
QF := QuadraticForm(2, Fraction Integer)
QuadraticForm (2, Fraction Integer)
Type: Domain
```

(1)

Define a matrix to be used to construct a quadratic form.

```
A := matrix [[-1, 1/2], [1/2, 1]]

$$\begin{bmatrix} -1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}$$

Type: Matrix Fraction Integer
```

(2)

Construct the quadratic form. A package call `$QF` is necessary since there are other `QuadraticForm` domains.

```
q : QF := quadraticForm(A)

$$\begin{bmatrix} -1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}$$

Type: QuadraticForm(2, Fraction Integer)
```

(3)

Looks like a matrix. Try computing the number of rows. AXIOM won't let you.

nrows q

There are 2 exposed and 0 unexposed library operations named nrows having 1 argument(s) but none was determined to be applicable. Use HyperDoc Browse, or issue  

$$\text{display op nrows}$$
to learn more about the available operations. Perhaps package-calling the operation or using coercions on the arguments will allow you to apply the operation.

Cannot find a definition or applicable library operation named nrows with argument type(s)  

$$\text{QuadraticForm}(2, \text{Fraction Integer})$$

Perhaps you should use "@" to indicate the required return type, or "\$" to specify which version of the function you need.

Create a direct product element v. A package call is again necessary, but AXIOM understands your list as denoting a vector.

$$v := \text{directProduct}([2, -1])\$ \text{DirectProduct}(2, \text{Fraction Integer})$$

$$[2, -1]$$
 (4)

Type: DirectProduct(2, Fraction Integer)

Compute the product  $v^T Av$ .

$$q \cdot v$$

$$-5$$
 (5)

Type: Fraction Integer

What is 3 times q minus q plus q?

$$3 * q - q + q$$

$$\begin{bmatrix} -3 & \frac{3}{2} \\ \frac{3}{2} & 3 \end{bmatrix}$$
 (6)

Type: QuadraticForm(2, Fraction Integer)

## 13.5 Browse

The Browse facility of HyperDoc is useful for investigating the properties of domains, packages, and categories. From the main HyperDoc menu, move your mouse to **Browse** and click on the left mouse button. This brings up the Browse first page. Now, with your mouse pointer somewhere in this window, enter the string "quadraticform" into the input area (all lower case letters will do). Move your mouse to **Constructors** and click. Up comes a page describing QuadraticForm.

From here, click on **Description**. This gives you a page that includes a part labeled by "*Description*:". You also see the types for arguments **n** and **K** displayed as well as the fact that QuadraticForm returns an Abelian-Group. You can go and experiment a bit by selecting **Field** with your

mouse. Eventually, use  several times to return to the first page on QuadraticForm.

Select **Operations** to get a list of operations for QuadraticForm. You can select an operation by clicking on it to get an individual page with information about that operation. Or you can select the buttons along the bottom to see alternative views or get additional information on the operations. Then return to the page on QuadraticForm.

Select **Cross Reference** to get another menu. This menu has buttons for **Parents**, **Ancestors**, and others. Clicking on **Parents**, you see that QuadraticForm has one parent AbelianMonoid.

## 13.6 Representation

---

The **Implementation** part of an AXIOM capsule for a domain constructor uses the special variable **Rep** to identify the lower level data type used to represent the objects of the domain. The **Rep** for quadratic forms is `SquareMatrix(n, K)`. This means that all objects of the domain are required to be  $n$  by  $n$  matrices with elements from  $K$ .

The code for **quadraticForm** in Figure 13.1 on page 677 checks that the matrix is symmetric and then converts it to “\$”, which means, as usual, “this domain.” Such explicit conversions are generally required by the compiler. Aside from checking that the matrix is symmetric, the code for this function essentially does nothing. The `m :: $` on line 28 coerces `m` to a quadratic form. In fact, the quadratic form you created in step (3) of Section 13.4 on page 678 is just the matrix you passed it in disguise! Without seeing this definition, you would not know that. Nor can you take advantage of this fact now that you do know! When we try in the next step of Section 13.4 on page 678 to regard `q` as a matrix by asking for **nrows**, the number of its rows, AXIOM gives you an error message saying, in effect, “Good try, but this won’t work!”

The definition for the **matrix** function could hardly be simpler: it just returns its argument after explicitly *coercing* its argument to a matrix. Since the argument is already a matrix, this coercion does no computation.

Within the context of a capsule, an object of “\$” is regarded both as a quadratic form *and* as a matrix.<sup>1</sup> This makes the definition of `q.v` easy—it just calls the **dot** product from `DirectProduct` to perform the indicated operation.

---

<sup>1</sup>In case each of “\$” and **Rep** have the same named operation available, the one from \$ takes precedence. Thus, if you want the one from “**Rep**”, you must package call it using a “\$**Rep**” suffix.

## 13.7 Multiple Representations

---

To write functions that implement the operations of a domain, you want to choose the most computationally efficient data structure to represent the elements of your domain.

A classic problem in computer algebra is the optimal choice for an internal representation of polynomials. If you create a polynomial, say  $3x^2 + 5$ , how does AXIOM hold this value internally? There are many ways. AXIOM has nearly a dozen different representations of polynomials, one to suit almost any purpose. Algorithms for solving polynomial equations work most efficiently with polynomials represented one way, whereas those for factoring polynomials are most efficient using another. One often-used representation is a list of terms, each term consisting of exponent-coefficient records written in the order of decreasing exponents. For example, the polynomial  $3x^2 + 5$  is represented by the list `[[e:2, c:3], [e:0, c:5]]`.

What is the optimal data structure for a matrix? It depends on the application. For large sparse matrices, a linked-list structure of records holding only the non-zero elements may be optimal. If the elements can be defined by a simple formula  $f(i, j)$ , then a compiled function for `f` may be optimal. Some programmers prefer to represent ordinary matrices as vectors of vectors. Others prefer to represent matrices by one big linear array where elements are accessed with linearly computable indexes.

While all these simultaneous structures tend to be confusing, AXIOM provides a helpful organizational tool for such a purpose: categories. `PolynomialCategory`, for example, provides a uniform user interface across all polynomial types. Each kind of polynomial implements functions for all these operations, each in its own way. If you use only the top-level operations in `PolynomialCategory` you usually do not care what kind of polynomial implementation is used.

Within a given domain, however, you define (at most) one representation.<sup>2</sup> If you want to have multiple representations (that is, several domains, each with its own representation), use a category to describe the **Exports**, then define separate domains for each representation.

## 13.8 Add Domain

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The capsule part of **Implementation** defines functions that implement the operations exported by the domain—usually only some of the operations. In our demo in Section 13.4 on page 678, we asked for the value of  $3*q-q+q$ . Where do the operations “\*”, “+”, and “-” come from? There is no definition for them in the capsule!

---

<sup>2</sup>You can make that representation a Union type, however. See Section 2.5 on page 108 for examples of unions.

The **Implementation** part of a definition can optionally specify an “add-domain” to the left of an **add** (for `QuadraticForm`, defines `SquareMatrix(n,K)` is the add-domain). The meaning of an add-domain is simply this: if the capsule part of the **Implementation** does not supply a function for an operation, AXIOM goes to the add-domain to find the function. So do “\*”, “+” and “-” come from `SquareMatrix(n,K)`?

## 13.9 Defaults

---

In Chapter 11, we saw that categories can provide default implementations for their operations. How and when are they used? When AXIOM finds that `QuadraticForm(2, Fraction Integer)` does not implement the operations “\*”, “+”, and “-”, it goes to `SquareMatrix(2, Fraction Integer)` to find it. As it turns out, `SquareMatrix(2, Fraction Integer)` does not implement *any* of these operations!

What does AXIOM do then? Here is its overall strategy. First, AXIOM looks for a function in the capsule for the domain. If it is not there, AXIOM looks in the add-domain for the operation. If that fails, AXIOM searches the add-domain of the add-domain, and so on. If all those fail, it then searches the default packages for the categories of which the domain is a member. In the case of `QuadraticForm`, it searches `AbelianGroup`, then its parents, grandparents, and so on. If this fails, it then searches the default packages of the add-domain. Whenever a function is found, the search stops immediately and the function is returned. When all fails, the system calls **error** to report this unfortunate news to you. To find out the actual order of constructors searched for `QuadraticForm`, consult Browse: from the `QuadraticForm`, click on **Cross Reference**, then on **Lineage**.

Let’s apply this search strategy for our example  $3q \cdot q + q$ . The scalar multiplication comes first. AXIOM finds a default implementation in `AbelianGroup`. Remember from Section 12.6 on page 668 that `SemiGroup` provides a default definition for  $x^n$  by repeated squaring? `AbelianGroup` similarly provides a definition for  $nx$  by repeated doubling.

But the search of the defaults for `QuadraticForm` fails to find any “+” or “\*” in the default packages for the ancestors of `QuadraticForm`. So it now searches among those for `SquareMatrix`. Category `MatrixCategory`, which provides a uniform interface for all matrix domains, is a grandparent of `SquareMatrix` and has a capsule defining many functions for matrices, including matrix addition, subtraction, and scalar multiplication. The default package `MatrixCategory` is where the functions for “+” and “-” come from.

You can use Browse to discover where the operations for `QuadraticForm` are implemented. First, get the page describing `QuadraticForm`. With



your mouse somewhere in this window, type a “2”, press the **Tab** key, and then enter “Fraction Integer” to indicate that you want the domain `QuadraticForm(2, Fraction Integer)`. Now click on **Operations** to get a table of operations and on “\*” to get a page describing the “\*” operation. Finally, click on **implementation** at the bottom.

## 13.10 Origins

---

Aside from the notion of where an operation is implemented, a useful notion is the *origin* or “home” of an operation. When an operation (such as **quadraticForm**) is explicitly exported by a domain (such as `QuadraticForm`), you can say that the origin of that operation is that domain. If an operation is not explicitly exported from a domain, it is inherited from, and has as origin, the (closest) category that explicitly exports it. The operations “+” and “-” of `QuadraticForm`, for example, are inherited from `AbelianMonoid`. As it turns out, `AbelianMonoid` is the origin of virtually every “+” operation in AXIOM!

Again, you can use Browse to discover the origins of operations. From the Browse page on `QuadraticForm`, click on **Operations**, then on **origins** at the bottom of the page.

The origin of the operation is the *only* place where on-line documentation is given. However, you can re-export an operation to give it special documentation. Suppose you have just invented the world’s fastest algorithm for inverting matrices using a particular internal representation for matrices. If your matrix domain just declares that it exports `MatrixCategory`, it exports the **inverse** operation, but the documentation the user gets from Browse is the standard one from `MatrixCategory`. To give your version of **inverse** the attention it deserves, simply export the operation explicitly with new documentation. This redundancy gives **inverse** a new origin and tells Browse to present your new documentation.

## 13.11 Short Forms

---

In AXIOM, a domain could be defined using only an add-domain and no capsule. Although we talk about rational numbers as quotients of integers, there is no type `RationalNumber` in AXIOM. To create such a type, you could compile the following “short-form” definition:

---

<code>RationalNumber() == Fraction(Integer)</code>	1
--	---

---

The **Exports** part of this definition is missing and is taken to be equivalent to that of `Fraction(Integer)`. Because of the add-domain philosophy, you get precisely what you want. The effect is to create a little stub of a domain. When a user asks to add two rational numbers, AXIOM would

ask RationalNumber for a function implementing this “+”. Since the domain has no capsule, the domain then immediately sends its request to Fraction (Integer).

The short form definition for domains is used to define such domains as MultivariatePolynomial:

---

MultivariatePolynomial(vl: List Symbol, R: Ring) ==	1
SparseMultivariatePolynomial(R,	2
OrderedVariableList vl)	3

---

## 13.12

### Example 1: Clifford Algebra

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Now that we have QuadraticForm available, let’s put it to use. Given some quadratic form  $Q$  described by an  $n$  by  $n$  matrix over a field  $K$ , the domain CliffordAlgebra( $n$ ,  $K$ ,  $Q$ ) defines a vector space of dimension  $2^n$  over  $K$ . This is an interesting domain since complex numbers, quaternions, exterior algebras and spin algebras are all examples of Clifford algebras.

The basic idea is this: the quadratic form  $Q$  defines a basis  $e_1, e_2, \dots, e_n$  for the vector space  $K^n$ —the direct product of  $K$  with itself  $n$  times. From this, the Clifford algebra generates a basis of  $2^n$  elements given by all the possible products of the  $e_i$  in order without duplicates, that is,  $1, e_1, e_2, e_1e_2, e_3, e_1e_3, e_2e_3, e_1e_2e_3$ , and so on.

The algebra is defined by the relations

$$\begin{aligned} e_i e_i &= Q(e_i) \\ e_i e_j &= -e_j e_i \quad \text{for } i \neq j \end{aligned}$$

Now look at the snapshot of its definition given in Figure 13.2. Lines 9-10 show part of the definitions of the **Exports**. A Clifford algebra over a field  $K$  is asserted to be a ring, an algebra over  $K$ , and a vector space over  $K$ . Its explicit exports include  $e(n)$ , which returns the  $n^{\text{th}}$  unit element.

The **Implementation** part begins by defining a local variable **Qeelist** to hold the list of all  $q.v$  where  $v$  runs over the unit vectors from 1 to the dimension  $n$ . Another local variable **dim** is set to  $2^n$ , computed once and for all. The representation for the domain is PrimitiveArray( $K$ ), which is a basic array of elements from domain  $K$ . Line 18 defines **New** as shorthand for the more lengthy expression **new(dim, 0\$K)\$Rep**, which computes a primitive array of length  $2^n$  filled with 0’s from domain  $K$ .

Lines 19-22 define the sum of two elements  $x$  and  $y$  straightforwardly. First, a new array of all 0’s is created, then filled with the sum of the corresponding elements. Indexing for primitive arrays starts at 0. The definition of the product of  $x$  and  $y$  first requires the definition of a local

```

NNI ==> NonNegativeInteger      1
PI  ==> PositiveInteger          2
                                   3
CliffordAlgebra(n,K,q): Exports == Implementation where  4
  n: PI                    5
  K: Field                 6
  q: QuadraticForm(n, K)   7
                                   8
Exports == Join(Ring,Algebra(K),VectorSpace(K)) with  9
  e: PI -> $              10
  ...                     11
                                   12
Implementation == add      13
  Qeelist :=              14
    [q.unitVector(i::PI) for i in 1..n]  15
  dim      := 2**n        16
  Rep      := PrimitiveArray K  17
  New ==> new(dim, 0$K)$Rep  18
  x + y ==              19
    z := New              20
    for i in 0..dim-1 repeat z.i := x.i + y.i  21
    z                                         22
  addMonomProd: (K, NNI, K, NNI, $) -> $  23
  addMonomProd(c1, b1, c2, b2, z) == ...  24
  x * y ==              25
    z := New              26
    for ix in 0..dim-1 repeat  27
      if x.ix ~= 0 then for iy in 0..dim-1 repeat  28
        if y.iy ~= 0  29
          then addMonomProd(x.ix,ix,y.iy,iy,z)  30
    z                                         31
    ...                                     32

```

Figure 13.2: Part of the CliffordAlgebra domain.

function **addMonomProd**. AXIOM knows it is local since it is not an exported function. The types of all local functions must be declared.

For a demonstration of CliffordAlgebra, see ‘CliffordAlgebra’ on page 378.

### 13.13

#### Example 2: Building A Query Facility

We now turn to an entirely different kind of application, building a query language for a database.

Here is the practical problem to solve. The Browse facility of AXIOM has a database for all operations and constructors which is stored on disk and accessed by HyperDoc. For our purposes here, we regard each line of this file as having eight fields: **class**, **name**, **type**, **nargs**, **exposed**, **kind**, **origin**, and **condition**. Here is an example entry:

```
o'determinant'$->R'1'x'd'Matrix(R)'has(R,commutative("")))
```

In English, the entry means:

The operation **determinant**:  $\$ \rightarrow R$  with 1 argument, is *exposed* and is exported by *domain* Matrix(R) if R **has** `commutative("*")`.

Our task is to create a little query language that allows us to get useful information from this database.

### 13.13.1 A Little Query Language

---

First we design a simple language for accessing information from the database. We have the following simple model in mind for its design. Think of the database as a box of index cards. There is only one search operation—it takes the name of a field and a predicate (a boolean-valued function) defined on the fields of the index cards. When applied, the search operation goes through the entire box selecting only those index cards for which the predicate is true. The result of a search is a new box of index cards. This process can be repeated again and again.

The predicates all have a particularly simple form: *symbol* = *pattern*, where *symbol* designates one of the fields, and *pattern* is a “search string”—a string that may contain a “\*” as a wildcard. Wildcards match any substring, including the empty string. Thus the pattern “\*ma\*t” matches “mat”, “doormat” and “smart”.

To illustrate how queries are given, we give you a sneak preview of the facility we are about to create.

Extract the database of all  
AXIOM operations.

```
ops := getDatabase("o")
6156
```

(1)  
Type: Database IndexCard

How many exposed  
three-argument **map** operations  
involving streams?

```
ops.(name="map").(nargs="3").(type="*Stream*")
3
```

(2)  
Type: Database IndexCard

As usual, the arguments of **elt** (“.”) associate to the left. The first **elt** produces the set of all operations with name **map**. The second **elt** produces the set of all map operations with three arguments. The third **elt** produces the set of all three-argument map operations having a type mentioning Stream.

Another thing we’d like to do is to extract one field from each of the index cards in the box and look at the result. Here is an example of that kind of request.

What constructors explicitly export a **determinant** operation?

```
elt(elt(elt(elt(ops,name="determinant"),origin),sort),unique)
["InnerMatrixLinearAlgebraFunctions", "MatrixCategory", "MatrixLinearAL,
```

Type: DataList String

The first **elt** produces the set of all index cards with name **determinant**. The second **elt** extracts the **origin** component from each index card. Each origin component is the name of a constructor which directly exports the operation represented by the index card. Extracting a component from each index card produces what we call a *datalist*. The third **elt**, **sort**, causes the datalist of origins to be sorted in alphabetic order. The fourth, **unique**, causes duplicates to be removed.

Before giving you a more extensive demo of this facility, we now build the necessary domains and packages to implement it.

### 13.13.2 The Database Constructor

We work from the top down. First, we define a database, our box of index cards, as an abstract datatype. For sake of illustration and generality, we assume that an index card is some type **S**, and that a database is a box of objects of type **S**. Here is the AXIOM program defining the Database domain.

Select by an equation.  
Select by a field name.  
Combine two databases.  
Subtract one from another.  
A brief database display.  
A full database display.  
A selective display.  
Display a database.

---

```
PI ==> PositiveInteger                                1
Database(S): Exports == Implementation where          2
  S: Object with                                       3
    elt: ($, Symbol) -> String                        4
    display: $ -> Void                                5
    fullDisplay: $ -> Void                             6
                                                         7
  Exports == with                                     8
    elt: ($,QueryEquation) -> $                       9
    elt: ($, Symbol) -> DataList String              10
    "+": ($,$) -> $                                   11
    "-": ($,$) -> $                                   12
    display: $ -> Void                                13
    fullDisplay: $ -> Void                             14
    fullDisplay: ($,PI,PI) -> Void                   15
    coerce: $ -> OutputForm                          16
  Implementation == add                              17
    ...                                              18
```

---

The domain constructor takes a parameter **S**, which stands for the class of index cards. We describe an index card later. Here think of an index card as a string which has the eight fields mentioned above.

First, we tell AXIOM what operations we are going to require from index cards. We need an **elt** to extract the contents of a field (such as **name**

and **type**) as a string. For example, `c.name` returns a string that is the content of the **name** field on the index card `c`. We need to display an index card in two ways: **display** shows only the name and type of an operation; **fullDisplay** displays all fields. The display operations return no useful information and thus have return type `Void`.

Next, we tell AXIOM what operations the user can apply to the database. This part defines our little query language. The most important operation is `db . field = pattern` which returns a new database, consisting of all index cards of `db` such that the **field** part of the index card is matched by the string pattern called **pattern**. The expression `field = pattern` is an object of type `QueryEquation` (defined in the next section).

Another **elt** is needed to produce a `DataList` object. Operation “+” is to merge two databases together; “-” is used to subtract away common entries in a second database from an initial database. There are three display functions. The **fullDisplay** function has two versions: one that prints all the records, the other that prints only a fixed number of records. A **coerce** to `OutputForm` creates a display object.

The **Implementation** part of Database is straightforward.

---

Implementation == add	1
s: Symbol	2
Rep := List S	3
elt(db,equation) == ...	4
elt(db,key) == [x.key for x in db]::DataList(String)	5
display(db) == for x in db repeat display x	6
fullDisplay(db) == for x in db repeat fullDisplay x	7
fullDisplay(db, n, m) == for x in db for i in 1..m	8
repeat	9
if i >= n then fullDisplay x	10
x+y == removeDuplicates! merge(x,y)	11
x-y == mergeDifference(copy(x::Rep),	12
y::Rep)\$MergeThing(S)	13
coerce(db): OutputForm == (#db)::OutputForm	14

---

The database is represented by a list of elements of `S` (index cards). We leave the definition of the first **elt** operation (on line 4) until the next section. The second **elt** collects all the strings with field name *key* into a list. The **display** function and first **fullDisplay** function simply call the corresponding functions from `S`. The second **fullDisplay** function provides an efficient way of printing out a portion of a large list. The “+” is defined by using the existing **merge** operation defined on lists, then removing duplicates from the result. The “-” operation requires writing a corresponding subtraction operation. A package `MergeThing` (not shown) provides this.

The **coerce** function converts the database to an OutputForm by computing the number of index cards. This is a good example of the independence of the representation of an AXIOM object from how it presents itself to the user. We usually do not want to look at a database—but do care how many “hits” we get for a given query. So we define the output representation of a database to be simply the number of index cards our query finds.

### 13.13.3 Query Equations

The predicate for our search is given by an object of type QueryEquation. AXIOM does not have such an object yet so we have to invent it.

---

```

QueryEquation(): Exports == Implementation where      1
  Exports == with                                     2
    equation: (Symbol, String) -> $                  3
    variable: $ -> Symbol                             4
    value:    $ -> String                             5
                                                    6
  Implementation == add                               7
    Rep := Record(var:Symbol, val:String)              8
    equation(x, s) == [x, s]                          9
    variable q == q.var                               10
    value    q == q.val                               11

```

---

AXIOM converts an input expression of the form  $a = b$  to `equation(a, b)`. Our equations always have a symbol on the left and a string on the right. The **Exports** part thus specifies an operation **equation** to create a query equation, and **variable** and **value** to select the left- and right-hand sides. The **Implementation** part uses `Record` for a space-efficient representation of an equation.

Here is the missing definition for the **elt** function of Database in the last section:

---

```

elt(db,eq) ==                                         1
  field := variable eq                               2
  value := value eq                                   3
  [x for x in db | matches?(value,x.field)]          4

```

---

Recall that a database is represented by a list. Line 4 simply runs over that list collecting all elements such that the pattern (that is, `value`) matches the selected field of the element.

### 13.13.4 DataLists

---

Type `DataList` is a new type invented to hold the result of selecting one field from each of the index cards in the box. It is useful to make datalists extensions of lists—lists that have special **elt** operations defined on them for sorting and removing duplicates.

---

```
DataList(S:OrderedSet) : Exports == Implementation where 1
  Exports == ListAggregate(S) with 2
    elt: ($,"unique") -> $ 3
    elt: ($,"sort") -> $ 4
    elt: ($,"count") -> NonNegativeInteger 5
    coerce: List S -> $ 6
  Implementation == List(S) add 7
  Rep := List S 8
  elt(x,"unique") == removeDuplicates(x) 9
  elt(x,"sort") == sort(x) 10
  elt(x,"count") == #x 11
  coerce(x:List S) == x :: $ 12
  13
```

---

The **Exports** part asserts that datalists belong to the category `ListAggregate`. Therefore, you can use all the usual list operations on datalists, such as **first**, **rest**, and **concat**. In addition, datalists have four explicit operations. Besides the three **elt** operations, there is a **coerce** operation that creates datalists from lists.

The **Implementation** part needs only to define four functions. All the rest are obtained from `List(S)`.

### 13.13.5 Index Cards

---

An index card comes from a file as one long string. We define functions that extract substrings from the long string. Each field has a name that is passed as a second argument to **elt**.

---

```
IndexCard() == Implementation where 1
  Exports == with 2
    elt: ($, Symbol) -> String 3
    display: $ -> Void 4
    fullDisplay: $ -> Void 5
    coerce: String -> $ 6
  Implementation == String add ... 7
```

---

We leave the **Implementation** part to the reader. All operations involve straightforward string manipulations.



### 13.13.6 Creating a Database

We must not forget one important operation: one that builds the database in the first place! We'll name it **getDatabase** and put it in a package. This function is implemented by calling the Common LISP function **getBrowseDatabase(s)** to get appropriate information from Browse. This operation takes a string indicating which lines you want from the database: "o" gives you all operation lines, and "k", all constructor lines. Similarly, "c", "d", and "p" give you all category, domain and package lines respectively.

---

OperationsQuery(): Exports == Implementation where	1
Exports == with	2
getDatabase: String -> Database(IndexCard)	3
	4
Implementation == add	5
getDatabase(s) == getBrowseDatabase(s)\$Lisp	6

---

We do not bother creating a special name for databases of index cards. Database(IndexCard) will do. Notice that we used the package OperationsQuery to create, in effect, a new kind of domain: Database(IndexCard).

### 13.13.7 Putting It All Together

To create the database facility, you put all these constructors into one file.<sup>3</sup> At the top of the file put **)abbrev** commands, giving the constructor abbreviations you created.

---

)abbrev domain ICARD	IndexCard	1
)abbrev domain QEQUAT	QueryEquation	2
)abbrev domain MTHING	MergeThing	3
)abbrev domain DLIST	DataList	4
)abbrev domain DBASE	Database	5
)abbrev package OPQUERY	OperationsQuery	6

---

With all this in **alql.spad**, for example, compile it using

```
)compile alql
```

and then load each of the constructors:

```
)load ICARD QEQUAT MTHING DLIST DBASE OPQUERY
```

You are ready to try some sample queries.

---

<sup>3</sup>You could use separate files, but we are putting them all together because, organizationally, that is the logical thing to do.

### 13.13.8 Example Queries

---

How many constructors does  
AXIOM have?

Our first set of queries give some statistics on constructors in the current  
AXIOM system.

```
ks := getDatabase "k"
```

```
1048
```

(1)

Type: Database IndexCard

Break this down into the  
number of categories, domains,  
and packages.

```
[ks.(kind=k) for k in ["c","d","p"]]
```

```
[199, 382, 467]
```

(2)

Type: List Database IndexCard

What are all the domain constructors that take no parameters?

```
elt(ks.(kind="d").(nargs="0"),name)
["AlgebraicNumber", "AnonymousFunction", "Any", "AttributeButtons", "Basi
"BinaryExpansion", "BinaryFile", "Bits", "Boolean", "CardinalNumber", "CI
"Color", "Commutator", "DecimalExpansion", "DoubleFloat", "DrawOption",
"FileName", "Float", "FortranCode", "FortranScalarType", "FortranTemplat
"GraphImage", "HexadecimalExpansion", "IVBaseColor", "IVBasicNode", "IVC
"IVFaceSet", "IVField", "IVGroup", "IVIndexedLineSet", "IVNodeConnection
"IVPointSet", "IVQuadMesh", "IVSeparator", "IVSimpleInnerNode", "IVUtili
"IndexCard", "InnerAlgebraicNumber", "InputForm", "Integer", "Integratio
"InventorRenderPackage", "InventorViewPort", "Library", "MachineComplex
"MachineInteger", "NagDiscreteFourierTransformInterfacePackage", "NagE
"NagOptimisationInterfacePackage", "NagQuadratureInterfacePackage", "N
"NagSpecialFunctionsInterfacePackage", "NonNegativeInteger", "None", "N
"NumericalODEProblem", "NumericalOptimizationProblem", "NumericalPDEPr
"OrdSetInts", "OutputForm", "Palette", "Partition", "Pi", "PlaneAlgebrai
"Plot", "PositiveInteger", "QueryEquation", "RenderTools", "Result", "Ro
"SExpression", "ScriptFormulaFormat", "SingleInteger", "SingletonAsOrde
"Switch", "SymbolTable", "Symbol", "TexFormat", "TextFile", "TheSymbolTa
"Timer", "TwoDimensionalViewport", "Void", "d01TransformFunctionType",
"d01akfAnnaType", "d01alfAnnaType", "d01amfAnnaType", "d01anfAnnaType",
"d01aqfAnnaType", "d01asfAnnaType", "d01fcfAnnaType", "d01gbfAnnaType",
"d02bhfAnnaType", "d02cjfAnnaType", "d02ejfAnnaType", "d03eefAnnaType",
"e04dgmfAnnaType", "e04fdfAnnaType", "e04gcfAnnaType", "e04jafAnnaType",
"e04nafAnnaType", "e04ucfAnnaType", "TexFormat"]
```

How many constructors have “Matrix” in their name?	<pre>mk := ks.(name="*Matrix*")</pre> <p>26</p> <p style="text-align: right;">(4)</p> <p style="text-align: right;">Type: Database IndexCard</p>
What are the names of those that are domains?	<pre>elt(mk.(kind="d"),name)</pre> <p>["DenavitHartenbergMatrix", "DirectProductMatrixModule", "IndexedMatrix", "Matrix", "RectangularMatrix", "SquareMatrix", "ThreeDimensionalMatrix"]</p> <p style="text-align: right;">Type: DataList String</p>
How many operations are there in the library?	<pre>o := getDatabase "o"</pre> <p>6156</p> <p style="text-align: right;">(6)</p> <p style="text-align: right;">Type: Database IndexCard</p>
Break this down into categories, domains, and packages.	<pre>[o.(kind=k) for k in ["c","d","p"]]</pre> <p>[1590, 1956, 2610]</p> <p style="text-align: right;">(7)</p> <p style="text-align: right;">Type: List Database IndexCard</p> <p>The query language is helpful in getting information about a particular operation you might like to apply. While this information can be obtained with Browse, the use of the query database gives you data that you can manipulate in the workspace.</p>
How many operations have “eigen” in the name?	<pre>eigens := o.(name="*eigen*")</pre> <p>4</p> <p style="text-align: right;">(8)</p> <p style="text-align: right;">Type: Database IndexCard</p>
What are their names?	<pre>elt(eigens,name)</pre> <p>["eigenMatrix", "eigenvalues", "eigenvector", "eigenvectors"]</p> <p style="text-align: right;">(9)</p> <p style="text-align: right;">Type: DataList String</p>
Where do they come from?	<pre>elt(elt(elt(eigens,origin),sort),unique)</pre> <p>["EigenPackage", "RadicalEigenPackage"]</p> <p style="text-align: right;">(10)</p> <p style="text-align: right;">Type: DataList String</p> <p>The operations “+” and “−” are useful for constructing small databases and combining them. However, remember that the only matching you can do is string matching. Thus a pattern such as “*Matrix*” on the type field matches any type containing Matrix, MatrixCategory, SquareMatrix, and so on.</p>
How many operations mention “Matrix” in their type?	<pre>tm := o.(type="*Matrix*")</pre> <p>353</p> <p style="text-align: right;">(11)</p> <p style="text-align: right;">Type: Database IndexCard</p>

How many operations come from constructors with “Matrix” in their name?	<code>fm := o.(origin="*Matrix*")</code> 192	(12)
		Type: Database IndexCard
How many operations are in <code>fm</code> but not in <code>tm</code> ?	<code>fm-tm</code> 146	(13)
		Type: Database IndexCard

Display the operations that both mention “Matrix” in their type and come from a constructor having “Matrix” in their name.

```
fullDisplay(fm-%)
** : (Matrix(R),NonNegativeInteger)->Matrix(R)
    from StorageEfficientMatrixOperations(R) (unexposed)
clearDenominator : (Matrix(Q))->Matrix(R)
    from MatrixCommonDenominator(R,Q)
coerceP :
(Vector(Matrix(R)))->Vector(Matrix(Polynomial(R)))
    from CoerceVectorMatrixPackage(R) (unexposed)
coerce
:
(Vector(Matrix(R)))-
>Vector(Matrix(Fraction(Polynomial(R))
))
    from CoerceVectorMatrixPackage(R) (unexposed)
coerce : (.$)->Matrix(R)
    from RectangularMatrix(m,n,R) (unexposed)
coerce : (.$)->Matrix(R) from SquareMatrix(ndim,R)
(unexposed)
coerce : (Matrix(MachineFloat))->.$ from
FortranMatrixCategory
commonDenominator : (Matrix(Q))->R
    from MatrixCommonDenominator(R,Q)
copy! : (Matrix(R),Matrix(R))->Matrix(R)
    from StorageEfficientMatrixOperations(R) (unexposed)
f01brf
:
(Integer,Integer,Integer,Integer,DoubleFloat,Boolean,Boole
an,List(Boolean),Matrix(DoubleFloat),Matrix(Integer),Matri
x(Integer),Integer)->Result
    from NagMatrixOperationsPackage
f01bsf
:
(Integer,Integer,Integer,Matrix(Integer),Matrix(Integer),M
atrix(Integer),Matrix(Integer),Boolean,DoubleFloat,Boolean
,Matrix(Integer),Matrix(DoubleFloat),Integer)->Result
    from NagMatrixOperationsPackage
f01maf
:
(Integer,Integer,Integer,Integer,List(Boolean),Matrix(Doub
leFloat),Matrix(Integer),Matrix(Integer),DoubleFloat,Doubl
eFloat,Integer)->Result
    from NagMatrixOperationsPackage
f01mcf
:
(Integer,Matrix(DoubleFloat),Integer,Matrix(Integer),Integ
er)->Result
    from NagMatrixOperationsPackage
f01qcf
:
(Integer,Integer,Integer,Matrix(DoubleFloat),Integer)-
>Res
ult
    from NagMatrixOperationsPackage
f01qdf
:
(String,String,Integer,Integer,Matrix(DoubleFloat),Integer
,Matrix(DoubleFloat),Integer,Integer,Matrix(DoubleFloat),I
nteger)->Result
    from NagMatrixOperationsPackage
f01qef
:
(String,Integer,Integer,Integer,Integer,Matrix(DoubleFloat
),Matrix(DoubleFloat),Integer)->Result
    from NagMatrixOperationsPackage
f01rcf
```

How many operations involve matrices?

```
m := tm+fm
```

```
499
```

(15)

Type: Database IndexCard

Display 4 of them.

```
fullDisplay(m, 202, 205)
```

```
elt : (_,List(Integer),List(Integer))->_
      from MatrixCategory(R,Row,Col)
```

```
elt : (_,Integer,Integer,R)->R
      from RectangularMatrixCategory(m,n,R,Row,Col)
```

```
elt
```

```
:
```

```
(_,NonNegativeInteger,NonNegativeInteger,NonNegativeInteger)->R
      from ThreeDimensionalMatrix(R)
```

```
eval
```

```
:
```

```
(Matrix(Expression(DoubleFloat)),List(Symbol),Vector(Expression(DoubleFloat)))->Matrix(Expression(DoubleFloat))
      from d02AgentsPackage
```

Type: Void

How many distinct names of operations involving matrices are there?

```
elt(elt(elt(m,name),unique),count)
```

```
317
```

(17)

Type: PositiveInteger





---

## CHAPTER 14

---

# Browse

This chapter discusses the Browse component of HyperDoc. We suggest you invoke AXIOM and work through this chapter, section by section, following our examples to gain some familiarity with Browse.

### 14.1 The Front Page: Searching the Library

---

To enter Browse, click on **Browse** on the top level page of HyperDoc to get the *front page* of Browse.

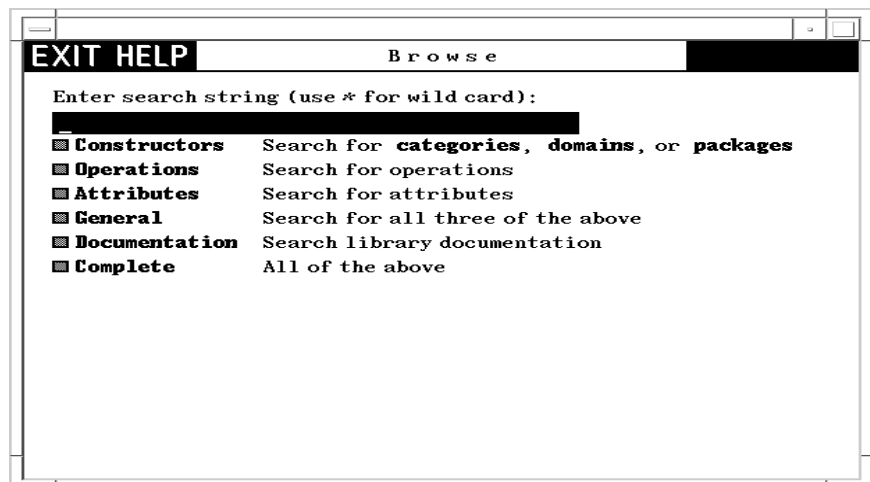



Figure 14.1: The Browse front page.

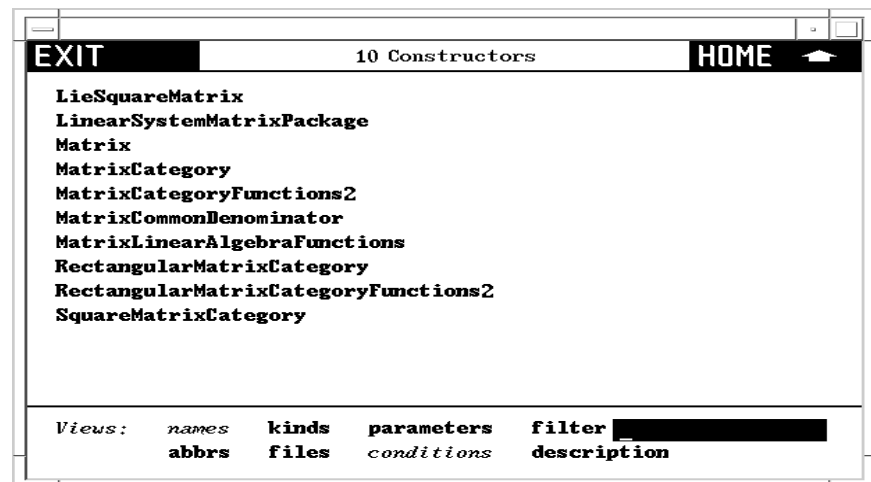
To use this page, you first enter a *search string* into the input area at the top, then click on one of the buttons below. We show the use of each of the buttons by example.

## Constructors

First enter the search string **Matrix** into the input area and click on **Constructors**. What you get is the *constructor page* for Matrix. We show and describe this page in detail in Section 14.2 on page 704. By convention, AXIOM does a case-insensitive search for a match. Thus **matrix** is just as good as **Matrix**, has the same effect as **MaTriX**, and so on. We recommend that you generally use small letters for names however. A search string with only capital letters has a special meaning (see Section 14.3.3 on page 719).

Click on  to return to the Browse front page.

Use the symbol “\*” in search strings as a *wild card*. A wild card matches any substring, including the empty string. For example, enter the search string **\*matrix\*** into the input area and click on **Constructors**.<sup>1</sup> What you get is a table of all constructors whose names contain the string “matrix.”





EXIT		10 Constructors		HOME 	
<b>LieSquareMatrix</b> <b>LinearSystemMatrixPackage</b> <b>Matrix</b> <b>MatrixCategory</b> <b>MatrixCategoryFunctions2</b> <b>MatrixCommonDenominator</b> <b>MatrixLinearAlgebraFunctions</b> <b>RectangularMatrixCategory</b> <b>RectangularMatrixCategoryFunctions2</b> <b>SquareMatrixCategory</b>					
<b>Views:</b> <i>names</i> <b>kinds</b> <b>parameters</b> <b>filter</b> <input type="text"/>					
<b>abbrs</b> <b>files</b> <i>conditions</i> <b>description</b>					


Figure 14.2: Table of exposed constructors matching **\*matrix\***.

All constructors containing the string are listed, whether *exposed* or *unexposed*. You can hide the names of the unexposed constructors by clicking on the **\*=unexposed** button in the *Views* panel at the bottom of the window. (The button will change to **exposed only**.)

One of the names in this table is Matrix. Click on Matrix. What you get is again the constructor page for Matrix. As you see, Browse gives you a large network of information in which there are many ways to reach the same pages.

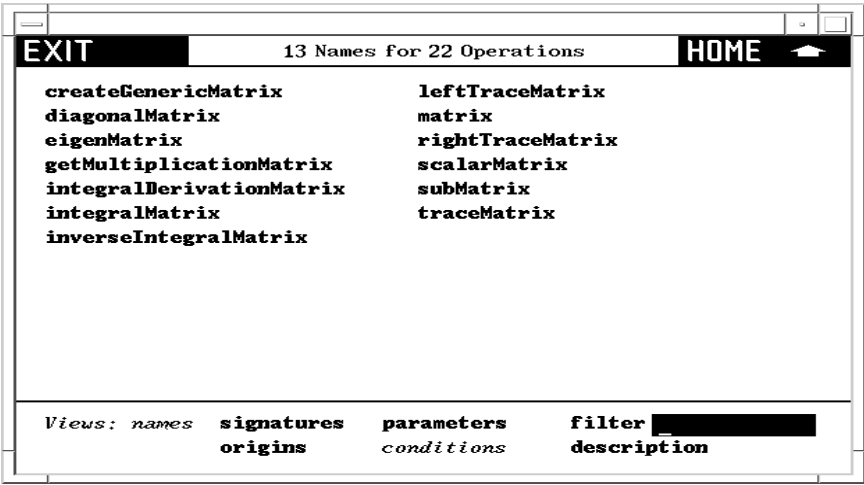
<sup>1</sup>To get only categories, domains, or packages, rather than all constructors, you can click on the corresponding button to the right of **Constructors**.


Again click on the  to return to the table of constructors whose names contain **matrix**. Below the table is a *Views* panel. This panel contains buttons that let you view constructors in different ways. To learn about views of constructors, skip to Section 14.2.3 on page 712.

Click on  to return to the Browse front page.

Operations

Enter **\*matrix** into the input area and click on **Operations**. This time you get a table of *operations* whose names end with **matrix** or **Matrix**.




EXIT		13 Names for 22 Operations		HOME 	
createGenericMatrix	leftTraceMatrix				
diagonalMatrix	matrix				
eigenMatrix	rightTraceMatrix				
getMultiplicationMatrix	scalarMatrix				
integralDerivationMatrix	subMatrix				
integralMatrix	traceMatrix				
inverseIntegralMatrix					

Views: names

signatures

parameters

filter 


origins

conditions

description


Figure 14.3: Table of operations matching **\*matrix** .

If you select an operation name, you go to a page describing all the operations in AXIOM of that name. At the bottom of an operation page is another kind of *Views* panel, one for operation pages. To learn more about these views, skip to Section 14.3.2 on page 715.

Click on  to return to the Browse front page.

Attributes

This button gives you a table of attribute names that match the search string. Enter the search string **\*** and click on **Attributes** to get a list of all system attributes.

Click on  to return to the Browse front page.

Again there is a *Views* panel at the bottom with buttons that let you view the attributes in different ways.

General

This button does a general search for all constructor, operation, and attribute names matching the search string. Enter the search string **\*matrix\*** into the input area. Click on **General** to find all constructs that have **matrix** as a part of their name.


EXIT		19 Names for 67 Attributes	HOME
<b>additiveValuation</b>		<b>finiteAggregate</b>	
<b>approximate</b>		<b>leftUnitary</b>	
<b>arbitraryExponent</b>		<b>multiplicativeValuation</b>	
<b>arbitraryPrecision</b>		<b>noetherian</b>	
<b>canonical</b>		<b>noZeroDivisors</b>	
<b>canonicalsClosed</b>		<b>partiallyOrderedSet</b>	
<b>canonicalUnitNormal</b>		<b>rightUnitary</b>	
<b>central</b>		<b>shallowlyMutable</b>	
<b>commutative</b>		<b>unitsKnown</b>	
<b>complex</b>			
Views: <i>names</i> <i>parameters</i> <b>filter</b> <input type="text"/>			
	<b>origins</b>	<b>conditions</b>	<b>description</b>

Figure 14.4: Table of AXIOM attributes.

EXIT		35 entries match *matrix*	HOME
■ 25 operations			
■ 3 categories			
<b>MatrixCategory</b>		<b>SquareMatrixCategory</b>	
<b>RectangularMatrixCategory</b>			
■ 2 domains			
<b>LieSquareMatrix</b>		<b>Matrix</b>	
■ 5 packages			
<b>LinearSystemMatrixPackage</b>			
<b>MatrixCategoryFunctions2</b>			
<b>MatrixCommonDenominator</b>			
<b>MatrixLinearAlgebraFunctions</b>			
<b>RectangularMatrixCategoryFunctions2</b>			

Figure 14.5: Table of all constructs matching \*matrix\*.

The summary gives you all the names under a heading when the number of entries is less than 10.

Click on  to return to the Browse front page.

Documentation

Again enter the search key `*matrix*` and this time click on **Documentation**. This search matches any constructor, operation, or attribute name whose documentation contains a substring matching `matrix`.

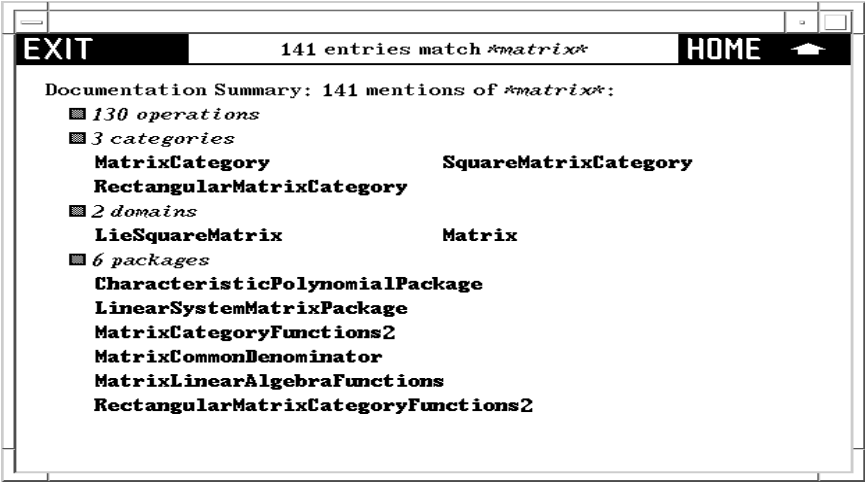



Figure 14.6: Table of constructs with documentation matching `*matrix*`.

Click on  to return to the Browse front page.

Complete

This search combines both **General** and **Documentation**.

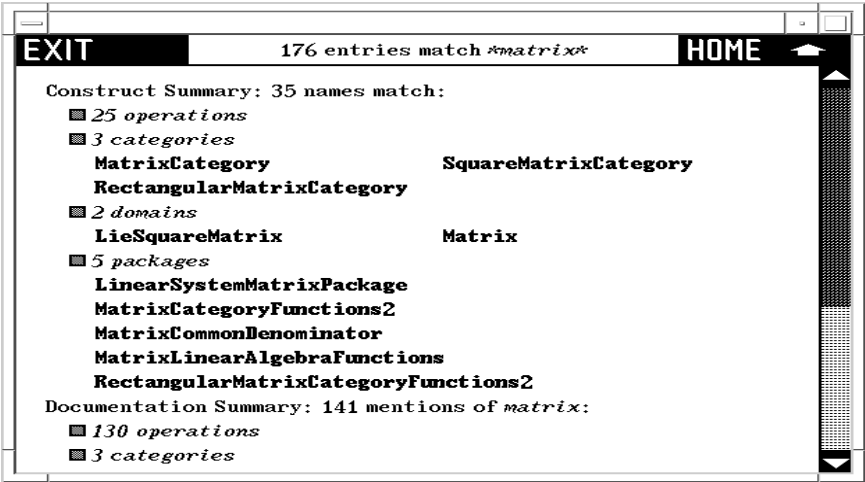


Figure 14.7: Table summarizing complete search for pattern `*matrix*`.

## 14.2 The Constructor Page

In this section we look in detail at a constructor page for domain Matrix. Enter `matrix` into the input area on the main Browse page and click on Constructors.

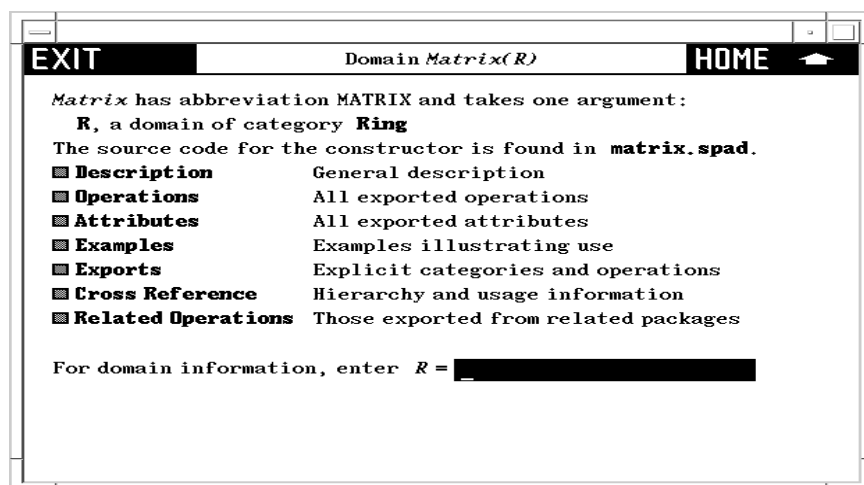


Figure 14.8: Constructor page for Matrix.

The header part tells you that Matrix has abbreviation MATRIX and one argument called R that must be a domain of category Ring. Just what domains can be arguments of Matrix? To find this out, click on the R on the second line of the heading. What you get is a table of all acceptable domain parameter values of R, or a table of *rings* in AXIOM.

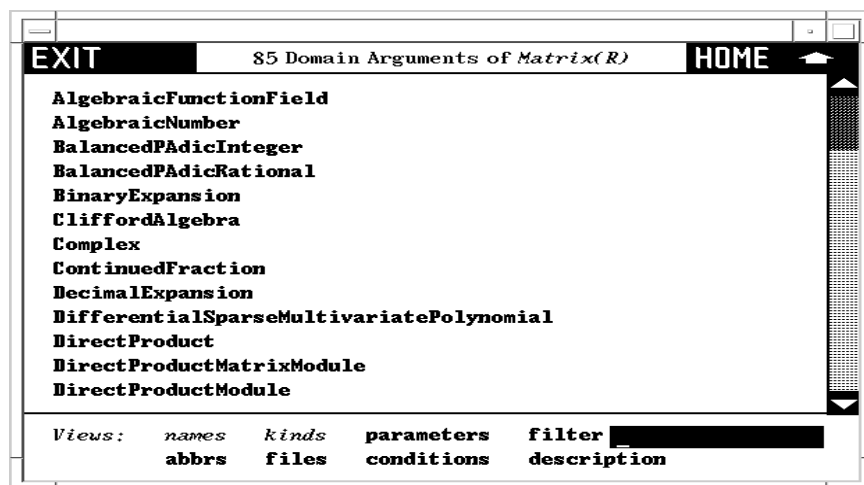

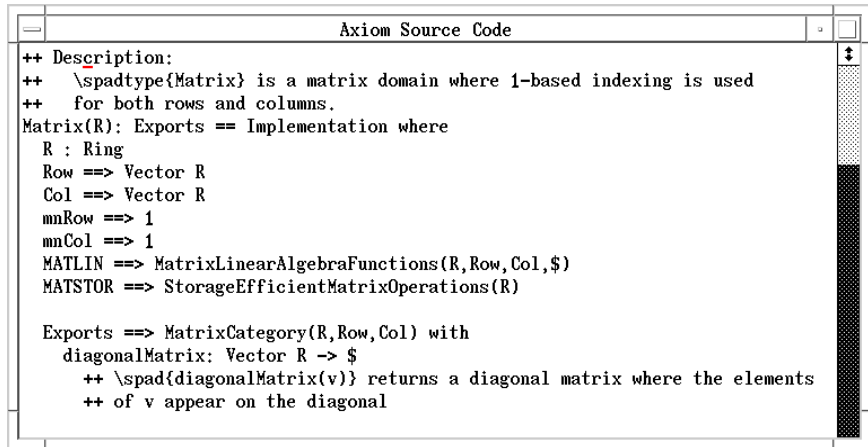


Figure 14.9: Table of acceptable domain parameters to Matrix.

Click on  to return to the constructor page for Matrix.

If you have access to the source code of AXIOM, the third line of the heading gives you the name of the source file containing the definition of Matrix. Click on it to pop up an editor window containing the source code of Matrix.



```

++ Description:
++ \spadtype{Matrix} is a matrix domain where 1-based indexing is used
++ for both rows and columns.
Matrix(R): Exports == Implementation where
  R : Ring
  Row ==> Vector R
  Col ==> Vector R
  mnRow ==> 1
  mnCol ==> 1
  MATLIN ==> MatrixLinearAlgebraFunctions(R,Row,Col,$)
  MATSTOR ==> StorageEfficientMatrixOperations(R)

Exports ==> MatrixCategory(R,Row,Col) with
  diagonalMatrix: Vector R -> $
  ++ \spad{diagonalMatrix(v)} returns a diagonal matrix where the elements
  ++ of v appear on the diagonal

```

Figure 14.10: Source code for Matrix.

We recommend that you leave the editor window up while working through this chapter as you occasionally may want to refer to it.

## 14.2.1 Constructor Page Buttons

Description

We examine each button on this page in order.

Click here to bring up a page with a brief description of constructor `Matrix`. If you have access to system source code, note that these comments can be found directly over the constructor definition.

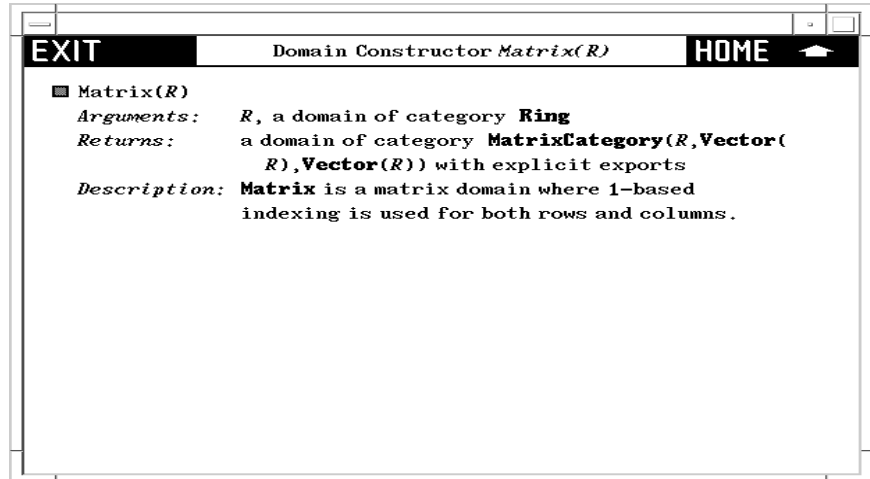


Figure 14.11: Description page for `Matrix`.

Operations

Click here to get a table of operations exported by `Matrix`. You may wish to widen the window to have multiple columns as below.

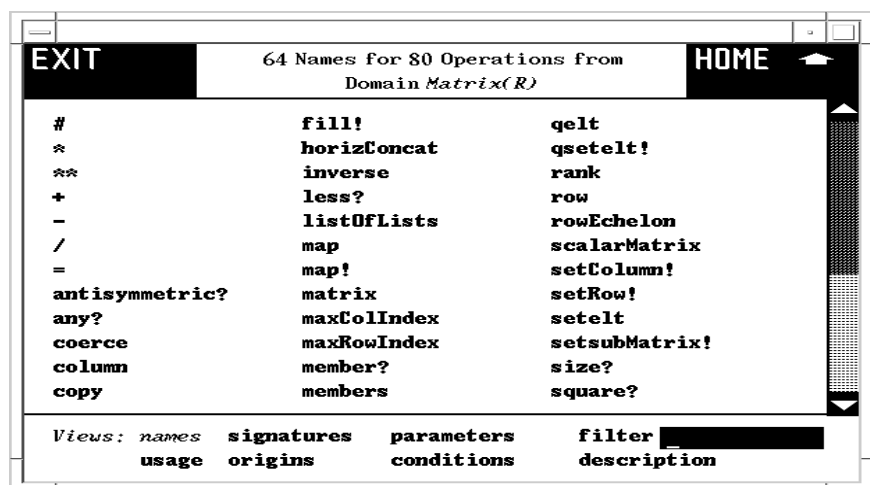


Figure 14.12: Table of operations from `Matrix`.

If you click on an operation name, you bring up a description page for



## Attributes

the operations. For a detailed description of these pages, skip to Section 14.3.2 on page 715.

Click here to get a table of the two attributes exported by Matrix: `finiteAggregate` and `shallowlyMutable`. These are two computational properties that result from Matrix being regarded as a data structure.

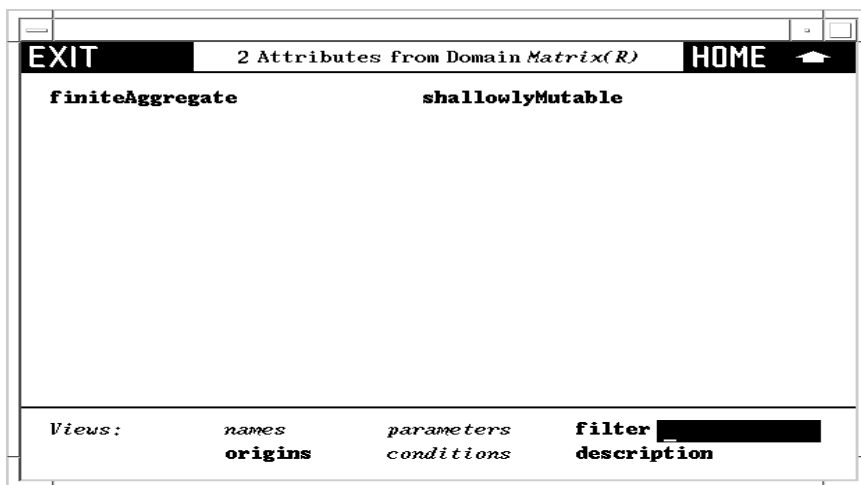


Figure 14.13: Attributes from Matrix.

## Examples

Click here to get an *examples page* with examples of operations to create and manipulate matrices.

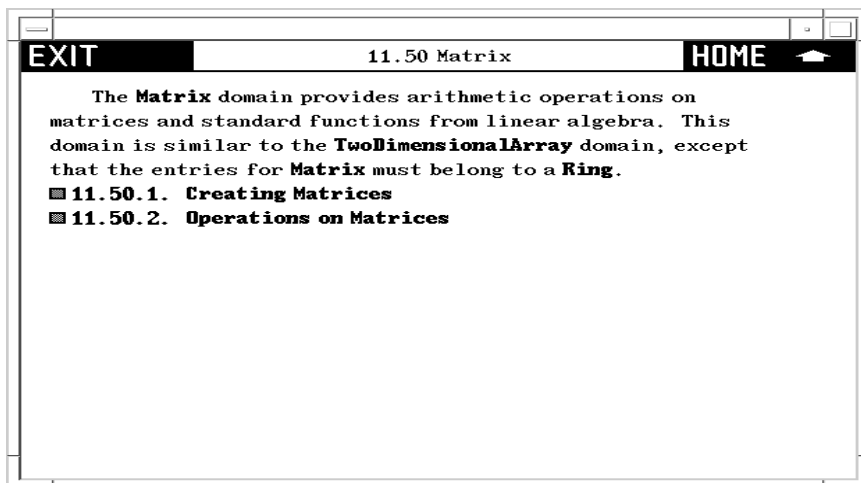


Figure 14.14: Example page for Matrix.

Read through this section. Try selecting the various buttons. Notice that if you click on an operation name, such as **new**, you bring up a description

## Exports

page for that operation from Matrix.

Example pages have several examples of AXIOM commands. Each example has an active button to its left. Click on it! A pre-computed answer is pasted into the page immediately following the command. If you click on the button a second time, the answer disappears. This button thus acts as a toggle: “now you see it; now you don’t.”

Note also that the AXIOM commands themselves are active. If you want to see AXIOM execute the command, then click on it! A new AXIOM window appears on your screen and the command is executed.

Click [here](#) to see a page describing the exports of Matrix exactly as described by the source code.

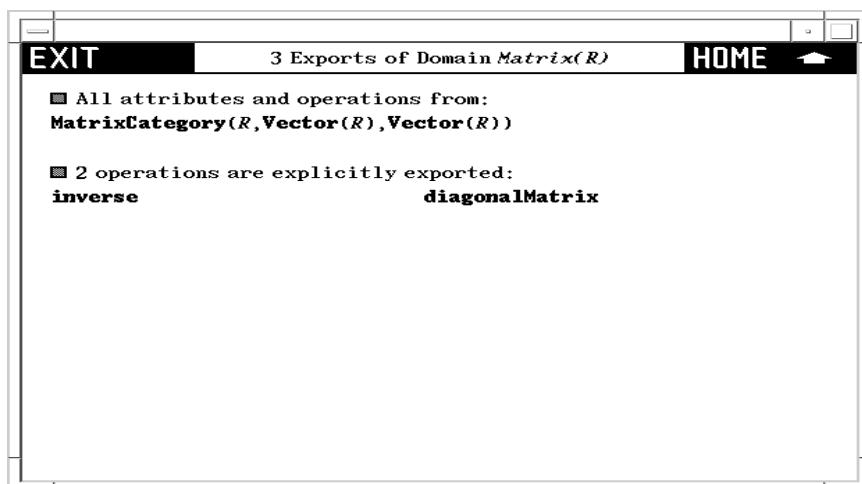




Figure 14.15: Exports of Matrix.

As you see, Matrix declares that it exports all the operations and attributes exported by category `MatrixCategory(R, Row, Col)`. In addition, two operations, **diagonalMatrix** and **inverse**, are explicitly exported.

To learn a little about the structure of AXIOM, we suggest you do the following exercise. Otherwise, go on to the next section. Matrix explicitly exports only two operations. The other operations are thus exports of `MatrixCategory`. In general, operations are usually not explicitly exported by a domain. Typically they are *inherited* from several different categories. Let’s find out from where the operations of Matrix come.

1. Click on **MatrixCategory**, then on **Exports**. Here you see that **MatrixCategory** explicitly exports many matrix operations. Also, it inherits its operations from `TwoDimensionalArrayCategory`.
2. Click on **TwoDimensionalArrayCategory**, then on **Exports**.

Here you see explicit operations dealing with rows and columns. In addition, it inherits operations from HomogeneousAggregate.

3. Click on  and then click on **Object**, then on **Exports**, where you see there are no exports.
4. Click on  repeatedly to return to the constructor page for Matrix.

## Related Operations

Click here bringing up a table of operations that are exported by *packages* but not by Matrix itself.

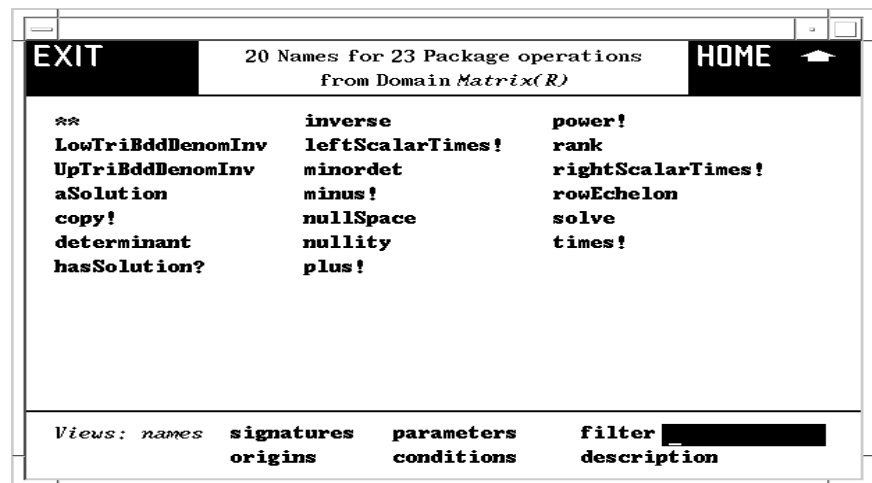


Figure 14.16: Related operations of Matrix.

To see a table of such packages, use the **Relatives** button on the **Cross Reference** page described next.

## 14.2.2 Cross Reference

### Parents

Click on the **Cross Reference** button on the main constructor page for Matrix. This gives you a page having various cross reference information stored under the respective buttons.

The parents of a domain are the same as the categories mentioned under the **Exports** button on the first page. Domain Matrix has only one parent but in general a domain can have any number.

### Ancestors

The *ancestors* of a constructor consist of its parents, the parents of its parents, and so on. Did you perform the exercise in the last section under **Exports**? If so, you see here all the categories you found while ascending the **Exports** chain for Matrix.

### Relatives

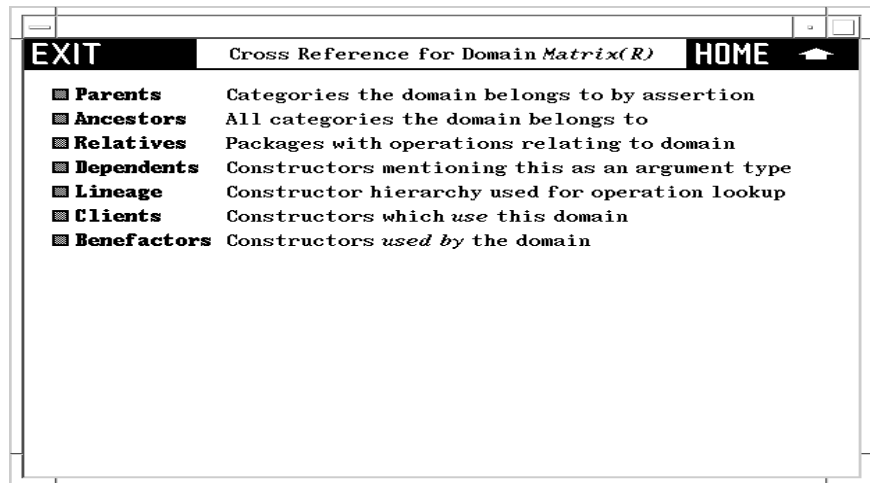




Figure 14.17: Cross-reference page for Matrix.

The *relatives* of a domain constructor are package constructors that provide operations in addition to those *exported* by the domain.

Try this exercise.

1. Click on **Relatives**, bringing up a list of *packages*.
2. Click on **LinearSystemMatrixPackage** bringing up its constructor page.<sup>2</sup>
3. Click on **Operations**. Here you see **rank**, an operation also exported by Matrix itself.
4. Click on **rank**. This **rank** has two arguments and thus is different from the **rank** from Matrix.
5. Click on  to return to the list of operations for the package LinearSystemMatrixPackage.
6. Click on **solve** to bring up a **solve** for linear systems of equations.
7. Click on  several times to return to the cross reference page for Matrix.

## Dependents

The *dependents* of a constructor are those *domains* or *packages* that mention that constructor either as an argument or in its *exports*.

If you click on **Dependents** two entries may surprise you: RectangularMatrix and SquareMatrix. This happens because Matrix, as it turns out, appears in signatures of operations exported by these domains.

<sup>2</sup>You may want to widen your HyperDoc window to make what follows more legible.

## Lineage

The term *lineage* refers to the *search order* for functions. If you are an expert user or curious about how the AXIOM system works, try the following exercise. Otherwise, you best skip this button and go on to **Clients**.

Clicking on **Lineage** gives you a list of domain constructors: `InnerIndexedTwoDimensionalArray`, `MatrixCategory&`, `TwoDimensionalArrayCategory&`, `HomogeneousAggregate&`, `Aggregate&`. What are these constructors and how are they used?

We explain by an example. Suppose you create a matrix using the interpreter, then ask for its **rank**. AXIOM must then find a function implementing the **rank** operation for matrices. The first place AXIOM looks for **rank** is in the Matrix domain.

If not there, the lineage of Matrix tells AXIOM where else to look. Associated with the matrix domain are five other lineage domains. Their order is important. AXIOM first searches the first one, `InnerIndexedTwoDimensionalArray`. If not there, it searches the second `MatrixCategory&`. And so on.

Where do these *lineage constructors* come from? The source code for Matrix contains this syntax for the *function body* of Matrix:<sup>3</sup>

```
InnerIndexedTwoDimensionalArray(R,mnRow,mnCol,Row,Col)
  add ...
```

where the “...” denotes all the code that follows. In English, this means: “The functions for matrices are defined as those from `InnerIndexedTwoDimensionalArray` domain augmented by those defined in ‘...’,” where the latter take precedence.

This explains `InnerIndexedTwoDimensionalArray`. The other names, those with names ending with an ampersand “&” are *default packages* for categories to which Matrix belongs. Default packages are ordered by the notion of “closest ancestor.”

## Clients

A client of Matrix is any constructor that uses Matrix in its implementation. For example, `Complex` is a client of Matrix; it exports several operations that take matrices as arguments or return matrices as values.<sup>4</sup>

## Benefactors

---

<sup>3</sup>`InnerIndexedTwoDimensionalArray` is a special domain implemented for matrix-like domains to provide efficient implementations of two-dimensional arrays. For example, domains of category `TwoDimensionalArrayCategory` can have any integer as their `minIndex`. Matrices and other members of this special “inner” array have their `minIndex` defined as 1.

<sup>4</sup>A constructor is a client of Matrix if it handles any matrix. For example, a constructor having internal (unexported) operations dealing with matrices is also a client.

A *benefactor* of Matrix is any constructor that Matrix uses in its implementation. This information, like that for clients, is gathered from run-time structures.<sup>5</sup>

Cross reference pages for categories have some different buttons on them. Starting with the constructor page of Matrix, click on Ring producing its constructor page. Click on **Cross Reference**, producing the cross-reference page for Ring. Here are buttons **Parents** and **Ancestors** similar to the notion for domains, except for categories the relationship between parent and child is defined through *category extension*.

Children

Category hierarchies go both ways. There are children as well as parents. A child can have any number of parents, but always at least one. Every category is therefore a descendant of exactly one category: Object.

Descendants

These are children, children of children, and so on.

Category hierarchies are complicated by the fact that categories take parameters. Where a parameterized category fits into a hierarchy *may* depend on values of its parameters. In general, the set of categories in AXIOM forms a *directed acyclic graph*, that is, a graph with directed arcs and no cycles.

Domains

This produces a table of all domain constructors that can possibly be rings (members of category Ring). Some domains are unconditional rings. Others are rings for some parameters and not for others. To find out which, select the **conditions** button in the views panel. For example, `DirectProduct(n, R)` is a ring if `R` is a ring.

### 14.2.3 Views Of Constructors

---

Below every constructor table page is a *Views* panel. As an example, click on **Cross Reference** from the constructor page of Matrix, then on **Benefactors** to produce a short table of constructor names.

The *Views* panel is at the bottom of the page. Two items, *names* and *conditions*, are in italics. Others are active buttons. The active buttons are those that give you useful alternative views on this table of constructors. Once you select a view, you notice that the button turns off (becomes italicized) so that you cannot reselect it.

names

This view gives you a table of names. Selecting any of these names brings up the constructor page for that constructor.

---

<sup>5</sup>The benefactors exclude constructors such as `PrimitiveArray` whose operations `macro-expand` and so vanish from sight!


abbrs	This view gives you a table of abbreviations, in the same order as the original constructor names. Abbreviations are in capitals and are limited to 7 characters. They can be used interchangeably with constructor names in input areas.
kinds	This view organizes constructor names into the three kinds: categories, domains and packages.
files	This view gives a table of file names for the source code of the constructors in alphabetic order after removing duplicates.
parameters	This view presents constructors with the arguments. This view of the benefactors of Matrix shows that Matrix uses as many as five different List domains in its implementation.
filter	This button is used to refine the list of names or abbreviations. Starting with the <i>names</i> view, enter <b>m*</b> into the input area and click on <b>filter</b> . You then get a shorter table with only the names beginning with <b>m</b> .
documentation	This gives you documentation for each of the constructors.
conditions	This page organizes the constructors according to predicates. The view is not available for your example page since all constructors are unconditional. For a table with conditions, return to the <b>Cross Reference</b> page for Matrix, click on <b>Ancestors</b> , then on <b>conditions</b> in the view panel. This page shows you that CoercibleTo(OutputForm) and SetCategory are ancestors of Matrix(R) only if R belongs to category SetCategory.

#### 14.2.4 Giving Parameters to Constructors

---

Notice the input area at the bottom of the constructor page. If you leave this blank, then the information you get is for the domain constructor Matrix(R), that is, Matrix for an arbitrary underlying domain R.

In general, however, the exports and other information *do* usually depend on the actual value of R. For example, Matrix exports the **inverse** operation only if the domain R is a Field. To see this, try this from the main constructor page:

1. Enter **Integer** into the input area at the bottom of the page.
2. Click on **Operations**, producing a table of operations. Note the number of operation names that appear at the top of the page.
3. Click on  to return to the constructor page.
4. Use the **Delete** or **Backspace** keys to erase **Integer** from the input area.
5. Click on **Operations** to produce a new table of operations. Look at the number of operations you get. This number is greater than what you had before. Find, for example, the operation **inverse**.

- Click on **inverse** to produce a page describing the operation **inverse**. At the bottom of the description, you notice that the **Conditions** line says “*R* has Field.” This operation is *not* exported by `Matrix(Integer)` since `Integer` is not a *field*.  
Try putting the name of a domain such as `Fraction Integer` (which is a field) into the input area, then clicking on **Operations**. As you see, the operation **inverse** is exported.

### 14.3 Miscellaneous Features of Browse

#### 14.3.1 The Description Page for Operations

From the constructor page of `Matrix`, click on **Operations** to bring up the table of operations for `Matrix`.  
Find the operation **inverse** in the table and click on it. This takes you to a page showing the documentation for this operation.

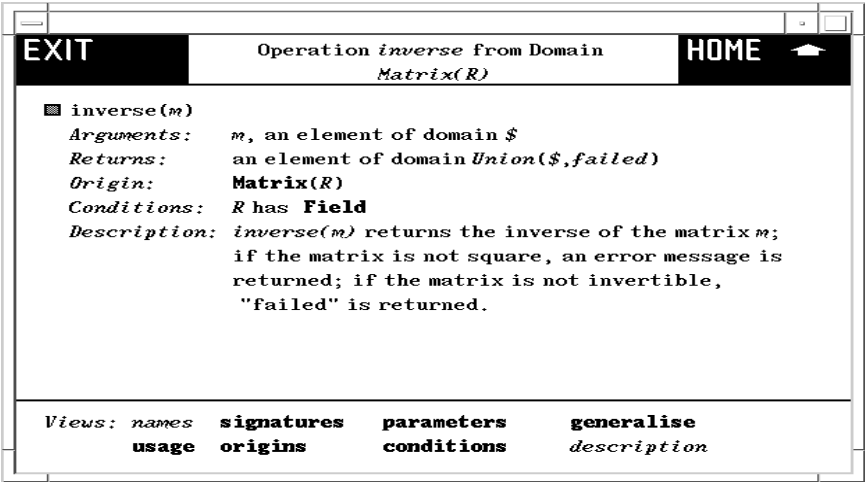


Figure 14.18: Operation **inverse** from `Matrix`.

Here is the significance of the headings you see.

Arguments	This lists each of the arguments of the operation in turn, paraphrasing the <i>signature</i> of the operation. As for signatures, a “\$” is used to designate <i>this domain</i> , that is, <code>Matrix(R)</code> .
Returns	This describes the return value for the operation, analogous to the <b>Arguments</b> part.



Origin	This tells you which domain or category explicitly exports the operation. In this example, the domain itself is the <i>Origin</i> .
Conditions	This tells you that the operation is exported by Matrix(R) only if “R has Field,” that is, “R is a member of category Field.” When no <b>Conditions</b> part is given, the operation is exported for all values of R.
Description	Here are the “++” comments that appear in the source code of its <i>Origin</i> , here Matrix. You find these comments in the source code for Matrix.

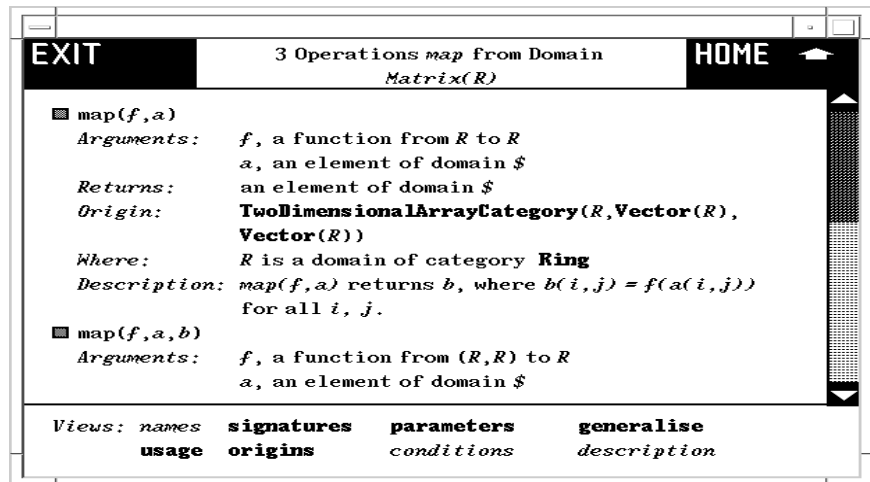



Figure 14.19: Operations **map** from Matrix.

Click on  to return to the table of operations. Click on **map**. Here you find three different operations named **map**. This should not surprise you. Operations are identified by name and *signature*. There are three operations named **map**, each with different signatures. What you see is the *descriptions* view of the operations. If you like, select the button in the heading of one of these descriptions to get *only* that operation.

Where	This part qualifies domain parameters mentioned in the arguments to the operation.
-------	--

### 14.3.2 Views of Operations

names	This view lists the names of the operations. Unlike constructors, however, there may be several operations with the same name. The heading for the page tells you the number of unique names and the number of distinct operations when these numbers are different.
-------	--

filter	<p>As for constructors, you can use this button to cut down the list of operations you are looking at. Enter, for example, <b>m*</b> into the input area to the right of <b>filter</b> then click on <b>filter</b>. As usual, any logical expression is permitted. For example, use</p> <pre>*! or *?</pre> <p>to get a list of destructive operations and predicates.</p>
documentation	This gives you the most information: a detailed description of all the operations in the form you have seen before. Every other button summarizes these operations in some form.
signatures	This views the operations by showing their signatures.
parameters	This views the operations by their distinct syntactic forms with parameters.
origins	This organizes the operations according to the constructor that explicitly exports them.
conditions	This view organizes the operations into conditional and unconditional operations.
usage	This button is only available if your user-level is set to <i>development</i> . The <b>usage</b> button produces a table of constructors that reference this operation. <sup>6</sup>
implementation	<p>This button is only available if your user-level is set to <i>development</i>. If you enter values for all domain parameters on the constructor page, then the <b>implementation</b> button appears in place of the <b>conditions</b> button. This button tells you what domains or packages actually implement the various operations.<sup>7</sup></p> <p>With your user-level set to <i>development</i>, we suggest you try this exercise. Return to the main constructor page for Matrix, then enter <b>Integer</b> into the input area at the bottom as the value of <b>R</b>. Then click on <b>Operations</b> to produce a table of operations. Note that the <b>conditions</b> part of the <i>Views</i> table is replaced by <b>implementation</b>. Click on <b>implementation</b>. After some delay, you get a page describing what implements each of the matrix operations, organized by the various domains and packages.</p>
generalize	This button only appears for an operation page of a constructor involving a unique operation name.

From an operations page for Matrix, select any operation name, say **rank**. In the views panel, the **filter** button is replaced by **generalize**. Click on

<sup>6</sup>AXIOM requires an especially long time to produce this table, so anticipate this when requesting this information.

<sup>7</sup>This button often takes a long time; expect a delay while you wait for an answer.

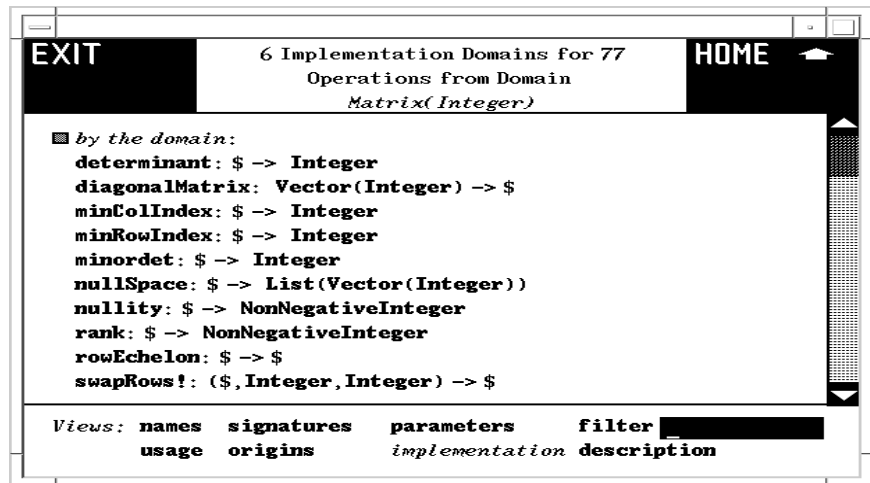


Figure 14.20: Implementation domains for Matrix.

it! What you get is a description of all AXIOM operations named **rank**.<sup>8</sup>

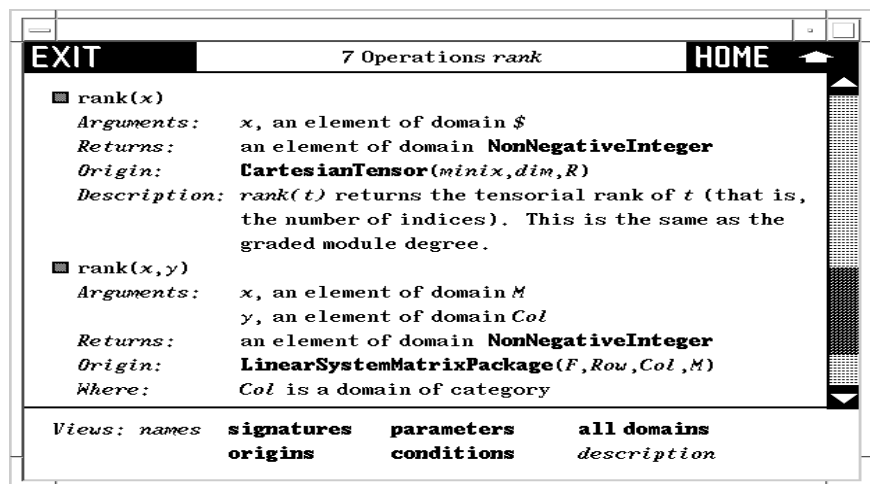


Figure 14.21: All operations named **rank** in AXIOM.

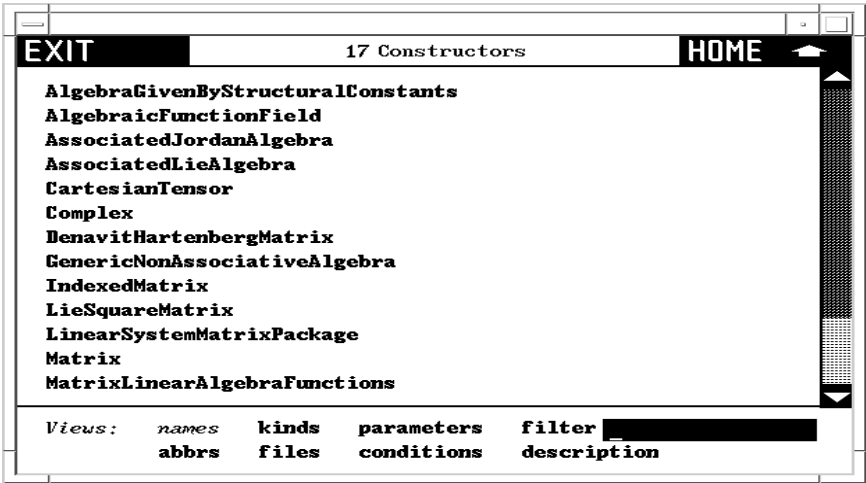
all domains

This button only appears on an operation page resulting from a search from the front page of Browse or from selecting **generalize** from an operation page for a constructor.

Note that the **filter** button in the *Views* panel is replaced by **all domains**. Click on it to produce a table of *all* domains or packages that export a

<sup>8</sup>If there were more than 10 operations of the name, you get instead a page with a *Views* panel at the bottom and the message to **Select a view below**. To get the descriptions of all these operations as mentioned above, select the **description** button.

rank operation.



EXIT	17 Constructors				HOME
AlgebraGivenByStructuralConstants					
AlgebraicFunctionField					
AssociatedJordanAlgebra					
AssociatedLieAlgebra					
CartesianTensor					
Complex					
DenavitHartenbergMatrix					
GenericNonAssociativeAlgebra					
IndexedMatrix					
LieSquareMatrix					
LinearSystemMatrixPackage					
Matrix					
MatrixLinearAlgebraFunctions					
Views: names kinds parameters filter					
abbrs files conditions description					

Figure 14.22: Table of all domains that export **rank**.

We note that this table specifically refers to all the **rank** operations shown in the preceding page. Return to the descriptions of all the **rank** operations and select one of them by clicking on the button in its heading. Select **all domains**. As you see, you have a smaller table of constructors. When there is only one constructor, you get the constructor page for that constructor.

### 14.3.3 Capitalization Convention

---

When entering search keys for constructors, you can use capital letters to search for abbreviations. For example, enter **UTS** into the input area and click on **Constructors**. Up comes a page describing `UnivariateTaylorSeries` whose abbreviation is **UTS**.

Constructor abbreviations always have three or more capital letters. For short constructor names (six letters or less), abbreviations are not generally helpful as their abbreviation is typically the constructor name in capitals. For example, the abbreviation for `Matrix` is **MATRIX**.

Abbreviations can also contain numbers. For example, **POLY2** is the abbreviation for constructor `PolynomialFunctions2`. For default packages, the abbreviation is the same as the abbreviation for the corresponding category with the “&” replaced by “-”. For example, for the category default package `MatrixCategory&` the abbreviation is **MATCAT-** since the corresponding category `MatrixCategory` has abbreviation **MATCAT**.



---

## APPENDICES

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# AXIOM System Commands

This chapter describes system commands, the command-line facilities used to control the AXIOM environment. The first section is an introduction and discusses the common syntax of the commands available.

## A.1 Introduction

---

System commands are used to perform AXIOM environment management. Among the commands are those that display what has been defined or computed, set up multiple logical AXIOM environments (frames), clear definitions, read files of expressions and commands, show what functions are available, and terminate AXIOM.

Some commands are restricted: the commands

```
)set userlevel interpreter  
)set userlevel compiler  
)set userlevel development
```

set the user-access level to the three possible choices. All commands are available at **development** level and the fewest are available at **interpreter** level. The default user-level is **interpreter**. In addition to the **)set** command (discussed in Section A.21 on page 741) you can use the HyperDoc settings facility to change the *user-level*.

Each command listing begins with one or more syntax pattern descriptions plus examples of related commands. The syntax descriptions are intended to be easy to read and do not necessarily represent the most compact way of specifying all possible arguments and options; the descriptions may occasionally be redundant.

All system commands begin with a right parenthesis which should be in the first available column of the input line (that is, immediately after the input prompt, if any). System commands may be issued directly to AXIOM or be included in **.input** files.

A system command *argument* is a word that directly follows the command name and is not followed or preceded by a right parenthesis. A system command *option* follows the system command and is directly preceded by a right parenthesis. Options may have arguments: they directly follow the option. This example may make it easier to remember what is an option and what is an argument:

```
)syscmd arg1 arg2 )opt1 opt1arg1 opt1arg2 )opt2 opt2arg1 ...
```

In the system command descriptions, optional arguments and options are enclosed in brackets (“[” and “]”). If an argument or option name is in italics, it is meant to be a variable and must have some actual value substituted for it when the system command call is made. For example, the syntax pattern description

```
)read fileName [)quietly]
```

would imply that you must provide an actual file name for *fileName* but need not use the `)quietly` option. Thus

```
)read matrix.input
```

is a valid instance of the above pattern.

System command names and options may be abbreviated and may be in upper or lower case. The case of actual arguments may be significant, depending on the particular situation (such as in file names). System command names and options may be abbreviated to the minimum number of starting letters so that the name or option is unique. Thus

```
)s Integer
```

is not a valid abbreviation for the `)set` command, because both `)set` and `)show` begin with the letter “s”. Typically, two or three letters are sufficient for disambiguating names. In our descriptions of the commands, we have used no abbreviations for either command names or options.

In some syntax descriptions we use a vertical line “|” to indicate that you must specify one of the listed choices. For example, in

```
)set output fortran on | off
```

only `on` and `off` are acceptable words for following `boot`. We also sometimes use “...” to indicate that additional arguments or options of the listed form are allowed. Finally, in the syntax descriptions we may also list the syntax of related commands.

## A.2 )abbreviation

---

**User Level Required:** compiler

**Command Syntax:**

```
)abbreviation query [nameOrAbbrev]  

)abbreviation category abbrev fullname [)quiet]  

)abbreviation domain abbrev fullname [)quiet]  

)abbreviation package abbrev fullname [)quiet]  

)abbreviation remove nameOrAbbrev
```

**Command Description:**

This command is used to query, set and remove abbreviations for category, domain and package constructors. Every constructor must have a unique abbreviation. This abbreviation is part of the name of the subdirectory under which the components of the compiled constructor are stored. Furthermore, by issuing this command you let the system know what file to load automatically if you use a new constructor. Abbreviations must start with a letter and then be followed by up to seven letters or digits. Any letters appearing in the abbreviation must be in uppercase.

When used with the **query** argument, this command may be used to list the name associated with a particular abbreviation or the abbreviation for a constructor. If no abbreviation or name is given, the names and corresponding abbreviations for *all* constructors are listed.

The following shows the abbreviation for the constructor List:

```
)abbreviation query List
```

The following shows the constructor name corresponding to the abbreviation NNI:

```
)abbreviation query NNI
```

The following lists all constructor names and their abbreviations.

```
)abbreviation query
```

To add an abbreviation for a constructor, use this command with **category**, **domain** or **package**. The following add abbreviations to the system for a category, domain and package, respectively:

```
)abbreviation domain SET Set
)abbreviation category COMPCAT ComplexCategory
)abbreviation package LIST2MAP ListToMap
```

If the **)quiet** option is used, no output is displayed from this command. You would normally only define an abbreviation in a library source file. If this command is issued for a constructor that has already been loaded, the constructor will be reloaded next time it is referenced. In particular, you can use this command to force the automatic reloading of constructors.

To remove an abbreviation, the **remove** argument is used. This is usually only used to correct a previous command that set an abbreviation for a constructor name. If, in fact, the abbreviation does exist, you are prompted for confirmation of the removal request. Either of the following commands will remove the abbreviation VECTOR2 and the constructor name VectorFunctions2 from the system:

```
)abbreviation remove VECTOR2
)abbreviation remove VectorFunctions2
```

**Also See:** '**)compile**' in Section A.7 on page 728 and

## A.3 )boot

---

**User Level Required:** development

**Command Syntax:**

```
)boot bootExpression
```

**Command Description:**

This command is used by AXIOM system developers to execute expressions written in the BOOT language. For example,

```
)boot times3(x) == 3*x
```

creates and compiles the Common LISP function “times3” obtained by translating the BOOT code.

**Also See:** ‘)fin’ in Section A.10 on page 733, ‘)lisp’ in Section A.15 on page 738, ‘)set’ in Section A.21 on page 741, and ‘)system’ in Section A.25 on page 743.

## A.4 )cd

---

**User Level Required:** interpreter

**Command Syntax:**

```
)cd directory
```

**Command Description:**

This command sets the AXIOM working current directory. The current directory is used for looking for input files (for )read), AXIOM library source files (for )compile), saved history environment files (for )history )restore), compiled AXIOM library files (for )library), and files to edit (for )edit). It is also used for writing spool files (via )spool), writing history input files (via )history )write) and history environment files (via )history )save), and compiled AXIOM library files (via )compile).

If issued with no argument, this command sets the AXIOM current directory to your home directory. If an argument is used, it must be a valid directory name. Except for the “)” at the beginning of the command, this has the same syntax as the operating system cd command.

**Also See:** ‘)compile’ in Section A.7 on page 728, ‘)edit’ in Section A.9 on page 733, ‘)history’ in Section A.13 on page 735, ‘)library’ in Section A.14 on page 737, ‘)read’ in Section A.20 on page 740, and ‘)spool’ in Section A.23 on page 742.

## A.5 )close

---

**User Level Required:** interpreter

**Command Syntax:**

```
)close  
)close )quietly
```

**Command Description:**

This command is used to close down interpreter client processes. Such processes are started by HyperDoc to run AXIOM examples when you click on their text. When you have finished examining or modifying the example and you do not want the extra window around anymore, issue

```
)close
```

to the AXIOM prompt in the window.

If you try to close down the last remaining interpreter client process, AXIOM will offer to close down the entire AXIOM session and return you to the operating system by displaying something like

```
This is the last AXIOM session. Do you want to kill AXIOM?
```

Type “y” (followed by the Return key) if this is what you had in mind. Type “n” (followed by the Return key) to cancel the command.

You can use the `)quietly` option to force AXIOM to close down the interpreter client process without closing down the entire AXIOM session.

**Also See:** ‘`)quit`’ in Section A.19 on page 740 and ‘`)pquit`’ in Section A.18 on page 739.

## A.6 `)clear`

---

**User Level Required:** interpreter

**Command Syntax:**

```
)clear all
)clear completely
)clear properties all
)clear properties obj1 [obj2 ...]
)clear value all
)clear value obj1 [obj2 ...]
)clear mode all
)clear mode obj1 [obj2 ...]
```

**Command Description:**

This command is used to remove function and variable declarations, definitions and values from the workspace. To empty the entire workspace and reset the step counter to 1, issue

```
)clear all
```

To remove everything in the workspace but not reset the step counter, issue

```
)clear properties all
```

To remove everything about the object **x**, issue

```
)clear properties x
```

To remove everything about the objects **x**, **y** and **f**, issue

```
)clear properties x y f
```

The word **properties** may be abbreviated to the single letter “**p**”.

```
)clear p all
)clear p x
)clear p x y f
```

All definitions of functions and values of variables may be removed by either

```
)clear value all
)clear v all
```

This retains whatever declarations the objects had. To remove definitions and values for the specific objects **x**, **y** and **f**, issue

```
)clear value x y f
)clear v x y f
```

To remove the declarations of everything while leaving the definitions and values, issue

```
)clear mode all
)clear m all
```

To remove declarations for the specific objects `x`, `y` and `f`, issue

```
)clear mode x y f  
)clear m x y f
```

The `)display names` and `)display properties` commands may be used to see what is currently in the workspace.

The command

```
)clear completely
```

does everything that `)clear all` does, and also clears the internal system function and constructor caches.

**Also See:** ‘`)display`’ in Section A.8 on page 732, ‘`)history`’ in Section A.13 on page 735, and ‘`)undo`’ in Section A.27 on page 747.

## A.7 **)compile**

---

**User Level Required:** compiler

**Command Syntax:**

```
)compile  
)compile fileName  
)compile fileName.as  
)compile directory/fileName.as  
)compile fileName.ao  
)compile directory/fileName.ao  
)compile fileName.al  
)compile directory/fileName.al  
)compile fileName.lsp  
)compile directory/fileName.lsp  
)compile fileName.spad  
)compile directory/fileName.spad  
)compile fileName )new  
)compile fileName )old  
)compile fileName )translate  
)compile fileName )quiet  
)compile fileName )noquiet  
)compile fileName )moreargs  
)compile fileName )onlyargs  
)compile fileName )break  
)compile fileName )nobreak  
)compile fileName )library  
)compile fileName )nolibrary  
)compile fileName )vartrace  
)compile fileName )constructor nameOrAbbrev
```

**Command Description:**

You use this command to invoke the new AXIOM library compiler or the old AXIOM system compiler. The `)compile` system command is actually a combination of AXIOM processing and a call to the Aldor compiler. It is performing double-duty, acting as a

front-end to both the Aldor compiler and the old AXIOM system compiler. (The old AXIOM system compiler was written in Lisp and was an integral part of the AXIOM environment. The Aldor compiler is written in C and executed by the operating system when called from within AXIOM.)

The command compiles files with file extensions *.as*, *.ao* and *.al* with the Aldor compiler and files with file extension *.spad* with the old AXIOM system compiler. It also can compile files with file extension *.lsp*. These are assumed to be Lisp files generated by the Aldor compiler. If you omit the file extension, the command looks to see if you have specified the `)new` or `)old` option. If you have given one of these options, the corresponding compiler is used. Otherwise, the command first looks in the standard system directories for files with extension *.as*, *.ao* and *.al* and then files with extension *.spad*. The first file found has the appropriate compiler invoked on it. If the command cannot find a matching file, an error message is displayed and the command terminates.

The `)translate` option is used to invoke a special version of the old system compiler that will translate a *.spad* file to a *.as* file. That is, the *.spad* file will be parsed and analyzed and a file using the new syntax will be created. By default, the *.as* file is created in the same directory as the *.spad* file. If that directory is not writable, the current directory is used. If the current directory is not writable, an error message is given and the command terminates. Note that `)translate` implies the `)old` option so the file extension can safely be omitted. If `)translate` is given, all other options are ignored. Please be aware that the translation is not necessarily one hundred percent complete or correct. You should attempt to compile the output with the Aldor compiler and make any necessary corrections.

We now describe the options for the new Aldor compiler.

The first thing `)compile` does is look for a source code filename among its arguments. Thus

```
)compile mycode.as
)compile /u/jones/as/mycode.as
)compile mycode
```

all invoke `)compiler` on the file `/u/jones/as/mycode.as` if the current AXIOM working directory is `/u/jones/as`. (Recall that you can set the working directory via the `)cd` command. If you don't set it explicitly, it is the directory from which you started AXIOM.)

This is frequently all you need to compile your file. This simple command:

1. Invokes the Aldor compiler and produces Lisp output.
2. Calls the Lisp compiler if the Aldor compilation was successful.
3. Uses the `)library` command to tell AXIOM about the contents of your compiled file and arrange to have those contents loaded on demand.

Should you not want the `)library` command automatically invoked, call `)compile` with the `)nolibrary` option. For example,

```
)compile mycode.as )nolibrary
```

The general description of Aldor command line arguments is in the Aldor documentation. The default options used by the `)compile` command can be viewed and set using the `)set compiler args` AXIOM system command. The current defaults are

```
-O -Fasy -Fao -Flsp -laxiom -Mno-AXL_WillObsolete -DAxiom
```

These options mean:

- `-O`: perform all optimizations,

- `-Fasy`: generate a `.asy` file,
- `-Fao`: generate a `.ao` file,
- `-Flsp`: generate a `.lsp` (Lisp) file,
- `-laxiom`: use the axiom library `libaxiom.al`,
- `-Mno-AXL_WillObsolete`: do not display messages about older generated files becoming obsolete, and
- `-DAxiom`: define the global assertion `Axiom` so that the Aldor libraries for generating stand-alone code are not accidentally used with AXIOM.

To supplement these default arguments, use the `)moreargs` option on `)compile`. For example,

```
)compile mycode.as )moreargs "-v"
```

uses the default arguments and appends the `-v` (verbose) argument flag. The additional argument specification **must be enclosed in double quotes**.

To completely replace these default arguments for a particular use of `)compile`, use the `)onlyargs` option. For example,

```
)compile mycode.as )onlyargs "-v -O"
```

only uses the `-v` (verbose) and `-O` (optimize) arguments. The argument specification **must be enclosed in double quotes**. In this example, Lisp code is not produced and so the compilation output will not be available to AXIOM.

To completely replace the default arguments for all calls to `)compile` within your AXIOM session, use `)set compiler args`. For example, to use the above arguments for all compilations, issue

```
)set compiler args "-v -O"
```

Make sure you include the necessary `-l` and `-Y` arguments along with those needed for Lisp file creation. As above, **the argument specification must be enclosed in double quotes**.

By default, the `)library` system command *exposes* all domains and categories it processes. This means that the AXIOM interpreter will consider those domains and categories when it is trying to resolve a reference to a function. Sometimes domains and categories should not be exposed. For example, a domain may just be used privately by another domain and may not be meant for top-level use. The `)library` command should still be used, though, so that the code will be loaded on demand. In this case, you should use the `)nolibrary` option on `)compile` and the `)noexpose` option in the `)library` command. For example,

```
)compile mycode.as )nolibrary
)library mycode )noexpose
```

Once you have established your own collection of compiled code, you may find it handy to use the `)dir` option on the `)library` command. This causes `)library` to process all compiled code in the specified directory. For example,

```
)library )dir /u/jones/as/quantum
```

You must give an explicit directory after `)dir`, even if you want all compiled code in the current working directory processed, e.g.

```
)library )dir .
```



The `)compile` command works with several file extensions. We saw above what happens when it is invoked on a file with extension `.as`. A `.ao` file is a portable binary compiled version of a `.as` file, and `)compile` simply passes the `.ao` file onto Aldor. The generated Lisp file is compiled and `)library` is automatically called, just as if you had specified a `.as` file.

A `.al` file is an archive file containing `.ao` files. The archive is created (on Unix systems) with the `ar` program. When `)compile` is given a `.al` file, it creates a directory whose name is based on that of the archive. For example, if you issue

```
)compile mylib.al
```

the directory `mylib.axldir` is created. All members of the archive are unarchived into the directory and `)compile` is called on each `.ao` file found. It is your responsibility to remove the directory and its contents, if you choose to do so.

A `.lsp` file is a Lisp source file, presumably, in our context, generated by Aldor when called with the `-Flsp` option. When `)compile` is used with a `.lsp` file, the Lisp file is compiled and `)library` is called. You must also have present a `.asy` generated from the same source file.

The following are descriptions of options for the old system compiler.

You can compile category, domain, and package constructors contained in files with file extension `.spad`. You can compile individual constructors or every constructor in a file.

The full filename is remembered between invocations of this command and `)edit` commands. The sequence of commands

```
)compile matrix.spad
)edit
)compile
```

will call the compiler, edit, and then call the compiler again on the file `matrix.spad`. If you do not specify a *directory*, the working current directory (see Section A.4 on page 726) is searched for the file. If the file is not found, the standard system directories are searched.

If you do not give any options, all constructors within a file are compiled. Each constructor should have an `)abbreviation` command in the file in which it is defined. We suggest that you place the `)abbreviation` commands at the top of the file in the order in which the constructors are defined. The list of commands serves as a table of contents for the file.

The `)library` option causes directories containing the compiled code for each constructor to be created in the working current directory. The name of such a directory consists of the constructor abbreviation and the `.NRLIB` file extension. For example, the directory containing the compiled code for the `MATRIX` constructor is called `MATRIX.NRLIB`. The `)nolibrary` option says that such files should not be created. The default is `)library`. Note that the semantics of `)library` and `)nolibrary` for the new Aldor compiler and for the old system compiler are completely different.

The `)vartrace` option causes the compiler to generate extra code for the constructor to support conditional tracing of variable assignments. (see Section A.26 on page 744). Without this option, this code is suppressed and one cannot use the `)vars` option for the trace command.

The `)constructor` option is used to specify a particular constructor to compile. All other constructors in the file are ignored. The constructor name or abbreviation follows `)constructor`. Thus either

```
)compile matrix.spad )constructor RectangularMatrix
```

or

```
)compile matrix.spad )constructor RMATRIX
```

compiles the `RectangularMatrix` constructor defined in **matrix.spad**.

The `)break` and `)nobreak` options determine what the old system compiler does when it encounters an error. `)break` is the default and it indicates that processing should stop at the first error. The value of the `)set break` variable then controls what happens.

**Also See:** ‘`)abbreviation`’ in Section A.2 on page 724, ‘`)edit`’ in Section A.9 on page 733, and ‘`)library`’ in Section A.14 on page 737.

## A.8 **)display**

---

**User Level Required:** interpreter

**Command Syntax:**

```
)display all
)display properties
)display properties all
)display properties [obj1 [obj2 ...]]
)display value all
)display value [obj1 [obj2 ...]]
)display mode all
)display mode [obj1 [obj2 ...]]
)display names
)display operations opName
```

**Command Description:**

This command is used to display the contents of the workspace and signatures of functions with a given name.<sup>1</sup>

The command

```
)display names
```

lists the names of all user-defined objects in the workspace. This is useful if you do not wish to see everything about the objects and need only be reminded of their names.

The commands

```
)display all
)display properties
)display properties all
```

all do the same thing: show the values and types and declared modes of all variables in the workspace. If you have defined functions, their signatures and definitions will also be displayed.

To show all information about a particular variable or user functions, for example, something named `d`, issue

```
)display properties d
```

To just show the value (and the type) of `d`, issue

---

<sup>1</sup>A *signature* gives the argument and return types of a function.

```
)display value d
```

To just show the declared mode of **d**, issue

```
)display mode d
```

All modemap for a given operation may be displayed by using **)display operations**. A *modemap* is a collection of information about a particular reference to an operation. This includes the types of the arguments and the return value, the location of the implementation and any conditions on the types. The modemap may contain patterns. The following displays the modemaps for the operation **complex**:

```
)d op complex
```

**Also See:** '**)clear**' in Section A.6 on page 727, '**)history**' in Section A.13 on page 735, '**)set**' in Section A.21 on page 741, '**)show**' in Section A.22 on page 741, and '**)what**' in Section A.28 on page 748.

## A.9 **)edit**

---

**User Level Required:** interpreter

**Command Syntax:**

```
)edit [filename]
```

**Command Description:**

This command is used to edit files. It works in conjunction with the **)read** and **)compile** commands to remember the name of the file on which you are working. By specifying the name fully, you can edit any file you wish. Thus

```
)edit /u/julius/matrix.input
```

will place you in an editor looking at the file **/u/julius/matrix.input**. By default, the editor is **vi**, but if you have an **EDITOR** shell environment variable defined, that editor will be used. When AXIOM is running under the X Window System, it will try to open a separate **xterm** running your editor if it thinks one is necessary. For example, under the Korn shell, if you issue

```
export EDITOR=emacs
```

then the emacs editor will be used by **)edit**.

If you do not specify a file name, the last file you edited, read or compiled will be used. If there is no "last file" you will be placed in the editor editing an empty unnamed file.

It is possible to use the **)system** command to edit a file directly. For example,

```
)system emacs /etc/rc.tcpip
```

calls **emacs** to edit the file.

**Also See:** '**)system**' in Section A.25 on page 743, '**)compile**' in Section A.7 on page 728, and '**)read**' in Section A.20 on page 740.

## A.10 **)fin**

---

**User Level Required:** development

**Command Syntax:**

```
)fin
```

## A.11 )frame

---

### Command Description:

This command is used by AXIOM developers to leave the AXIOM system and return to the underlying Common LISP system. To return to AXIOM, issue the “(|spad|)” function call to Common LISP.

**Also See:** ‘)pquit’ in Section A.18 on page 739 and ‘)quit’ in Section A.19 on page 740.

**User Level Required:** interpreter

### Command Syntax:

```
)frame new frameName
)frame drop [frameName]
)frame next
)frame last
)frame names
)frame import frameName [objectName1 [objectName2 ...]]
)set message frame on | off
)set message prompt frame
```

### Command Description:

A *frame* can be thought of as a logical session within the physical session that you get when you start the system. You can have as many frames as you want, within the limits of your computer’s storage, paging space, and so on. Each frame has its own *step number*, *environment* and *history*. You can have a variable named **a** in one frame and it will have nothing to do with anything that might be called **a** in any other frame.

Some frames are created by the HyperDoc program and these can have pretty strange names, since they are generated automatically. To find out the names of all frames, issue

```
)frame names
```

It will indicate the name of the current frame.

You create a new frame “**quark**” by issuing

```
)frame new quark
```

The history facility can be turned on by issuing either `)set history on` or `)history on`. If the history facility is on and you are saving history information in a file rather than in the AXIOM environment then a history file with filename **quark.axh** will be created as you enter commands. If you wish to go back to what you were doing in the “**initial**” frame, use

```
)frame next
```

or

```
)frame last
```

to cycle through the ring of available frames to get back to “**initial**”.

If you want to throw away a frame (say “**quark**”), issue

```
)frame drop quark
```

If you omit the name, the current frame is dropped.

If you do use frames with the history facility on and writing to a file, you may want to delete some of the older history files. These are directories, so you may want to issue a command like `rm -r quark.axh` to the operating system.

You can bring things from another frame by using `)frame import`. For example, to bring the `f` and `g` from the frame “**quark**” to the current frame, issue

```
)frame import quark f g
```

If you want everything from the frame “**quark**”, issue

```
)frame import quark
```

You will be asked to verify that you really want everything.

There are two `)set` flags to make it easier to tell where you are.

```
)set message frame on | off
```

will print more messages about frames when it is set on. By default, it is off.

```
)set message prompt frame
```

will give a prompt that looks like

```
initial (1) ->
```

when you start up. In this case, the frame name and step make up the prompt.

**Also See:** ‘`)history`’ in Section A.13 on page 735 and ‘`)set`’ in Section A.21 on page 741.

## A.12 )help

**User Level Required:** interpreter

**Command Syntax:**

```
)help
```

```
)help commandName
```

**Command Description:**

This command displays help information about system commands. If you issue

```
)help
```

then this very text will be shown. You can also give the name or abbreviation of a system command to display information about it. For example,

```
)help clear
```

will display the description of the `)clear` system command.

All this material is available in the AXIOM User Guide and in HyperDoc. In HyperDoc, choose the **Commands** item from the **Reference** menu.

## A.13 )history

**User Level Required:** interpreter

**Command Syntax:**

```

)history )on
)history )off
)history )write historyInputFileName
)history )show [n] [both]
)history )save savedHistoryName
)history )restore [savedHistoryName]
)history )reset
)history )change n
)history )memory
)history )file
%
%%(n)
)set history on | off

```

#### Command Description:

The *history* facility within AXIOM allows you to restore your environment to that of another session and recall previous computational results. Additional commands allow you to review previous input lines and to create an **.input** file of the lines typed to AXIOM.

AXIOM saves your input and output if the history facility is turned on (which is the default). This information is saved if either of

```

)set history on
)history )on

```

has been issued. Issuing either

```

)set history off
)history )off

```

will discontinue the recording of information.

Whether the facility is disabled or not, the value of “%” in AXIOM always refers to the result of the last computation. If you have not yet entered anything, “%” evaluates to an object of type `Variable('%)`. The function “%%” may be used to refer to other previous results if the history facility is enabled. In that case, %%(*n*) is the output from step *n* if *n* > 0. If *n* < 0, the step is computed relative to the current step. Thus %%(-1) is also the previous step, %%(-2), is the step before that, and so on. If an invalid step number is given, AXIOM will signal an error.

The *environment* information can either be saved in a file or entirely in memory (the default). Each frame (Section A.11 on page 734) has its own history database. When it is kept in a file, some of it may also be kept in memory for efficiency. When the information is saved in a file, the name of the file is of the form **FRAME.axh** where “**FRAME**” is the name of the current frame. The history file is placed in the current working directory (see Section A.4 on page 726). Note that these history database files are not text files (in fact, they are directories themselves), and so are not in human-readable format.

The options to the `)history` command are as follows:

```

)change n will set the number of steps that are saved in memory to n. This option
only has effect when the history data is maintained in a file. If you have issued
)history )memory (or not changed the default) there is no need to use )history
)change.

```

**)on** will start the recording of information. If the workspace is not empty, you will be asked to confirm this request. If you do so, the workspace will be cleared and history data will begin being saved. You can also turn the facility on by issuing **)set history on**.

**)off** will stop the recording of information. The **)history )show** command will not work after issuing this command. Note that this command may be issued to save time, as there is some performance penalty paid for saving the environment data. You can also turn the facility off by issuing **)set history off**.

**)file** indicates that history data should be saved in an external file on disk.

**)memory** indicates that all history data should be kept in memory rather than saved in a file. Note that if you are computing with very large objects it may not be practical to keep this data in memory.

**)reset** will flush the internal list of the most recent workspace calculations so that the data structures may be garbage collected by the underlying Common LISP system. Like **)history )change**, this option only has real effect when history data is being saved in a file.

**)restore** [*savedHistoryName*] completely clears the environment and restores it to a saved session, if possible. The **)save** option below allows you to save a session to a file with a given name. If you had issued **)history )save jacobi** the command **)history )restore jacobi** would clear the current workspace and load the contents of the named saved session. If no saved session name is specified, the system looks for a file called **last.axh**.

**)save** *savedHistoryName* is used to save a snapshot of the environment in a file. This file is placed in the current working directory (see Section A.4 on page 726). Use **)history )restore** to restore the environment to the state preserved in the file. This option also creates an input file containing all the lines of input since you created the workspace frame (for example, by starting your AXIOM session) or last did a **)clear all** or **)clear completely**.

**)show** [*n*] [*both*] can show previous input lines and output results. **)show** will display up to twenty of the last input lines (fewer if you haven't typed in twenty lines). **)show n** will display up to *n* of the last input lines. **)show both** will display up to five of the last input lines and output results. **)show n both** will display up to *n* of the last input lines and output results.

**)write** *historyInputFile* creates an **.input** file with the input lines typed since the start of the session/frame or the last **)clear all** or **)clear completely**. If *historyInputFileName* does not contain a period (".") in the filename, **.input** is appended to it. For example, **)history )write chaos** and **)history )write chaos.input** both write the input lines to a file called **chaos.input** in your current working directory. If you issued one or more **)undo** commands, **)history )write** eliminates all input lines backtracked over as a result of **)undo**. You can edit this file and then use **)read** to have AXIOM process the contents.

**Also See:** '**)frame**' in Section A.11 on page 734, '**)read**' in Section A.20 on page 740, '**)set**' in Section A.21 on page 741, and '**)undo**' in Section A.27 on page 747.

## A.14 )library

---

**User Level Required:** interpreter

**Command Syntax:**

```
)library libName1 [libName2 ...]
)library )dir dirName
)library )only objName1 [objlib2 ...]
```

`)library )noexpose`

**Command Description:**

This command replaces the `)load` system command that was available in AXIOM releases before version 2.0. The `)library` command makes available to AXIOM the compiled objects in the libraries listed.

For example, if you `)compile dopler.as` in your home directory, issue `)library dopler` to have AXIOM look at the library, determine the category and domain constructors present, update the internal database with various properties of the constructors, and arrange for the constructors to be automatically loaded when needed. If the `)noexpose` option has not been given, the constructors will be exposed (that is, available) in the current frame.

If you compiled a file with the old system compiler, you will have an *NRLIB* present, for example, *DOPLER.NRLIB*, where *DOPLER* is a constructor abbreviation. The command `)library DOPLER` will then do the analysis and database updates as above.

To tell the system about all libraries in a directory, use `)library )dir dirName` where *dirName* is an explicit directory. You may specify “.” as the directory, which means the current directory from which you started the system or the one you set via the `)cd` command. The directory name is required.

You may only want to tell the system about particular constructors within a library. In this case, use the `)only` option. The command `)library dopler )only Test1` will only cause the *Test1* constructor to be analyzed, autoloaded, etc..

Finally, each constructor in a library are usually automatically exposed when the `)library` command is used. Use the `)noexpose` option if you not want them exposed. At a later time you can use `)set expose add constructor` to expose any hidden constructors.

**Note for AXIOM beta testers:** At various times this command was called `)local` and `)with` before the name `)library` became the official name.

**Also See:** ‘`)cd`’ in Section A.4 on page 726, ‘`)compile`’ in Section A.7 on page 728, ‘`)frame`’ in Section A.11 on page 734, and ‘`)set`’ in Section A.21 on page 741.

## A.15 `)lisp`

---

**User Level Required:** development

**Command Syntax:**

`)lisp [lispExpression]`

**Command Description:**

This command is used by AXIOM system developers to have single expressions evaluated by the Common LISP system on which AXIOM is built. The *lispExpression* is read by the Common LISP reader and evaluated. If this expression is not complete (unbalanced parentheses, say), the reader will wait until a complete expression is entered.

Since this command is only useful for evaluating single expressions, the `)fin` command may be used to drop out of AXIOM into Common LISP.

**Also See:** ‘`)system`’ in Section A.25 on page 743, ‘`)boot`’ in Section A.3 on page 725, and ‘`)fin`’ in Section A.10 on page 733.



## A.16 )**load**

---

**User Level Required:** interpreter

**Command Description:**

This command is obsolete. Use `)library` instead.

## A.17 )**ltrace**

---

**User Level Required:** development

**Command Syntax:**

This command has the same arguments as options as the `)trace` command.

**Command Description:**

This command is used by AXIOM system developers to trace Common LISP or BOOT functions. It is not supported for general use.

**Also See:** '`)boot`' in Section A.3 on page 725, '`)lisp`' in Section A.15 on page 738, and '`)trace`' in Section A.26 on page 744.

## A.18 )**pquit**

---

**User Level Required:** interpreter

**Command Syntax:**

`)pquit`

**Command Description:**

This command is used to terminate AXIOM and return to the operating system. Other than by redoing all your computations or by using the `)history )restore` command to try to restore your working environment, you cannot return to AXIOM in the same state.

`)pquit` differs from the `)quit` in that it always asks for confirmation that you want to terminate AXIOM (the "p" is for "protected"). When you enter the `)pquit` command, AXIOM responds

Please enter **y** or **yes** if you really want to leave the interactive  
environment and return to the operating system:

If you respond with **y** or **yes**, you will see the message

You are now leaving the AXIOM interactive environment.  
Issue the command **axiom** to the operating system to start a new session.

and AXIOM will terminate and return you to the operating system (or the environment from which you invoked the system). If you responded with something other than **y** or **yes**, then the message

You have chosen to remain in the AXIOM interactive environment.

will be displayed and, indeed, AXIOM would still be running.

**Also See:** '`)fin`' in Section A.10 on page 733, '`)history`' in Section A.13 on page 735, '`)close`' in Section A.5 on page 726, '`)quit`' in Section A.19 on page 740, and '`)system`' in Section A.25 on page 743.

## A.19 **)quit**

---

**User Level Required:** interpreter

**Command Syntax:**

```
)quit  
)set quit protected | unprotected
```

**Command Description:**

This command is used to terminate AXIOM and return to the operating system. Other than by redoing all your computations or by using the **)history** **)restore** command to try to restore your working environment, you cannot return to AXIOM in the same state.

**)quit** differs from the **)pquit** in that it asks for confirmation only if the command

```
)set quit protected
```

has been issued. Otherwise, **)quit** will make AXIOM terminate and return you to the operating system (or the environment from which you invoked the system).

The default setting is **)set quit protected** so that **)quit** and **)pquit** behave in the same way. If you do issue

```
)set quit unprotected
```

we suggest that you do not (somehow) assign **)quit** to be executed when you press, say, a function key.

**Also See:** '**)fin**' in Section A.10 on page 733, '**)history**' in Section A.13 on page 735, '**)close**' in Section A.5 on page 726, '**)pquit**' in Section A.18 on page 739, and '**)system**' in Section A.25 on page 743.

## A.20 **)read**

---

**User Level Required:** interpreter

**Command Syntax:**

```
)read [fileName]  
)read [fileName] [quiet] [ifthere]
```

**Command Description:**

This command is used to read **.input** files into AXIOM. The command

```
)read matrix.input
```

will read the contents of the file **matrix.input** into AXIOM. The ".input" file extension is optional. See Section 4.1 on page 139 for more information about **.input** files.

This command remembers the previous file you edited, read or compiled. If you do not specify a file name, the previous file will be read.

The **)ifthere** option checks to see whether the **.input** file exists. If it does not, the **)read** command does nothing. If you do not use this option and the file does not exist, you are asked to give the name of an existing **.input** file.

The **)quiet** option suppresses output while the file is being read.

**Also See:** '**)compile**' in Section A.7 on page 728, '**)edit**' in Section A.9 on page 733, and '**)history**' in Section A.13 on page 735.

## A.21

### **)set**

---

**User Level Required:** interpreter

**Command Syntax:**

```
)set  
)set label1 [... labelN]  
)set label1 [... labelN] newValue
```

**Command Description:**

The **)set** command is used to view or set system variables that control what messages are displayed, the type of output desired, the status of the history facility, the way AXIOM user functions are cached, and so on. Since this collection is very large, we will not discuss them here. Rather, we will show how the facility is used. We urge you to explore the **)set** options to familiarize yourself with how you can modify your AXIOM working environment. There is a HyperDoc version of this same facility available from the main HyperDoc menu.

The **)set** command is command-driven with a menu display. It is tree-structured. To see all top-level nodes, issue **)set** by itself.

```
)set
```

Variables with values have them displayed near the right margin. Subtrees of selections have “...” displayed in the value field. For example, there are many kinds of messages, so issue **)set message** to see the choices.

```
)set message
```

The current setting for the variable that displays whether computation times are displayed is visible in the menu displayed by the last command. To see more information, issue

```
)set message time
```

This shows that time printing is on now. To turn it off, issue

```
)set message time off
```

As noted above, not all settings have so many qualifiers. For example, to change the **)quit** command to being unprotected (that is, you will not be prompted for verification), you need only issue

```
)set quit unprotected
```

**Also See:** ‘**)quit**’ in Section A.19 on page 740.

## A.22

### **)show**

---

**User Level Required:** interpreter

**Command Syntax:**

```
)show nameOrAbbrev  
)show nameOrAbbrev )operations  
)show nameOrAbbrev )attributes
```

**Command Description:** This command displays information about AXIOM domain, package and category *constructors*. If no options are given, the **)operations** option is assumed. For example,

```

)show POLY
)show POLY )operations
)show Polynomial
)show Polynomial )operations

```

each display basic information about the Polynomial domain constructor and then provide a listing of operations. Since Polynomial requires a Ring (for example, Integer) as argument, the above commands all refer to a unspecified ring R. In the list of operations, “\$” means Polynomial(R).

The basic information displayed includes the *signature* of the constructor (the name and arguments), the constructor *abbreviation*, the *exposure status* of the constructor, and the name of the *library source file* for the constructor.

If operation information about a specific domain is wanted, the full or abbreviated domain name may be used. For example,

```

)show POLY INT
)show POLY INT )operations
)show Polynomial Integer
)show Polynomial Integer )operations

```

are among the combinations that will display the operations exported by the domain Polynomial(Integer) (as opposed to the general *domain constructor* Polynomial). Attributes may be listed by using the **)attributes** option.

**Also See:** ‘)display’ in Section A.8 on page 732, ‘)set’ in Section A.21 on page 741, and ‘)what’ in Section A.28 on page 748.

## A.23 )spool

---

**User Level Required:** interpreter

**Command Syntax:**

```

)spool [fileName]
)spool

```

**Command Description:**

This command is used to save (*spool*) all AXIOM input and output into a file, called a *spool file*. You can only have one spool file active at a time. To start spool, issue this command with a filename. For example,

```

)spool integrate.out

```

To stop spooling, issue **)spool** with no filename.

If the filename is qualified with a directory, then the output will be placed in that directory. If no directory information is given, the spool file will be placed in the *current directory*. The current directory is the directory from which you started AXIOM or is the directory you specified using the **)cd** command.

**Also See:** ‘)cd’ in Section A.4 on page 726.

## A.24

### )synonym

User Level Required: interpreter

Command Syntax:

```
)synonym
)synonym synonym fullCommand
)what synonyms
```

Command Description:

This command is used to create short synonyms for system command expressions. For example, the following synonyms might simplify commands you often use.

```
)synonym save          history )save
)synonym restore       history )restore
)synonym mail          system mail
)synonym ls            system ls
)synonym fortran       set output fortran
```

Once defined, synonyms can be used in place of the longer command expressions. Thus

```
)fortran on
```

is the same as the longer

```
)set fortran output on
```

To list all defined synonyms, issue either of

```
)synonyms
)what synonyms
```

To list, say, all synonyms that contain the substring “ap”, issue

```
)what synonyms ap
```

**Also See:** ‘)set’ in Section A.21 on page 741 and ‘)what’ in Section A.28 on page 748.

## A.25

### )system

User Level Required: interpreter

Command Syntax:

```
)system cmdExpression
```

Command Description:

This command may be used to issue commands to the operating system while remaining in AXIOM. The *cmdExpression* is passed to the operating system for execution.

To get an operating system shell, issue, for example, `)system sh`. When you enter the key combination, `[Ctrl] [D]` (pressing and holding the `[Ctrl]` key and then pressing the `[D]` key) the shell will terminate and you will return to AXIOM. We do not recommend this way of creating a shell because Common LISP may field some interrupts instead of the shell. If possible, use a shell running in another window.

If you execute programs that misbehave you may not be able to return to AXIOM. If this happens, you may have no other choice than to restart AXIOM and restore the environment via `)history )restore`, if possible.

**Also See:** ‘)boot’ in Section A.3 on page 725, ‘)fin’ in Section A.10 on page 733, ‘)lisp’ in Section A.15 on page 738, ‘)pquit’ in Section A.18 on page 739, and ‘)quit’ in Section A.19 on page 740.

## A.26

### **)trace**

---

**User Level Required:** interpreter

**Command Syntax:**

```
)trace
)trace )off
)trace function [options]
)trace constructor [options]
)trace domainOrPackage [options]
```

where options can be one or more of

```
)after S-expression
)before S-expression
)break after
)break before
)cond S-expression
)count
)count n
)depth n
)local op1 [... opN]
)nonquietly
)nt
)off
)only listOfDataToDisplay
)ops
)ops op1 [... opN]
)restore
)stats
)stats reset
)timer
)varbreak
)varbreak var1 [... varN]
)vars
)vars var1 [... varN]
)within executingFunction
```

**Command Description:**

This command is used to trace the execution of functions that make up the AXIOM system, functions defined by users, and functions from the system library. Almost all options are available for each type of function but exceptions will be noted below.

To list all functions, constructors, domains and packages that are traced, simply issue

```
)trace
```

To untrace everything that is traced, issue

```
)trace )off
```

When a function is traced, the default system action is to display the arguments to the function and the return value when the function is exited. Note that if a function is left via an action such as a **THROW**, no return value will be displayed. Also, optimization of tail recursion may decrease the number of times a function is actually invoked and so may cause less trace information to be displayed. Other information can be displayed or collected when a function is traced and this is controlled by the various options. Most options will be of interest only to AXIOM system developers. If a domain or package is traced, the default action is to trace all functions exported.

Individual interpreter, lisp or boot functions can be traced by listing their names after **)trace**. Any options that are present must follow the functions to be traced.

```
)trace f
```

traces the function **f**. To untrace **f**, issue

```
)trace f )off
```

Note that if a function name contains a special character, it will be necessary to escape the character with an underscore

```
)trace _/D_,1
```

To trace all domains or packages that are or will be created from a particular constructor, give the constructor name or abbreviation after **)trace**.

```
)trace MATRIX
)trace List Integer
```

The first command traces all domains currently instantiated with **Matrix**. If additional domains are instantiated with this constructor (for example, if you have used **Matrix(Integer)** and **Matrix(Float)**), they will be automatically traced. The second command traces **List(Integer)**. It is possible to trace individual functions in a domain or package. See the **)ops** option below.

The following are the general options for the **)trace** command.

**)break after** causes a Common LISP break loop to be entered after exiting the traced function.

**)break before** causes a Common LISP break loop to be entered before entering the traced function.

**)break** is the same as **)break before**.

**)count** causes the system to keep a count of the number of times the traced function is entered. The total can be displayed with **)trace )stats** and cleared with **)trace )stats reset**.

**)count n** causes information about the traced function to be displayed for the first  $n$  executions. After the  $n^{\text{th}}$  execution, the function is untraced.

**)depth n** causes trace information to be shown for only  $n$  levels of recursion of the traced function. The command

```
)trace fib )depth 10
```

will cause the display of only 10 levels of trace information for the recursive execution of a user function **fib**.

**)math** causes the function arguments and return value to be displayed in the AXIOM monospace two-dimensional math format.

**)nonquietly** causes the display of additional messages when a function is traced.

**)nt** This suppresses all normal trace information. This option is useful if the **)count** or **)timer** options are used and you are interested in the statistics but not the function calling information.

**)off** causes untracing of all or specific functions. Without an argument, all functions, constructors, domains and packages are untraced. Otherwise, the given functions and other objects are untraced. To immediately retrace the untraced functions, issue **)trace )restore**.

**)only** *listOfDataToDisplay* causes only specific trace information to be shown. The items are listed by using the following abbreviations:

- a** display all arguments
- v** display return value
- 1** display first argument
- 2** display second argument
- 15** display the 15th argument, and so on

**)restore** causes the last untraced functions to be retraced. If additional options are present, they are added to those previously in effect.

**)stats** causes the display of statistics collected by the use of the **)count** and **)timer** options.

**)stats reset** resets to 0 the statistics collected by the use of the **)count** and **)timer** options.

**)timer** causes the system to keep a count of execution times for the traced function. The total can be displayed with **)trace )stats** and cleared with **)trace )stats reset**.

**)varbreak** *var1* [... *varN*] causes a Common LISP break loop to be entered after the assignment to any of the listed variables in the traced function.

**)vars** causes the display of the value of any variable after it is assigned in the traced function. Note that library code must have been compiled (see Section A.7 on page 728) using the **)vartrace** option in order to support this option.

**)vars** *var1* [... *varN*] causes the display of the value of any of the specified variables after they are assigned in the traced function. Note that library code must have been compiled (see Section A.7 on page 728) using the **)vartrace** option in order to support this option.

**)within** *executingFunction* causes the display of trace information only if the traced function is called when the given *executingFunction* is running.

The following are the options for tracing constructors, domains and packages.

**)local** [*op1* [... *opN*]] causes local functions of the constructor to be traced. Note that to untrace an individual local function, you must use the fully qualified internal name, using the escape character “\_” before the semicolon.

```
)trace FRAC )local
)trace FRAC_;cancelGcd )off
```

**)ops** *op1* [... *opN*] By default, all operations from a domain or package are traced when the domain or package is traced. This option allows you to specify that only particular operations should be traced. The command

```
)trace Integer )ops min max _+ _-
```



traces four operations from the domain `Integer`. Since `+` and `-` are special characters, it is necessary to escape them with an underscore.

**Also See:** `'boot'` in Section A.3 on page 725, `'lisp'` in Section A.15 on page 738, and `'ltrace'` in Section A.17 on page 739.

## A.27 **)undo**

---

**User Level Required:** interpreter

**Command Syntax:**

```
)undo
)undo integer
)undo integer [option]
)undo )redo
```

where *option* is one of

```
)after
)before
```

**Command Description:**

This command is used to restore the state of the user environment to an earlier point in the interactive session. The argument of an `)undo` is an integer which must designate some step number in the interactive session.

```
)undo n
)undo n )after
```

These commands return the state of the interactive environment to that immediately after step *n*. If *n* is a positive number, then *n* refers to step number *n*. If *n* is a negative number, it refers to the *n*<sup>th</sup> previous command (that is, undoes the effects of the last  $-n$  commands).

A `)clear all` resets the `)undo` facility. Otherwise, an `)undo` undoes the effect of `)clear` with options `properties`, `value`, and `mode`, and that of a previous `undo`. If any such system commands are given between steps *n* and *n* + 1 (*n* > 0), their effect is undone for `)undo m` for any  $0 < m \leq n$ .

The command `)undo` is equivalent to `)undo -1` (it undoes the effect of the previous user expression). The command `)undo 0` undoes any of the above system commands issued since the last user expression.

```
)undo n )before
```

This command returns the state of the interactive environment to that immediately before step *n*. Any `)undo` or `)clear` system commands given before step *n* will not be undone.

```
)undo )redo
```

This command reads the file `redo.input`, created by the last `)undo` command. This file consists of all user input lines, excluding those backtracked over due to a previous `)undo`.

**Also See:** `'history'` in Section A.13 on page 735. The command `)history )write` will eliminate the “undone” command lines of your program.

## A.28

### )what

---

User Level Required: interpreter

Command Syntax:

```
)what categories pattern1 [pattern2 ...]  
)what commands pattern1 [pattern2 ...]  
)what domains pattern1 [pattern2 ...]  
)what operations pattern1 [pattern2 ...]  
)what packages pattern1 [pattern2 ...]  
)what synonym pattern1 [pattern2 ...]  
)what things pattern1 [pattern2 ...]  
)apropos pattern1 [pattern2 ...]
```

Command Description:

This command is used to display lists of things in the system. The patterns are all strings and, if present, restrict the contents of the lists. Only those items that contain one or more of the strings as substrings are displayed. For example,

```
)what synonym
```

displays all command synonyms,

```
)what synonym ver
```

displays all command synonyms containing the substring “**ver**”,

```
)what synonym ver pr
```

displays all command synonyms containing the substring “**ver**” or the substring “**pr**”. Output similar to the following will be displayed

```
----- System Command Synonyms -----  
  
user-defined synonyms satisfying patterns:  
    ver pr  
  
    )apr ..... )what things  
    )apropos ..... )what things  
    )prompt ..... )set message prompt  
    )version ..... )lisp *yearweek*
```

Several other things can be listed with the **)what** command:

**categories** displays a list of category constructors.

**commands** displays a list of system commands available at your user-level. Your user-level is set via the **)set userlevel** command. To get a description of a particular command, such as “**)what**”, issue **)help what**.

**domains** displays a list of domain constructors.

**operations** displays a list of operations in the system library. It is recommended that you qualify this command with one or more patterns, as there are thousands of operations available. For example, say you are looking for functions that involve computation of eigenvalues. To find their names, try **)what operations eig**. A rather large list of operations is loaded into the workspace when this command is first issued. This list will be deleted when you clear the workspace via **)clear all** or **)clear completely**. It will be re-created if it is needed again.

**packages** displays a list of package constructors.

**synonym** lists system command synonyms.

**things** displays all of the above types for items containing the pattern strings as substrings. The command synonym **)apropos** is equivalent to **)what things**.

**Also See:** '**)display**' in Section A.8 on page 732, '**)set**' in Section A.21 on page 741, and '**)show**' in Section A.22 on page 741.



[



# Categories

This is a listing of all categories in the AXIOM library at the time this book was produced. Use the Browse facility (described in Chapter 14) to get more information about these constructors.

This sample entry will help you read the following table:

CategoryName{CategoryAbbreviation}:	
Category <sub>1</sub> ...Category <sub>N</sub> with operation <sub>1</sub> ...operation <sub>M</sub>	
where	
CategoryName	is the full category name, for example, CommutativeRing.
CategoryAbbreviation	is the category abbreviation, for example, COM-RING.
Category <sub>i</sub>	is a category to which the category belongs.
operation <sub>j</sub>	is an operation explicitly exported by the category.

AbelianGroup{ABELGRP}: CancellationAbelianMonoid with \* -

AbelianMonoidRing{AMR}: Algebra BiModule  
CharacteristicNonZero CharacteristicZero CommutativeRing  
IntegralDomain Ring with / coefficient degree  
leadingCoefficient leadingMonomial map monomial

monomial? reductum

AbelianMonoid{ABELMON}: AbelianSemiGroup with \*  
Zero zero?

AbelianSemiGroup{ABELSG}: SetCategory with \* +

Aggregate{AGG}: Object with # copy empty empty? eq?  
less? more? size?

AlgebraicallyClosedField{ACF}: Field RadicalCategory with  
rootOf rootsOf zeroOf zerosOf

AlgebraicallyClosedFunctionSpace{ACFS}:  
AlgebraicallyClosedField FunctionSpace with rootOf rootsOf  
zeroOf zerosOf

Algebra{ALGEBRA}: Module Ring with coerce

ArcHyperbolicFunctionCategory{AHYP}: with acosh  
acoth acsch asech asinh atanh

ArcTrigonometricFunctionCategory{ATRIG}: with acos  
acot acsc asec asin atan

AssociationListAggregate{ALAGG}: ListAggregate  
TableAggregate with assoc

AttributeRegistry{ATTREG}: with

BagAggregate{BGAGG}: HomogeneousAggregate with bag  
extract! insert! inspect

BiModule{BMODULE}: LeftModule RightModule with

BinaryRecursiveAggregate{BRAGG}: RecursiveAggregate  
with elt left right setelt setleft! setright!

BinaryTreeCategory{BTCAT}: BinaryRecursiveAggregate  
with node

BitAggregate{BTAGG}: OneDimensionalArrayAggregate  
OrderedSet with ^ and nand nor not or xor

CachableSet{CACHSET}: OrderedSet with position

setPosition  
 CancellationAbelianMonoid{CABMON}: AbelianMonoid *with* -  
 CharacteristicNonZero{CHARNZ}: Ring *with* charthRoot  
 CharacteristicZero{CHARZ}: Ring *with*  
 CoercibleTo{KOERCE}: *with* coerce  
 Collection{CLAGG}: ConvertibleTo HomogeneousAggregate *with* construct find reduce remove removeDuplicates select  
 CombinatorialFunctionCategory{CFCAT}: *with* binomial factorial permutation  
 CombinatorialOpsCategory{COMBOPC}:  
 CombinatorialFunctionCategory *with* factorials product summation  
 CommutativeRing{COMRING}: BiModule Ring *with*  
 ComplexCategory{COMPCAT}: CharacteristicNonZero  
 CharacteristicZero CommutativeRing ConvertibleTo  
 DifferentialExtension EuclideanDomain Field  
 FullyEvalableOver FullyLinearlyExplicitRingOver  
 FullyRetractableTo IntegralDomain MonogenicAlgebra  
 OrderedSet PolynomialFactorizationExplicit RadicalCategory  
 TranscendentalFunctionCategory *with* abs argument complex conjugate exquo imag imaginary norm polarCoordinates rational rational? rationalIfCan real  
 ConvertibleTo{KONVERT}: *with* convert  
 DequeueAggregate{DQAGG}: QueueAggregate  
 StackAggregate *with* bottom! dequeue extractBottom! extractTop! height insertBottom! insertTop! reverse! top!  
 DictionaryOperations{DIOPS}: BagAggregate Collection *with* dictionary remove! select!  
 Dictionary{DIAGG}: DictionaryOperations *with*  
 DifferentialExtension{DIFEXT}: DifferentialRing  
 PartialDifferentialRing Ring *with* D differentiate  
 DifferentialPolynomialCategory{DPOLCAT}:  
 DifferentialExtension Evalable InnerEvalable  
 PolynomialCategory RetractableTo *with* degree  
 differentialVariables initial isobaric? leader makeVariable  
 order separant weight weights  
 DifferentialRing{DIFRING}: Ring *with* D differentiate  
 DifferentialVariableCategory{DVARCAT}: OrderedSet  
 RetractableTo *with* D coerce differentiate makeVariable  
 order variable weight  
 DirectProductCategory{DIRPCAT}: AbelianSemiGroup  
 Algebra BiModule CancellationAbelianMonoid CoercibleTo  
 CommutativeRing DifferentialExtension Finite  
 FullyLinearlyExplicitRingOver FullyRetractableTo  
 IndexedAggregate OrderedAbelianMonoidSup OrderedRing  
 VectorSpace *with* \* directProduct dot unitVector  
 DivisionRing{DIVRING}: Algebra EntireRing *with* \*\* inv  
 DoublyLinkedAggregate{DLAGG}: RecursiveAggregate  
*with* concat! head last next previous setnext! setprevious!  
 tail  
 ElementaryFunctionCategory{ELEMFUN}: *with* \*\* exp  
 log  
 EltableAggregate{ELTAGG}: Eltable *with* elt qelt qsetelt!  
 setelt  
 Eltable{ELTAB}: *with* elt  
 EntireRing{ENTIRER}: BiModule Ring *with*  
 EuclideanDomain{EUCDOM}: PrincipalIdealDomain *with*  
 divide euclideanSize extendedEuclidean multiEuclidean  
 quo rem sizeLess?  
 Evalable{EVALAB}: *with* eval  
 ExpressionSpace{ES}: Evalable InnerEvalable OrderedSet  
 RetractableTo *with* belong? box definingPolynomial  
 distribute elt eval freeOf? height is? kernel kernels  
 mainKernel map minPoly operator operators paren subst  
 tower  
 ExtensibleLinearAggregate{ELAGG}: LinearAggregate *with*  
 concat! delete! insert! merge! remove! removeDuplicates!  
 select!  
 ExtensionField{XF}: CharacteristicZero Field  
 FieldOfPrimeCharacteristic RetractableTo VectorSpace *with*  
 Frobenius algebraic? degree extensionDegree  
 inGroundField? transcendenceDegree transcendent?  
 FieldOfPrimeCharacteristic{FPC}: CharacteristicNonZero  
 Field *with* discreteLog order primeFrobenius  
 Field{FIELD}: DivisionRing EuclideanDomain  
 UniqueFactorizationDomain *with* /  
 FileCategory{FILECAT}: SetCategory *with* close! iomode  
 name open read! reopen! write!  
 FileNameCategory{FNCAT}: SetCategory *with* coerce  
 directory exists? extension filename name new readable?  
 writable?  
 FiniteAbelianMonoidRing{FAMR}: AbelianMonoidRing  
 FullyRetractableTo *with* coefficients content exquo ground  
 ground? mapExponents minimumDegree  
 numberOfMonomials primitivePart  
 FiniteAlgebraicExtensionField{FAXF}: ExtensionField  
 FiniteFieldCategory RetractableTo *with* basis coordinates  
 createNormalElement definingPolynomial degree  
 extensionDegree generator minimalPolynomial norm  
 normal? normalElement represents trace  
 FiniteFieldCategory{FFIELDC}: FieldOfPrimeCharacteristic  
 Finite StepThrough *with* charthRoot conditionP  
 createPrimitiveElement discreteLog  
 factorsOfCyclicGroupSize order primitive?



primitiveElement representationType  
 tableForDiscreteLogarithm  
 FiniteLinearAggregate{FLAGG}: LinearAggregate  
 OrderedSet *with* copyInto! merge position reverse reverse!  
 sort sort! sorted?  
 FiniteRankAlgebra{FINRAlg}: Algebra  
 CharacteristicNonZero CharacteristicZero *with*  
 characteristicPolynomial coordinates discriminant  
 minimalPolynomial norm rank regularRepresentation  
 represents trace traceMatrix  
 FiniteRankNonAssociativeAlgebra{FINAALG}:  
 NonAssociativeAlgebra *with* JacobiIdentity?  
 JordanAlgebra? alternative? antiAssociative?  
 antiCommutative? associative? associatorDependence  
 commutative? conditionsForIdempotents coordinates  
 flexible? jordanAdmissible? leftAlternative?  
 leftCharacteristicPolynomial leftDiscriminant  
 leftMinimalPolynomial leftNorm leftRecip  
 leftRegularRepresentation leftTrace leftTraceMatrix  
 leftUnit leftUnits lieAdmissible? lieAlgebra?  
 noncommutativeJordanAlgebra? powerAssociative? rank  
 recip represents rightAlternative?  
 rightCharacteristicPolynomial rightDiscriminant  
 rightMinimalPolynomial rightNorm rightRecip  
 rightRegularRepresentation rightTrace rightTraceMatrix  
 rightUnit rightUnits someBasis structuralConstants unit  
 FiniteSetAggregate{FSAGG}: Dictionary Finite  
 SetAggregate *with* cardinality complement max min  
 universe  
 Finite{FINITE}: SetCategory *with* index lookup random size  
 FloatingPointSystem{FPS}: RealNumberSystem *with* base  
 bits decreasePrecision digits exponent float  
 increasePrecision mantissa max order precision  
 FramedAlgebra{FRAMALG}: FiniteRankAlgebra *with* basis  
 convert coordinates discriminant regularRepresentation  
 represents traceMatrix  
 FramedNonAssociativeAlgebra{FRNAALG}:  
 FiniteRankNonAssociativeAlgebra *with* apply basis  
 conditionsForIdempotents convert coordinates elt  
 leftDiscriminant leftRankPolynomial  
 leftRegularRepresentation leftTraceMatrix represents  
 rightDiscriminant rightRankPolynomial  
 rightRegularRepresentation rightTraceMatrix  
 structuralConstants  
 FreeAbelianMonoidCategory{FAMONC}:  
 CancellationAbelianMonoid RetractableTo *with* \* +  
 coefficient highCommonTerms mapCoef mapGen nthCoef  
 nthFactor size terms  
 FullyEvalableOver{FEVALAB}: Eltable Evalable  
 InnerEvalable *with* map  
 FullyLinearlyExplicitRingOver{FLINEXP}:  
 LinearlyExplicitRingOver *with*  
 FullyPatternMatchable{FPATMAB}: Object  
 PatternMatchable *with*  
 FullyRetractableTo{FRETRCT}: RetractableTo *with*  
 FunctionFieldCategory{FFCAT}: MonogenicAlgebra *with* D  
 absolutelyIrreducible? branchPoint?  
 branchPointAtInfinity? complementaryBasis differentiate  
 elt genus integral? integralAtInfinity? integralBasis  
 integralBasisAtInfinity integralCoordinates  
 integralDerivationMatrix integralMatrix  
 integralMatrixAtInfinity integralRepresents  
 inverseIntegralMatrix inverseIntegralMatrixAtInfinity  
 nonSingularModel normalizeAtInfinity  
 numberOfComponents primitivePart ramified?  
 ramifiedAtInfinity? rationalPoint? rationalPoints  
 reduceBasisAtInfinity represents singular?  
 singularAtInfinity? yCoordinates  
 FunctionSpace{FS}: AbelianGroup AbelianMonoid Algebra  
 CharacteristicNonZero CharacteristicZero ConvertibleTo  
 ExpressionSpace Field FullyLinearlyExplicitRingOver  
 FullyPatternMatchable FullyRetractableTo Group Monoid  
 PartialDifferentialRing Patternable RetractableTo Ring *with*  
 \*\* / applyQuote coerce convert denom denominator eval  
 ground ground? isExpt isMult isPlus isPower isTimes  
 numer numerator univariate variables  
 GcdDomain{GCDDOM}: IntegralDomain *with* gcd lcm  
 GradedAlgebra{GRALG}: GradedModule *with* One product  
 GradedModule{GRMOD}: RetractableTo SetCategory *with*  
 \* + - Zero degree  
 Group{GROUP}: Monoid *with* \*\* / commutator conjugate  
 inv  
 HomogeneousAggregate{HOAGG}: Aggregate  
 SetCategory *with* any? count every? map map! member?  
 members parts  
 HyperbolicFunctionCategory{HYPCAT}: *with* cosh coth  
 csch sech sinh tanh  
 IndexedAggregate{IXAGG}: EltableAggregate  
 HomogeneousAggregate *with* entries entry? fill! first index?  
 indices maxIndex minIndex swap!  
 IndexedDirectProductCategory{IDPC}: SetCategory *with*  
 leadingCoefficient leadingSupport map monomial reductum  
 InnerEvalable{IEVALAB}: *with* eval  
 IntegerNumberSystem{INS}: CharacteristicZero  
 CombinatorialFunctionCategory ConvertibleTo  
 DifferentialRing EuclideanDomain LinearlyExplicitRingOver  
 OrderedRing PatternMatchable RealConstant RetractableTo  
 StepThrough UniqueFactorizationDomain *with* addmod base  
 bit? copy dec even? hash inc invmod length mask mulmod  
 odd? positiveRemainder powmod random rational

rational? rationalIfCan shift submod symmetricRemainder  
 IntegralDomain{INTDOM}: Algebra CommutativeRing  
 EntireRing *with* associates? exquo unit? unitCanonical  
 unitNormal  
 KeyedDictionary{KDAGG}: Dictionary *with* key? keys  
 remove! search  
 LazyStreamAggregate{LZSTAGG}: StreamAggregate *with*  
 complete explicitEntries? explicitlyEmpty? extend first  
 lazy? lazyEvaluate numberOfComputedEntries remove rst  
 select  
 LeftAlgebra{LALG}: LeftModule Ring *with* coerce  
 LeftModule{LMODULE}: AbelianGroup *with* \*  
 LinearAggregate{LNAGG}: Collection IndexedAggregate  
*with* concat delete elt insert map new setelt  
 LinearlyExplicitRingOver{LINEXP}: Ring *with*  
 reducedSystem  
 LiouvillianFunctionCategory{LFCAT}:  
 PrimitiveFunctionCategory TranscendentalFunctionCategory  
*with* Ci Ei Si dilog erf li  
 ListAggregate{LSAGG}: ExtensibleLinearAggregate  
 FiniteLinearAggregate StreamAggregate *with* list  
 MatrixCategory{MATCAT}: TwoDimensionalArrayCategory  
*with* \* \*\* + - / antisymmetric? coerce determinant  
 diagonal? diagonalMatrix elt exquo horizConcat inverse  
 listOfLists matrix minordet nullSpace nullity rank  
 rowEchelon scalarMatrix setelt setsubMatrix! square?  
 squareTop subMatrix swapColumns! swapRows!  
 symmetric? transpose vertConcat zero  
 Module{MODULE}: BiModule *with*  
 MonadWithUnit{MONADWU}: Monad *with* \*\* One  
 leftPower leftRecip one? recip rightPower rightRecip  
 Monad{MONAD}: SetCategory *with* \* \*\* leftPower  
 rightPower  
 MonogenicAlgebra{MONOGEN}: CommutativeRing  
 ConvertibleTo DifferentialExtension Field Finite  
 FiniteFieldCategory FramedAlgebra  
 FullyLinearlyExplicitRingOver FullyRetractableTo *with*  
 convert definingPolynomial derivationCoordinates  
 generator lift reduce  
 MonogenicLinearOperator{MLO}: Algebra BiModule Ring  
*with* coefficient degree leadingCoefficient minimumDegree  
 monomial reductum  
 Monoid{MONOID}: SemiGroup *with* \*\* One one? recip  
 MultiDictionary{MDAGG}: DictionaryOperations *with*  
 duplicates insert! removeDuplicates!  
 MultiSetAggregate{MSAGG}: MultiDictionary  
 SetAggregate *with*

MultivariateTaylorSeriesCategory{MTSCAT}: Evaluable  
 InnerEvaluable PartialDifferentialRing PowerSeriesCategory  
 RadicalCategory TranscendentalFunctionCategory *with*  
 coefficient extend integrate monomial order polynomial  
 NonAssociativeAlgebra{NAALG}: Module  
 NonAssociativeRng *with* plenaryPower  
 NonAssociativeRing{NASRING}: MonadWithUnit  
 NonAssociativeRng *with* characteristic coerce  
 NonAssociativeRng{NARNG}: AbelianGroup Monad *with*  
 antiCommutator associator commutator  
 Object{OBJECT}: *with*  
 OctonionCategory{OC}: Algebra CharacteristicNonZero  
 CharacteristicZero ConvertibleTo Finite FullyEvaluableOver  
 FullyRetractableTo OrderedSet *with* abs conjugate imagE  
 imagI imagJ imagK imagi imagj imagk inv norm octon  
 rational rational? rationalIfCan real  
 OneDimensionalArrayAggregate{A1AGG}:  
 FiniteLinearAggregate *with*  
 OrderedAbelianGroup{OAGROUP}: AbelianGroup  
 OrderedCancellationAbelianMonoid *with*  
 OrderedAbelianMonoidSup{OAMONS}:  
 OrderedCancellationAbelianMonoid *with* sup  
 OrderedAbelianMonoid{OAMON}: AbelianMonoid  
 OrderedAbelianSemiGroup *with*  
 OrderedAbelianSemiGroup{OASGP}: AbelianMonoid  
 OrderedSet *with*  
 OrderedCancellationAbelianMonoid{OCAMON}:  
 CancellationAbelianMonoid OrderedAbelianMonoid *with*  
 OrderedFinite{ORDFIN}: Finite OrderedSet *with*  
 OrderedMonoid{ORDMON}: Monoid OrderedSet *with*  
 OrderedMultiSetAggregate{OMAGG}: MultiSetAggregate  
 PriorityQueueAggregate *with* min  
 OrderedRing{ORDRING}: OrderedAbelianGroup  
 OrderedMonoid Ring *with* abs negative? positive? sign  
 OrderedSet{ORDSET}: SetCategory *with* < max min  
 PAdicIntegerCategory{PADICCT}: CharacteristicZero  
 EuclideanDomain *with* approximate complete digits extend  
 moduloP modulus order quotientByP sqrt  
 PartialDifferentialRing{PDRING}: Ring *with* D differentiate  
 PartialTranscendentalFunctions{PTRANFN}: *with*  
 acosIfCan acoshIfCan acotIfCan acothIfCan acscIfCan  
 asecIfCan asechIfCan asinIfCan asinhIfCan  
 atanIfCan atanhIfCan cosIfCan coshIfCan cotIfCan  
 cothIfCan csclIfCan cschIfCan expIfCan logIfCan  
 nthRootIfCan secIfCan sechIfCan sinIfCan sinhIfCan  
 tanIfCan tanhIfCan

Patternable{PATAB}: ConvertibleTo Object *with*  
 PatternMatchable{PATMAB}: SetCategory *with*  
 patternMatch  
 PermutationCategory{PERMCAT}: Group OrderedSet *with*  
 < cycle cycles elt eval orbit  
 PlottablePlaneCurveCategory{PPCURVE}: CoercibleTo  
*with* listBranches xRange yRange  
 PlottableSpaceCurveCategory{PSCURVE}: CoercibleTo  
*with* listBranches xRange yRange zRange  
 PointCategory{PTCAT}: VectorCategory *with* convert  
 cross dimension extend length point  
 PolynomialCategory{POLYCAT}: ConvertibleTo Evaluable  
 FiniteAbelianMonoidRing FullyLinearlyExplicitRingOver  
 GcdDomain InnerEvaluable OrderedSet PartialDifferentialRing  
 PatternMatchable PolynomialFactorizationExplicit  
 RetractableTo *with* coefficient content degree discriminant  
 isExpt isPlus isTimes mainVariable minimumDegree  
 monicDivide monomial monomials multivariate  
 primitiveMonomials primitivePart resultant squareFree  
 squareFreePart totalDegree univariate variables  
 PolynomialFactorizationExplicit{PFECAT}:  
 UniqueFactorizationDomain *with* charthRoot conditionP  
 factorPolynomial factorSquareFreePolynomial  
 gcdPolynomial solveLinearPolynomialEquation  
 squareFreePolynomial  
 PowerSeriesCategory{PSCAT}: AbelianMonoidRing *with*  
 complete monomial pole? variables  
 PrimitiveFunctionCategory{PRIMCAT}: *with* integral  
 PrincipalIdealDomain{PID}: GcdDomain *with*  
 expressIdealMember principalIdeal  
 PriorityQueueAggregate{PRQAGG}: BagAggregate *with*  
 max merge merge!  
 QuaternionCategory{QUATCAT}: Algebra  
 CharacteristicNonZero CharacteristicZero ConvertibleTo  
 DifferentialExtension DivisionRing EntireRing  
 FullyEvaluableOver FullyLinearlyExplicitRingOver  
 FullyRetractableTo OrderedSet *with* abs conjugate imagI  
 imagJ imagK norm quatern rational rational?  
 rationalIfCan real  
 QueueAggregate{QUAGG}: BagAggregate *with* back  
 dequeue! enqueue! front length rotate!  
 QuotientFieldCategory{QFCAT}: Algebra  
 CharacteristicNonZero CharacteristicZero ConvertibleTo  
 DifferentialExtension Field FullyEvaluableOver  
 FullyLinearlyExplicitRingOver FullyPatternMatchable  
 OrderedRing OrderedSet Patternable  
 PolynomialFactorizationExplicit RealConstant RetractableTo  
 StepThrough *with* / ceiling denom denominator floor  
 fractionPart numer numerator random wholePart  
 RadicalCategory{RADCAT}: *with* \*\* nthRoot sqrt  
 RealConstant{REAL}: ConvertibleTo *with*  
 RealNumberSystem{RNS}: CharacteristicZero  
 ConvertibleTo Field OrderedRing PatternMatchable  
 RadicalCategory RealConstant RetractableTo *with* abs ceiling  
 floor fractionPart norm round truncate wholePart  
 RectangularMatrixCategory{RMATCAT}: BiModule  
 HomogeneousAggregate Module *with* / antisymmetric?  
 column diagonal? elt exquo listOfLists map matrix  
 maxColIndex maxRowIndex minColIndex minRowIndex  
 ncols nrows nullSpace nullity qelt rank row rowEchelon  
 square? symmetric?  
 RecursiveAggregate{RCAGG}: HomogeneousAggregate  
*with* children cyclic? elt leaf? leaves node? nodes  
 setchildren! setelt setvalue! value  
 RetractableTo{RETRACT}: *with* coerce retract  
 retractIfCan  
 RightModule{RMODULE}: AbelianGroup *with* \*  
 Ring{RING}: LeftModule Monoid Rng *with* characteristic  
 coerce  
 Rng{RNG}: AbelianGroup SemiGroup *with*  
 SegmentCategory{SEGCAT}: SetCategory *with* BY  
 SEGMENT convert hi high incr lo low segment  
 SegmentExpansionCategory{SEGXCAT}:  
 SegmentCategory *with* expand map  
 SemiGroup{SGROUP}: SetCategory *with* \* \*\*  
 SetAggregate{SETAGG}: Collection SetCategory *with* <  
 brace difference intersect subset? symmetricDifference  
 union  
 SetCategory{SETCAT}: CoercibleTo Object *with* =  
 SExpressionCategory{SEXCAT}: SetCategory *with* #  
 atom? car cdr convert destruct elt eq expr float float?  
 integer integer? list? null? pair? string string? symbol  
 symbol? uequal  
 SpecialFunctionCategory{SPFCAT}: *with* Beta Gamma  
 abs airyAi airyBi bessellJ bessellK bessellY digamma  
 polygamma  
 SquareMatrixCategory{SMATCAT}: Algebra BiModule  
 DifferentialExtension FullyLinearlyExplicitRingOver  
 FullyRetractableTo Module RectangularMatrixCategory *with*  
 \* \*\* determinant diagonal diagonalMatrix  
 diagonalProduct inverse minordet scalarMatrix trace  
 StackAggregate{SKAGG}: BagAggregate *with* depth pop!  
 push! top  
 StepThrough{STEP}: SetCategory *with* init nextItem  
 StreamAggregate{STAGG}: LinearAggregate  
 UnaryRecursiveAggregate *with* explicitlyFinite?

possiblyInfinite?

StringAggregate{SRAGG}: OneDimensionalArrayAggregate  
with coerce elt leftTrim lowerCase lowerCase! match  
match? position prefix? replace rightTrim split substring?  
suffix? trim upperCase upperCase!

StringCategory{STRICAT}: StringAggregate with string

TableAggregate{TBAGG}: IndexedAggregate  
KeyedDictionary with map setelt table

ThreeSpaceCategory{SPACEC}: SetCategory with check  
closedCurve closedCurve? coerce components composite  
composites copy create3Space curve curve? enterPointData  
l1lip l1lp l1prop l1p merge mesh mesh?  
modifyPointData numberOfComponents  
numberOfComposites objects point point? polygon  
polygon? subspace

TranscendentalFunctionCategory{TRANFUN}:

ArcHyperbolicFunctionCategory

ArcTrigonometricFunctionCategory

ElementaryFunctionCategory HyperbolicFunctionCategory

TrigonometricFunctionCategory with pi

TrigonometricFunctionCategory{TRIGCAT}: with cos  
cot csc sec sin tan

TwoDimensionalArrayCategory{ARR2CAT}:

HomogeneousAggregate with column elt fill! map map!  
maxColIndex maxRowIndex minColIndex minRowIndex  
ncols new nrows parts qelt qsetelt! row setColumn!  
setRow! setelt

UnaryRecursiveAggregate{URAGG}: RecursiveAggregate  
with concat concat! cycleEntry cycleLength cycleSplit!  
cycleTail elt first last rest second setelt setfirst! setlast!  
setrest! split! tail third

UniqueFactorizationDomain{UFD}: GcdDomain with  
factor prime? squareFree squareFreePart

UnivariateLaurentSeriesCategory{ULSCAT}: Field

RadicalCategory TranscendentalFunctionCategory

UnivariatePowerSeriesCategory with integrate  
multiplyCoefficients rationalFunction

UnivariateLaurentSeriesConstructorCategory

{ULSCCAT}: QuotientFieldCategory RetractableTo

UnivariateLaurentSeriesCategory with coerce degree laurent  
removeZeroes taylor taylorIfCan taylorRep

UnivariatePolynomialCategory{UPOLYC}:

DifferentialExtension DifferentialRing Eltable

EuclideanDomain PolynomialCategory StepThrough with D  
composite differentiate discriminant divideExponents elt  
integrate makeSUP monicDivide multiplyExponents order  
pseudoDivide pseudoQuotient pseudoRemainder resultant  
separate subResultantGcd unmakeSUP vectorise

UnivariatePowerSeriesCategory{UPSCAT}:

DifferentialRing Eltable PowerSeriesCategory with

approximate center elt eval extend multiplyExponents  
order series terms truncate variable

UnivariatePuisseuxSeriesCategory{UPXSCAT}: Field

RadicalCategory TranscendentalFunctionCategory

UnivariatePowerSeriesCategory with integrate  
multiplyExponents

UnivariatePuisseuxSeriesConstructorCategory

{UPXSCCA}: RetractableTo UnivariatePuisseuxSeriesCategory

with coerce degree laurent laurentIfCan laurentRep  
puisseux rationalPower

UnivariateTaylorSeriesCategory{UTSCAT}:

RadicalCategory TranscendentalFunctionCategory

UnivariatePowerSeriesCategory with \*\* coefficients integrate  
multiplyCoefficients polynomial quoByVar series

VectorCategory{VECTCAT}:

OneDimensionalArrayAggregate with \* + - dot zero

VectorSpace{VSPACE}: Module with / dimension

[



# Domains

This is a listing of all domains in the AXIOM library at the time this book was produced. Use the Browse facility (described in Chapter 14) to get more information about these constructors.

This sample entry will help you read the following table:

DomainName{DomainAbbreviation}:

Category<sub>1</sub>... Category<sub>N</sub> *with* operation<sub>1</sub> ... operation<sub>M</sub>

where

DomainName is the full domain name, for example, Integer.

DomainAbbreviation is the domain abbreviation, for example, INT.

Category<sub>i</sub> is a category to which the domain belongs.

operation<sub>j</sub> is an operation exported by the domain.

AlgebraGivenByStructuralConstants{ALGSC}:  
 FramedNonAssociativeAlgebra LeftModule *with* 0 \* \*\* + - =  
 JacobiIdentity? JordanAlgebra? alternative?  
 antiAssociative? antiCommutative? antiCommutator  
 apply associative? associator associatorDependence basis  
 coerce commutative? commutator  
 conditionsForIdempotents convert coordinates elt flexible?  
 jordanAdmissible? leftAlternative?  
 leftCharacteristicPolynomial leftDiscriminant  
 leftMinimalPolynomial leftNorm leftPower

leftRankPolynomial leftRecip leftRegularRepresentation  
 leftTrace leftTraceMatrix leftUnit leftUnits lieAdmissible?  
 lieAlgebra? noncommutativeJordanAlgebra? plenaryPower  
 powerAssociative? rank recip represents rightAlternative?  
 rightCharacteristicPolynomial rightDiscriminant  
 rightMinimalPolynomial rightNorm rightPower  
 rightRankPolynomial rightRecip  
 rightRegularRepresentation rightTrace rightTraceMatrix  
 rightUnit rightUnits someBasis structuralConstants unit  
 zero?

AlgebraicFunctionField{ALGFF}: FunctionFieldCategory  
*with* 0 1 \* \*\* + - / = D absolutelyIrreducible? associates?  
 basis branchPoint? branchPointAtInfinity? characteristic  
 characteristicPolynomial charthRoot coerce  
 complementaryBasis convert coordinates  
 definingPolynomial derivationCoordinates differentiate  
 discriminant divide elt euclideanSize expressIdealMember  
 exquo extendedEuclidean factor gcd generator genus  
 integral? integralAtInfinity? integralBasis  
 integralBasisAtInfinity integralCoordinates  
 integralDerivationMatrix integralMatrix  
 integralMatrixAtInfinity integralRepresents inv  
 inverseIntegralMatrix inverseIntegralMatrixAtInfinity  
 knownInfBasis lcm lift minimalPolynomial multiEuclidean  
 nonSingularModel norm normalizeAtInfinity  
 numberOfComponents one? prime? primitivePart  
 principalIdeal quo ramified? ramifiedAtInfinity? rank  
 rationalPoint? rationalPoints recip reduce  
 reduceBasisAtInfinity reducedSystem  
 regularRepresentation rem represents retract retractIfCan  
 singular? singularAtInfinity? sizeLess? squareFree  
 squareFreePart trace traceMatrix unit? unitCanonical  
 unitNormal yCoordinates zero?

AlgebraicNumber{AN}: AlgebraicallyClosedField  
 CharacteristicZero ConvertibleTo DifferentialRing  
 ExpressionSpace LinearlyExplicitRingOver RealConstant  
 RetractableTo *with* 0 1 \* \*\* + - / < = D associates? belong?

box characteristic coerce convert definingPolynomial  
denom differentiate distribute divide elt euclideanSize eval  
expressIdealMember exquo extendedEuclidean factor  
freeOf? gcd height inv is? kernel kernels lcm mainKernel  
map max min minPoly multiEuclidean nthRoot numer  
one? operator operators paren prime? principalIdeal quo  
recip reduce reducedSystem rem retract retractIfCan  
rootOf rootsOf sizeLess? sqrt squareFree squareFreePart  
subst tower unit? unitCanonical unitNormal zero? zeroOf  
zerosOf

AnonymousFunction{ANON}: SetCategory *with* = coerce

AntiSymm{ANTISYM}: LeftAlgebra RetractableTo *with* 0 1  
\* \*\* + - = characteristic coefficient coerce degree exp  
generator homogeneous? leadingBasisTerm  
leadingCoefficient map one? recip reductum retract  
retractIfCan retractable? zero?

Any{ANY}: SetCategory *with* = any coerce domain  
domainOf obj objectOf showTypeInOutPut

ArrayStack{ASTACK}: StackAggregate *with* # = any?  
arrayStack bag coerce copy count depth empty empty? eq?  
every? extract! insert! inspect less? map map! member?  
members more? parts pop! push! size? top

AssociatedJordanAlgebra{JORDAN}: CoercibleTo  
FiniteRankNonAssociativeAlgebra

FramedNonAssociativeAlgebra NonAssociativeAlgebra *with* 0  
\* \*\* + - = JacobiIdentity? JordanAlgebra? alternative?  
antiAssociative? antiCommutative? antiCommutator  
apply associative? associator associatorDependence basis  
coerce commutative? commutator  
conditionsForIdempotents convert coordinates elt flexible?  
jordanAdmissible? leftAlternative?  
leftCharacteristicPolynomial leftDiscriminant  
leftMinimalPolynomial leftNorm leftPower  
leftRankPolynomial leftRecip leftRegularRepresentation  
leftTrace leftTraceMatrix leftUnit leftUnits lieAdmissible?  
lieAlgebra? noncommutativeJordanAlgebra? plenaryPower  
powerAssociative? rank recip represents rightAlternative?  
rightCharacteristicPolynomial rightDiscriminant  
rightMinimalPolynomial rightNorm rightPower  
rightRankPolynomial rightRecip  
rightRegularRepresentation rightTrace rightTraceMatrix  
rightUnit rightUnits someBasis structuralConstants unit  
zero?

AssociatedLieAlgebra{LIE}: CoercibleTo  
FiniteRankNonAssociativeAlgebra  
FramedNonAssociativeAlgebra NonAssociativeAlgebra *with* 0  
\* \*\* + - = JacobiIdentity? JordanAlgebra? alternative?  
antiAssociative? antiCommutative? antiCommutator  
apply associative? associator associatorDependence basis  
coerce commutative? commutator  
conditionsForIdempotents convert coordinates elt flexible?  
jordanAdmissible? leftAlternative?  
leftCharacteristicPolynomial leftDiscriminant

leftMinimalPolynomial leftNorm leftPower  
leftRankPolynomial leftRecip leftRegularRepresentation  
leftTrace leftTraceMatrix leftUnit leftUnits lieAdmissible?  
lieAlgebra? noncommutativeJordanAlgebra? plenaryPower  
powerAssociative? rank recip represents rightAlternative?  
rightCharacteristicPolynomial rightDiscriminant  
rightMinimalPolynomial rightNorm rightPower  
rightRankPolynomial rightRecip  
rightRegularRepresentation rightTrace rightTraceMatrix  
rightUnit rightUnits someBasis structuralConstants unit  
zero?

AssociationList{ALIST}: AssociationListAggregate *with* # =  
any? assoc bag child? children coerce concat concat!  
construct copy copyInto! count cycleEntry cycleLength  
cycleSplit! cycleTail cyclic? delete delete! dictionary  
distance elt empty empty? entries entry? eq? every?  
explicitlyFinite? extract! fill! find first index? indices  
insert insert! inspect key? keys last leaf? less? list map  
map! maxIndex member? members merge merge!  
minIndex more? new node? nodes parts position  
possiblyInfinite? qelt qsetelt! reduce remove remove!  
removeDuplicates removeDuplicates! rest reverse reverse!  
search second select select! setchildren! setelt setfirst!  
setlast! setrest! setvalue! size? sort sort! sorted? split!  
swap! table tail third value

BalancedBinaryTree{BBTREE}: BinaryTreeCategory *with* #  
= any? balancedBinaryTree children coerce copy count  
cyclic? elt empty empty? eq? every? leaf? leaves left less?  
map map! mapDown! mapUp! member? members more?  
node node? nodes parts right setchildren! setelt setleaves!  
setleft! setright! setvalue! size? value

BalancedPAdicInteger{BPADIC}: PAdicIntegerCategory  
*with* 0 1 \* \*\* + - = approximate associates? characteristic  
coerce complete digits divide euclideanSize  
expressIdealMember exquo extend extendedEuclidean gcd  
lcm moduloP modulus multiEuclidean one? order  
principalIdeal quo quotientByP recip rem sizeLess? sqrt  
unit? unitCanonical unitNormal zero?

BalancedPAdicRational{BPADICRT}:  
QuotientFieldCategory *with* 0 1 \* \*\* + - / = D approximate  
associates? characteristic coerce continuedFraction denom  
denominator differentiate divide euclideanSize  
expressIdealMember exquo extendedEuclidean factor  
fractionPart gcd inv lcm map multiEuclidean numer  
numerator one? prime? principalIdeal quo recip  
reducedSystem rem removeZeroes retract retractIfCan  
sizeLess? squareFree squareFreePart unit? unitCanonical  
unitNormal wholePart zero?

BasicOperator{BOP}: OrderedSet *with* < = arity assert  
coerce comparison copy deleteProperty! display equality  
has? input is? max min name nary? nullary? operator  
properties property setProperties setProperty unary?  
weight



**BinaryExpansion{BINARY}**: **QuotientFieldCategory** *with* 0 1 \* \*\* + - / < = D abs associates? binary ceiling characteristic coerce convert denom denominator differentiate divide euclideanSize expressIdealMember exquo extendedEuclidean factor floor fractionPart gcd init inv lcm map max min multiEuclidean negative? nextItem numer numerator one? patternMatch positive? prime? principalIdeal quo random recip reducedSystem rem retract retractIfCan sign sizeLess? squareFree squareFreePart unit? unitCanonical unitNormal wholePart zero?

**BinarySearchTree{BSTREE}**: **BinaryTreeCategory** *with* # = any? binarySearchTree children coerce copy count cyclic? elt empty empty? eq? every? insert! insertRoot! leaf? leaves left less? map map! member? members more? node node? nodes parts right setchildren! setelt setleft! setright! setvalue! size? split value

**BinaryTournament{BTOURN}**: **BinaryTreeCategory** *with* # = any? binaryTournament children coerce copy count cyclic? elt empty empty? eq? every? insert! leaf? leaves left less? map map! member? members more? node node? nodes parts right setchildren! setelt setleft! setright! setvalue! size? value

**BinaryTree{BTREE}**: **BinaryTreeCategory** *with* # = any? binaryTree children coerce copy count cyclic? elt empty empty? eq? every? leaf? leaves left less? map map! member? members more? node node? nodes parts right setchildren! setelt setleft! setright! setvalue! size? value

**Bits{BITS}**: **BitAggregate** *with* # < = ^ and any? bits coerce concat construct convert copy copyInto! count delete elt empty empty? entries entry? eq? every? fill! find first index? indices insert less? map map! max maxIndex member? members merge min minIndex more? nand new nor not or parts position qelt qsetelt! reduce remove removeDuplicates reverse reverse! select setelt size? sort sort! sorted? swap! xor

**Boolean{BOOLEAN}**: **ConvertibleTo Finite OrderedSet** *with* < = ^ and coerce convert false implies index lookup max min nand nor not or random size true xor

**CardinalNumber{CARD}**: **CancellationAbelianMonoid Monoid OrderedSet RetractableTo** *with* 0 1 \* \*\* + - < = Aleph coerce countable? finite? generalizedContinuumHypothesisAssumed generalizedContinuumHypothesisAssumed? max min one? recip retract retractIfCan zero?

**CartesianTensor{CARTEN}**: **GradedAlgebra** *with* 0 1 \* + - = coerce contract degree elt kroneckerDelta leviCivitaSymbol product rank ravel reindex retract retractIfCan transpose unravel

**CharacterClass{CCLASS}**: **ConvertibleTo FiniteSetAggregate SetCategory** *with* # < = alphabetic alphanumeric any? bag brace cardinality charClass coerce complement construct convert copy count dictionary

difference digit empty empty? eq? every? extract! find hexDigit index insert! inspect intersect less? lookup lowerCase map map! max member? members min more? parts random reduce remove remove! removeDuplicates select select! size size? subset? symmetricDifference union universe upperCase

**Character{CHAR}**: **OrderedFinite** *with* < = alphabetic? alphanumeric? char coerce digit? escape hexDigit? index lookup lowerCase lowerCase? max min ord quote random size space upperCase upperCase?

**CliffordAlgebra{CLIF}**: **Algebra Ring VectorSpace** *with* 0 1 \* \*\* + - / = characteristic coefficient coerce dimension e monomial one? recip zero?

**Color{COLOR}**: **AbelianSemiGroup** *with* \* + = blue coerce color green hue numberOffHues red yellow

**Commutator{COMM}**: **SetCategory** *with* = coerce mkcomm

**Complex{COMPLEX}**: **ComplexCategory** *with* 0 1 \* \*\* + - / < = D abs acos acosh acot acoth acsc acsch argument asec asech asin asinh associates? atan atanh basis characteristic characteristicPolynomial charthRoot coerce complex conditionP conjugate convert coordinates cos cosh cot coth createPrimitiveElement csc csch definingPolynomial derivationCoordinates differentiate discreteLog discriminant divide elt euclideanSize eval exp expressIdealMember exquo extendedEuclidean factor factorPolynomial factorSquareFreePolynomial factorsOfCyclicGroupSize gcd gcdPolynomial generator imag imaginary index init inv lcm lift log lookup map max min minimalPolynomial multiEuclidean nextItem norm nthRoot one? order pi polarCoordinates prime? primeFrobenius primitive? primitiveElement principalIdeal quo random rank rational rational? rationalIfCan real recip reduce reducedSystem regularRepresentation rem representationType represents retract retractIfCan sec sech sin sinh size sizeLess? solveLinearPolynomialEquation sqrt squareFree squareFreePart squareFreePolynomial tableForDiscreteLogarithm tan tanh trace traceMatrix unit? unitCanonical unitNormal zero?

**ContinuedFraction{CONTFRAC}**: **Algebra Field** *with* 0 1 \* \*\* + - / = approximants associates? characteristic coerce complete continuedFraction convergents denominators divide euclideanSize expressIdealMember exquo extend extendedEuclidean factor gcd inv lcm multiEuclidean numerators one? partialDenominators partialNumerators partialQuotients prime? principalIdeal quo recip reducedContinuedFraction reducedForm rem sizeLess? squareFree squareFreePart unit? unitCanonical unitNormal wholePart zero?

**Database{DBASE}**: **SetCategory** *with* + - = coerce display elt fullDisplay

**DoubleFloat{DFLOAT}**: **ConvertibleTo DifferentialRing**

**FloatingPointSystem TranscendentalFunctionCategory** *with* 0 1 \* \*\* + - / < = D abs acos acosh acot acoth acsc acsch asec asech asin asinh associates? atan atanh base bits ceiling characteristic coerce convert cos cosh cot coth csc csch decreasePrecision differentiate digits divide euclideanSize exp exp1 exponent expressIdealMember exquo extendedEuclidean factor float floor fractionPart gcd hash increasePrecision inv lcm log log10 log2 mantissa max min multiEuclidean negative? norm nthRoot one? order patternMatch pi positive? precision prime? principalIdeal quo rationalApproximation recip rem retract retractIfCan round sec sech sign sin sinh sizeLess? sqrt squareFree squareFreePart tan tanh truncate unit? unitCanonical unitNormal wholePart zero?

**DataList{DLIST}**: **ListAggregate** *with* # < = any? children coerce concat concat! construct convert copy copyInto! count cycleEntry cycleLength cycleSplit! cycleTail cyclic? datalist delete delete! elt empty empty? entries entry? eq? every? explicitlyFinite? fill! find first index? indices insert insert! last leaf? leaves less? list map map! max maxIndex member? members merge merge! min minIndex more? new node? nodes parts position possiblyInfinite? qelt qsetelt! reduce remove remove! removeDuplicates removeDuplicates! rest reverse reverse! second select select! setchildren! setelt setfirst! setlast! setrest! setvalue! size? sort sort! sorted? split! swap! tail third value

**DecimalExpansion{DECIMAL}**: **QuotientFieldCategory** *with* 0 1 \* \*\* + - / < = D abs associates? ceiling characteristic coerce convert decimal denom denominator differentiate divide euclideanSize expressIdealMember exquo extendedEuclidean factor floor fractionPart gcd init inv lcm map max min multiEuclidean negative? nextItem numer numerator one? patternMatch positive? prime? principalIdeal quo random recip reducedSystem rem retract retractIfCan sign sizeLess? squareFree squareFreePart unit? unitCanonical unitNormal wholePart zero?

**DenavitHartenbergMatrix{DHMATRIX}**: **MatrixCategory** *with* # \* \*\* + - / = antisymmetric? any? coerce column copy count determinant diagonal? diagonalMatrix elt empty empty? eq? every? exquo fill! horizConcat identity inverse less? listOfLists map map! matrix maxColIndex maxRowIndex member? members minColIndex minRowIndex minordet more? ncols new nrows nullSpace nullity parts qelt qsetelt! rank rotatex rotatey rotatez row rowEchelon scalarMatrix scale setColumn! setRow! setelt setsubMatrix! size? square? squareTop subMatrix swapColumns! swapRows! symmetric? translate transpose vertConcat zero

**Dequeue{DEQUEUE}**: **DequeueAggregate** *with* # = any? back bag bottom! coerce copy count depth dequeue dequeue! empty empty? enqueue! eq? every? extract! extractBottom! extractTop! front height insert! insertBottom! insertTop! inspect length less? map map! member? members more? parts pop! push! reverse!

rotate! size? top top!

**DeRhamComplex{DERHAM}**: **LeftAlgebra RetractableTo** *with* 0 1 \* \*\* + - = characteristic coefficient coerce degree exteriorDifferential generator homogeneous? leadingBasisTerm leadingCoefficient map one? recip reductum retract retractIfCan retractable? totalDifferential zero?

**DifferentialSparseMultivariatePolynomial{DSMP}**: **DifferentialPolynomialCategory RetractableTo** *with* 0 1 \* \*\* + - / < = D associates? characteristic charthRoot coefficient coefficients coerce conditionP content convert degree differentialVariables differentiate discriminant eval exquo factor factorPolynomial factorSquareFreePolynomial gcd gcdPolynomial ground ground? initial isExpt isPlus isTimes isobaric? lcm leader leadingCoefficient leadingMonomial mainVariable makeVariable map mapExponents max min minimumDegree monic monicDivide monomial monomial? monomials multivariate numberOfMonomials one? order patternMatch prime? primitiveMonomials primitivePart recip reducedSystem reductum resultant retract retractIfCan separant solveLinearPolynomialEquation squareFree squareFreePart squareFreePolynomial totalDegree unit? unitCanonical unitNormal univariate variables weight weights zero?

**DirectProductMatrixModule{DPMM}**: **DirectProductCategory LeftModule** *with* 0 1 # \* \*\* + - / < = D abs any? characteristic coerce copy count differentiate dimension directProduct dot elt empty empty? entries entry? eq? every? fill! first index index? indices less? lookup map map! max maxIndex member? members min minIndex more? negative? one? parts positive? qelt qsetelt! random recip reducedSystem retract retractIfCan setelt sign size size? sup swap! unitVector zero?

**DirectProductModule{DPMO}**: **DirectProductCategory LeftModule** *with* 0 1 # \* \*\* + - / < = D abs any? characteristic coerce copy count differentiate dimension directProduct dot elt empty empty? entries entry? eq? every? fill! first index index? indices less? lookup map map! max maxIndex member? members min minIndex more? negative? one? parts positive? qelt qsetelt! random recip reducedSystem retract retractIfCan setelt sign size size? sup swap! unitVector zero?

**DirectProduct{DIRPROD}**: **DirectProductCategory** *with* 0 1 # \* \*\* + - / < = D abs any? characteristic coerce copy count differentiate dimension directProduct dot elt empty empty? entries entry? eq? every? fill! first index index? indices less? lookup map map! max maxIndex member? members min minIndex more? negative? one? parts positive? qelt qsetelt! random recip reducedSystem retract retractIfCan setelt sign size size? sup swap! unitVector zero?

**DistributedMultivariatePolynomial{DMP}**: **PolynomialCategory** *with* 0 1 \* \*\* + - / < = D associates?

characteristic charthRoot coefficient coefficients coerce  
conditionP const content convert degree differentiate  
discriminant eval exquo factor factorPolynomial  
factorSquareFreePolynomial gcd gcdPolynomial ground  
ground? isExpt isPlus isTimes lcm leadingCoefficient  
leadingMonomial mainVariable map mapExponents max  
min minimumDegree monicDivide monomial monomial?  
monomials multivariate numberOfMonomials one? prime?  
primitiveMonomials primitivePart recip reducedSystem  
reductum reorder resultant retract retractIfCan  
solveLinearPolynomialEquation squareFree squareFreePart  
squareFreePolynomial totalDegree unit? unitCanonical  
unitNormal univariate variables zero?

DrawOption{DROPT}: SetCategory *with* = adaptive clip  
coerce colorFunction coordinate coordinates curveColor  
option option? pointColor range ranges space style title  
toScale tubePoints tubeRadius unit var1Steps var2Steps

ElementaryFunctionsUnivariateLaurentSeries{EFULS}:  
PartialTranscendentalFunctions *with* \*\* acos acosIfCan  
acosh acoshIfCan acot acotIfCan acoth acothIfCan acsc  
acscIfCan acsch acschIfCan asec asecIfCan asech  
asechIfCan asin asinIfCan asinh asinhIfCan atan atanIfCan  
atanh atanhIfCan cos cosIfCan cosh coshIfCan cot  
cotIfCan coth cothIfCan csc cscIfCan csch cschIfCan exp  
expIfCan log logIfCan nthRootIfCan sec secIfCan sech  
sechIfCan sin sinIfCan sinh sinhIfCan tan tanIfCan tanh  
tanhIfCan

ElementaryFunctionsUnivariatePuisseuxSeries{EFUPXS}:  
PartialTranscendentalFunctions *with* \*\* acos acosIfCan  
acosh acoshIfCan acot acotIfCan acoth acothIfCan acsc  
acscIfCan acsch acschIfCan asec asecIfCan asech  
asechIfCan asin asinIfCan asinh asinhIfCan atan atanIfCan  
atanh atanhIfCan cos cosIfCan cosh coshIfCan cot  
cotIfCan coth cothIfCan csc cscIfCan csch cschIfCan exp  
expIfCan log logIfCan nthRootIfCan sec secIfCan sech  
sechIfCan sin sinIfCan sinh sinhIfCan tan tanIfCan tanh  
tanhIfCan

EqTable{EQTBL}: TableAggregate *with* # = any? bag  
coerce construct copy count dictionary elt empty empty?  
entries entry? eq? every? extract! fill! find first index?  
indices insert! inspect key? keys less? map map! maxIndex  
member? members minIndex more? parts qelt qsetelt!  
reduce remove remove! removeDuplicates search select  
select! setelt size? swap! table

Equation{EQ}: CoercibleTo InnerEvalable Object  
SetCategory *with* \* \*\* + - = coerce equation eval lhs map  
rhs

EuclideanModularRing{EMR}: EuclideanDomain *with* 0 1  
\* \*\* + - = associates? characteristic coerce divide  
euclideanSize exQuo expressIdealMember exquo  
extendedEuclidean gcd inv lcm modulus multiEuclidean  
one? principalIdeal quo recip reduce rem sizeLess? unit?  
unitCanonical unitNormal zero?

Exit{EXIT}: SetCategory *with* = coerce

Expression{EXPR}: AlgebraicallyClosedFunctionSpace  
CombinatorialOpsCategory FunctionSpace  
LiouvillianFunctionCategory RetractableTo  
SpecialFunctionCategory TranscendentalFunctionCategory  
*with* 0 1 \* \*\* + - / < = Beta Ci D Ei Gamma Si abs acos  
acosh acot acoth acsc acsch airyAi airyBi applyQuote asec  
asech asin asinh associates? atan atanh belong? bessell  
besselJ besselK besselY binomial box characteristic  
charthRoot coerce commutator conjugate convert cos cosh  
cot coth csc csch definingPolynomial denom denominator  
differentiate digamma dilog distribute divide elt erf  
euclideanSize eval exp expressIdealMember exquo  
extendedEuclidean factor factorial factorials freeOf? gcd  
ground ground? height integral inv is? isExpt isMult  
isPlus isPower isTimes kernel kernels lcm li log mainKernel  
map max min minPoly multiEuclidean nthRoot numer  
numerator one? operator operators paren patternMatch  
permutation pi polygamma prime? principalIdeal product  
quo recip reduce reducedSystem rem retract retractIfCan  
rootOf rootsOf sec sech sin sinh sizeLess? sqrt squareFree  
squareFreePart subst summation tan tanh tower unit?  
unitCanonical unitNormal univariate variables zero?  
zeroOf zerosOf

ExtAlgBasis{EAB}: OrderedSet *with* < = Nul coerce degree  
exponents max min

Factored{FR}: Algebra DifferentialExtension Eltable  
Evalable FullyEvalableOver FullyRetractableTo GcdDomain  
InnerEvalable IntegralDomain RealConstant  
UniqueFactorizationDomain *with* 0 1 \* \*\* + - = D associates?  
characteristic coerce convert differentiate elt eval expand  
exponent exquo factor factorList factors flagFactor gcd  
irreducibleFactor lcm makeFR map nilFactor nthExponent  
nthFactor nthFlag numberOfFactors one? prime?  
primeFactor rational rational? rationalIfCan recip retract  
retractIfCan sqrFactor squareFree squareFreePart unit  
unit? unitCanonical unitNormal unitNormalize zero?

FileName{FNAME}: FileNameCategory *with* = coerce  
directory exists? extension filename name new readable?  
writable?

File{FILE}: FileCategory *with* = close! coerce iomode name  
open read! readIfCan! reopen! write!

FiniteDivisor{FDIV}: AbelianGroup *with* 0 \* + - = algsplit  
coerce divisor finiteBasis generator ideal lSpaceBasis  
mkBasicDiv principal? reduce zero?

FiniteFieldCyclicGroupExtensionByPolynomial{FFCGP}:  
FiniteAlgebraicExtensionField *with* 0 1 \* \*\* + - / = Frobenius  
algebraic? associates? basis characteristic charthRoot  
coerce conditionP coordinates createNormalElement  
createPrimitiveElement definingPolynomial degree  
dimension discreteLog divide euclideanSize  
expressIdealMember exquo extendedEuclidean  
extensionDegree factor factorsOfCyclicGroupSize gcd

generator getZechTable inGroundField? index init inv lcm  
lookup minimalPolynomial multiEuclidean nextItem norm  
normal? normalElement one? order prime?  
primeFrobenius primitive? primitiveElement principalIdeal  
quo random recip rem representationType represents  
retract retractIfCan size sizeLess? squareFree  
squareFreePart tableForDiscreteLogarithm trace  
transcendenceDegree transcendent? unit? unitCanonical  
unitNormal zero?

FiniteFieldCyclicGroupExtension{FFCGX}:  
FiniteAlgebraicExtensionField *with* 0 1 \* \*\* + - / = Frobenius  
algebraic? associates? basis characteristic charthRoot  
coerce conditionP coordinates createNormalElement  
createPrimitiveElement definingPolynomial degree  
dimension discreteLog divide euclideanSize  
expressIdealMember exquo extendedEuclidean  
extensionDegree factor factorsOfCyclicGroupSize gcd  
generator getZechTable inGroundField? index init inv lcm  
lookup minimalPolynomial multiEuclidean nextItem norm  
normal? normalElement one? order prime?  
primeFrobenius primitive? primitiveElement principalIdeal  
quo random recip rem representationType represents  
retract retractIfCan size sizeLess? squareFree  
squareFreePart tableForDiscreteLogarithm trace  
transcendenceDegree transcendent? unit? unitCanonical  
unitNormal zero?

FiniteFieldCyclicGroup{FFCG}:  
FiniteAlgebraicExtensionField *with* 0 1 \* \*\* + - / = Frobenius  
algebraic? associates? basis characteristic charthRoot  
coerce conditionP coordinates createNormalElement  
createPrimitiveElement definingPolynomial degree  
dimension discreteLog divide euclideanSize  
expressIdealMember exquo extendedEuclidean  
extensionDegree factor factorsOfCyclicGroupSize gcd  
generator getZechTable inGroundField? index init inv lcm  
lookup minimalPolynomial multiEuclidean nextItem norm  
normal? normalElement one? order prime?  
primeFrobenius primitive? primitiveElement principalIdeal  
quo random recip rem representationType represents  
retract retractIfCan size sizeLess? squareFree  
squareFreePart tableForDiscreteLogarithm trace  
transcendenceDegree transcendent? unit? unitCanonical  
unitNormal zero?

FiniteFieldExtensionByPolynomial{FFP}:  
FiniteAlgebraicExtensionField *with* 0 1 \* \*\* + - / =  
Frobenius algebraic? associates? basis characteristic  
charthRoot coerce conditionP coordinates  
createNormalElement createPrimitiveElement  
definingPolynomial degree dimension discreteLog divide  
euclideanSize expressIdealMember exquo  
extendedEuclidean extensionDegree factor  
factorsOfCyclicGroupSize gcd generator inGroundField?  
index init inv lcm lookup minimalPolynomial  
multiEuclidean nextItem norm normal? normalElement  
one? order prime? primeFrobenius primitive?

primitiveElement principalIdeal quo random recip rem  
representationType represents retract retractIfCan size  
sizeLess? squareFree squareFreePart  
tableForDiscreteLogarithm trace transcendenceDegree  
transcendent? unit? unitCanonical unitNormal zero?

FiniteFieldExtension{FFX}: FiniteAlgebraicExtensionField  
*with* 0 1 \* \*\* + - / = Frobenius algebraic? associates? basis  
characteristic charthRoot coerce conditionP coordinates  
createNormalElement createPrimitiveElement  
definingPolynomial degree dimension discreteLog divide  
euclideanSize expressIdealMember exquo  
extendedEuclidean extensionDegree factor  
factorsOfCyclicGroupSize gcd generator inGroundField?  
index init inv lcm lookup minimalPolynomial  
multiEuclidean nextItem norm normal? normalElement  
one? order prime? primeFrobenius primitive?  
primitiveElement principalIdeal quo random recip rem  
representationType represents retract retractIfCan size  
sizeLess? squareFree squareFreePart  
tableForDiscreteLogarithm trace transcendenceDegree  
transcendent? unit? unitCanonical unitNormal zero?

FiniteFieldNormalBasisExtensionByPolynomial{FFNBP}:  
FiniteAlgebraicExtensionField *with* 0 1 \* \*\* + - / = Frobenius  
algebraic? associates? basis characteristic charthRoot  
coerce conditionP coordinates createNormalElement  
createPrimitiveElement definingPolynomial degree  
dimension discreteLog divide euclideanSize  
expressIdealMember exquo extendedEuclidean  
extensionDegree factor factorsOfCyclicGroupSize gcd  
generator getMultiplicationMatrix getMultiplicationTable  
inGroundField? index init inv lcm lookup  
minimalPolynomial multiEuclidean nextItem norm  
normal? normalElement one? order prime?  
primeFrobenius primitive? primitiveElement principalIdeal  
quo random recip rem representationType represents  
retract retractIfCan size sizeLess? sizeMultiplication  
squareFree squareFreePart tableForDiscreteLogarithm  
trace transcendenceDegree transcendent? unit?  
unitCanonical unitNormal zero?

FiniteFieldNormalBasisExtension{FFNBX}:  
FiniteAlgebraicExtensionField *with* 0 1 \* \*\* + - / = Frobenius  
algebraic? associates? basis characteristic charthRoot  
coerce conditionP coordinates createNormalElement  
createPrimitiveElement definingPolynomial degree  
dimension discreteLog divide euclideanSize  
expressIdealMember exquo extendedEuclidean  
extensionDegree factor factorsOfCyclicGroupSize gcd  
generator getMultiplicationMatrix getMultiplicationTable  
inGroundField? index init inv lcm lookup  
minimalPolynomial multiEuclidean nextItem norm  
normal? normalElement one? order prime?  
primeFrobenius primitive? primitiveElement principalIdeal  
quo random recip rem representationType represents  
retract retractIfCan size sizeLess? sizeMultiplication  
squareFree squareFreePart tableForDiscreteLogarithm

trace transcendenceDegree transcendent? unit?  
unitCanonical unitNormal zero?

**FiniteFieldNormalBasis{FFNB}:**

**FiniteAlgebraicExtensionField** *with* 0 1 \* \*\* + - / = Frobenius algebraic? associates? basis characteristic charthRoot coerce conditionP coordinates createNormalElement createPrimitiveElement definingPolynomial degree dimension discreteLog divide euclideanSize expressIdealMember exquo extendedEuclidean extensionDegree factor factorsOfCyclicGroupSize gcd generator getMultiplicationMatrix getMultiplicationTable inGroundField? index init inv lcm lookup minimalPolynomial multiEuclidean nextItem norm normal? normalElement one? order prime? primeFrobenius primitive? primitiveElement principalIdeal quo random recip rem representationType represents retract retractIfCan size sizeLess? sizeMultiplication squareFree squareFreePart tableForDiscreteLogarithm trace transcendenceDegree transcendent? unit? unitCanonical unitNormal zero?

**FiniteField{FF}:** **FiniteAlgebraicExtensionField** *with* 0 1 \* \*\* + - / = Frobenius algebraic? associates? basis characteristic charthRoot coerce conditionP coordinates createNormalElement createPrimitiveElement definingPolynomial degree dimension discreteLog divide euclideanSize expressIdealMember exquo extendedEuclidean extensionDegree factor factorsOfCyclicGroupSize gcd generator inGroundField? index init inv lcm lookup minimalPolynomial multiEuclidean nextItem norm normal? normalElement one? order prime? primeFrobenius primitive? primitiveElement principalIdeal quo random recip rem representationType represents retract retractIfCan size sizeLess? squareFree squareFreePart tableForDiscreteLogarithm trace transcendenceDegree transcendent? unit? unitCanonical unitNormal zero?

**FlexibleArray{FARRAY}:** **ExtensibleLinearAggregate OneDimensionalArrayAggregate** *with* # <= any? coerce concat concat! construct convert copy copyInto! count delete delete! elt empty empty? entries entry? eq? every? fill! find first flexibleArray index? indices insert insert! less? map map! max maxIndex member? members merge merge! min minIndex more? new parts physicalLength physicalLength! position qelt qsetelt! reduce remove remove! removeDuplicates removeDuplicates! reverse reverse! select select! setelt shrinkable size? sort sort! sorted? swap!

**Float{FLOAT}:** **CoercibleTo ConvertibleTo DifferentialRing FloatingPointSystem TranscendentalFunctionCategory** *with* 0 1 \* \*\* + - / <= D abs acos acosh acot acoth acsc acsch asec asech asin asinh associates? atan atanh base bits ceiling characteristic coerce convert cos cosh cot coth csc csch decreasePrecision differentiate digits divide euclideanSize exp exp1 exponent expressIdealMember

exquo extendedEuclidean factor float floor fractionPart gcd increasePrecision inv lcm log log10 log2 mantissa max min multiEuclidean negative? norm normalize nthRoot one? order outputFixed outputFloating outputGeneral outputSpacing patternMatch pi positive? precision prime? principalIdeal quo rationalApproximation recip reerror rem retract retractIfCan round sec sech shift sign sin sinh sizeLess? sqrt squareFree squareFreePart tan tanh truncate unit? unitCanonical unitNormal wholePart zero?

**FractionalIdeal{FRIDEAL}:** **Group** *with* 1 \* \*\* / = basis coerce commutator conjugate denom ideal inv minimize norm numer one? randomLC recip

**Fraction{FRAC}:** **QuotientFieldCategory** *with* 0 1 \* \*\* + - / <= D abs associates? ceiling characteristic charthRoot coerce conditionP convert denom denominator differentiate divide elt euclideanSize eval expressIdealMember exquo extendedEuclidean factor factorPolynomial factorSquareFreePolynomial floor fractionPart gcd gcdPolynomial init inv lcm map max min multiEuclidean negative? nextItem numer numerator one? patternMatch positive? prime? principalIdeal quo random recip reducedSystem rem retract retractIfCan sign sizeLess? solveLinearPolynomialEquation squareFree squareFreePart squareFreePolynomial unit? unitCanonical unitNormal wholePart zero?

**FramedModule{FRMOD}:** **Monoid** *with* 1 \* \*\* = basis coerce module norm one? recip

**FreeAbelianGroup{FAGROUP}:** **AbelianGroup**  
**FreeAbelianMonoidCategory Module OrderedSet** *with* 0 \* + - <= coefficient coerce highCommonTerms mapCoef mapGen max min nthCoef nthFactor retract retractIfCan size terms zero?

**FreeAbelianMonoid{FAMONOID}:**  
**FreeAbelianMonoidCategory** *with* 0 \* + - = coefficient coerce highCommonTerms mapCoef mapGen nthCoef nthFactor retract retractIfCan size terms zero?

**FreeGroup{FGROUP}:** **Group RetractableTo** *with* 1 \* \*\* / = coerce commutator conjugate factors inv mapExpon mapGen nthExpon nthFactor one? recip retract retractIfCan size

**FreeModule{FM}:** **BiModule IndexedDirectProductCategory Module** *with* 0 \* + - = coerce leadingCoefficient leadingSupport map monomial reductum zero?

**FreeMonoid{FMONOID}:** **Monoid OrderedSet RetractableTo** *with* 1 \* \*\* <= coerce divide factors helf hcrf lquo mapExpon mapGen max min nthExpon nthFactor one? overlap recip retract retractIfCan rquo size

**FreeNilpotentLie{FNLA}:** **NonAssociativeAlgebra** *with* 0 \* \*\* + - = antiCommutator associator coerce commutator deepExpand dimension generator leftPower rightPower shallowExpand zero?

FunctionCalled{FUNCTION}: SetCategory *with* = coerce  
name

GeneralDistributedMultivariatePolynomial{GDMP}:  
PolynomialCategory *with* 0 1 \* \*\* + - / < = D associates?  
characteristic charthRoot coefficient coefficients coerce  
conditionP const content convert degree differentiate  
discriminant eval exquo factor factorPolynomial  
factorSquareFreePolynomial gcd gcdPolynomial ground  
ground? isExpt isPlus isTimes lcm leadingCoefficient  
leadingMonomial mainVariable map mapExponents max  
min minimumDegree monicDivide monomial monomial?  
monomials multivariate numberOfMonomials one? prime?  
primitiveMonomials primitivePart recip reducedSystem  
reductum reorder resultant retract retractIfCan  
solveLinearPolynomialEquation squareFree squareFreePart  
squareFreePolynomial totalDegree unit? unitCanonical  
unitNormal univariate variables zero?

GeneralSparseTable{GSTBL}: TableAggregate *with* # =  
any? bag coerce construct copy count dictionary elt empty  
empty? entries entry? eq? every? extract! fill! find first  
index? indices insert! inspect key? keys less? map map!  
maxIndex member? members minIndex more? parts qelt  
qsetelt! reduce remove remove! removeDuplicates search  
select select! setelt size? swap! table

GenericNonAssociativeAlgebra{GCNAALG}:  
FramedNonAssociativeAlgebra LeftModule *with* 0 \* \*\* + - =  
JacobiIdentity? JordanAlgebra? alternative?  
antiAssociative? antiCommutative? antiCommutator  
apply associative? associator associatorDependence basis  
coerce commutative? commutator  
conditionsForIdempotents convert coordinates elt flexible?  
generic genericLeftDiscriminant  
genericLeftMinimalPolynomial genericLeftNorm  
genericLeftTrace genericLeftTraceForm  
genericRightDiscriminant genericRightMinimalPolynomial  
genericRightNorm genericRightTrace  
genericRightTraceForm jordanAdmissible? leftAlternative?  
leftCharacteristicPolynomial leftDiscriminant  
leftMinimalPolynomial leftNorm leftPower  
leftRankPolynomial leftRecip leftRegularRepresentation  
leftTrace leftTraceMatrix leftUnit leftUnits lieAdmissible?  
lieAlgebra? noncommutativeJordanAlgebra? plenaryPower  
powerAssociative? rank recip represents rightAlternative?  
rightCharacteristicPolynomial rightDiscriminant  
rightMinimalPolynomial rightNorm rightPower  
rightRankPolynomial rightRecip  
rightRegularRepresentation rightTrace rightTraceMatrix  
rightUnit rightUnits someBasis structuralConstants unit  
zero?

GraphImage{GRIMAGE}: SetCategory *with* = appendPoint  
coerce component graphImage key makeGraphImage point  
pointLists putColorInfo ranges units

HashTable{HASHTBL}: TableAggregate *with* # = any? bag  
coerce construct copy count dictionary elt empty empty?

entries entry? eq? every? extract! fill! find first index?  
indices insert! inspect key? keys less? map map! maxIndex  
member? members minIndex more? parts qelt qsetelt!  
reduce remove remove! removeDuplicates search select  
select! setelt size? swap! table

Heap{HEAP}: PriorityQueueAggregate *with* # = any? bag  
coerce copy count empty empty? eq? every? extract! heap  
insert! inspect less? map map! max member? members  
merge merge! more? parts size?

HexadecimalExpansion{HEXADEC}:  
QuotientFieldCategory *with* 0 1 \* \*\* + - / < = D abs  
associates? ceiling characteristic coerce convert denom  
denominator differentiate divide euclideanSize  
expressIdealMember exquo extendedEuclidean factor floor  
fractionPart gcd hex init inv lcm map max min  
multiEuclidean negative? nextItem numer numerator one?  
patternMatch positive? prime? principalIdeal quo random  
recip reducedSystem rem retract retractIfCan sign  
sizeLess? squareFree squareFreePart unit? unitCanonical  
unitNormal wholePart zero?

IndexCard{ICARD}: OrderedSet *with* < = coerce display elt  
fullDisplay max min

IndexedBits{IBITS}: BitAggregate *with* # < = And Not Or  
~ and any? coerce concat construct convert copy copyInto!  
count delete elt empty empty? entries entry? eq? every?  
fill! find first index? indices insert less? map map! max  
maxIndex member? members merge min minIndex more?  
nand new nor not or parts position qelt qsetelt! reduce  
remove removeDuplicates reverse reverse! select setelt size?  
sort sort! sorted? swap! xor

IndexedDirectProductAbelianGroup{IDPAG}:  
AbelianGroup IndexedDirectProductCategory *with* 0 \* + - =  
coerce leadingCoefficient leadingSupport map monomial  
reductum zero?

IndexedDirectProductAbelianMonoid{IDPAM}:  
AbelianMonoid IndexedDirectProductCategory *with* 0 \* + =  
coerce leadingCoefficient leadingSupport map monomial  
reductum zero?

IndexedDirectProductObject{IDPO}:  
IndexedDirectProductCategory *with* = coerce  
leadingCoefficient leadingSupport map monomial reductum  
IndexedDirectProductOrderedAbelianMonoidSup  
{IDPOAMS}: IndexedDirectProductCategory  
OrderedAbelianMonoidSup *with* 0 \* + - < = coerce  
leadingCoefficient leadingSupport map max min monomial  
reductum sup zero?

IndexedDirectProductOrderedAbelianMonoid{IDPOAM}:  
IndexedDirectProductCategory OrderedAbelianMonoid *with* 0  
\* + < = coerce leadingCoefficient leadingSupport map max  
min monomial reductum zero?

IndexedExponents{INDE}: IndexedDirectProductCategory

**OrderedAbelianMonoidSup** *with* 0 \* + - < = coerce  
leadingCoefficient leadingSupport map max min monomial  
reductum sup zero?

**IndexedFlexibleArray**{IFARRAY}:

**ExtensibleLinearAggregate OneDimensionalArrayAggregate**  
*with* # < = any? coerce concat concat! construct convert  
copy copyInto! count delete delete! elt empty empty?  
entries entry? eq? every? fill! find first flexibleArray  
index? indices insert insert! less? map map! max  
maxIndex member? members merge merge! min minIndex  
more? new parts physicalLength physicalLength! position  
qelt qsetelt! reduce remove remove! removeDuplicates  
removeDuplicates! reverse reverse! select select! setelt  
shrinkable size? sort sort! sorted? swap!

**IndexedList**{ILIST}: **ListAggregate** *with* # < = any? child?  
children coerce concat concat! construct convert copy  
copyInto! count cycleEntry cycleLength cycleSplit!  
cycleTail cyclic? delete delete! distance elt empty empty?  
entries entry? eq? every? explicitlyFinite? fill! find first  
index? indices insert insert! last leaf? less? list map map!  
max maxIndex member? members merge merge! min  
minIndex more? new node? nodes parts position  
possiblyInfinite? qelt qsetelt! reduce remove remove!  
removeDuplicates removeDuplicates! rest reverse reverse!  
second select select! setchildren! setelt setfirst! setlast!  
setrest! setvalue! size? sort sort! sorted? split! swap! tail  
third value

**IndexedMatrix**{IMATRIX}: **MatrixCategory** *with* # \* \*\* + -  
/ = antisymmetric? any? coerce column copy count  
determinant diagonal? diagonalMatrix elt empty empty?  
eq? every? exquo fill! horizConcat inverse less? listOfLists  
map map! matrix maxColIndex maxRowIndex member?  
members minColIndex minRowIndex minordet more?  
ncols new nrows nullSpace nullity parts qelt qsetelt! rank  
row rowEchelon scalarMatrix setColumn! setRow! setelt  
setsubMatrix! size? square? squareTop subMatrix  
swapColumns! swapRows! symmetric? transpose  
vertConcat zero

**IndexedOneDimensionalArray**{IARRAY1}:

**OneDimensionalArrayAggregate** *with* # < = any? coerce  
concat construct convert copy copyInto! count delete elt  
empty empty? entries entry? eq? every? fill! find first  
index? indices insert less? map map! max maxIndex  
member? members merge min minIndex more? new parts  
position qelt qsetelt! reduce remove removeDuplicates  
reverse reverse! select setelt size? sort sort! sorted? swap!

**IndexedString**{ISTRING}: **StringAggregate** *with* # < = any?  
coerce concat construct copy copyInto! count delete elt  
empty empty? entries entry? eq? every? fill! find first hash  
index? indices insert leftTrim less? lowerCase lowerCase!  
map map! match? max maxIndex member? members  
merge min minIndex more? new parts position prefix? qelt  
qsetelt! reduce remove removeDuplicates replace reverse  
reverse! rightTrim select setelt size? sort sort! sorted?

split substring? suffix? swap! trim upperCase upperCase!

**IndexedTwoDimensionalArray**{IARRAY2}:

**TwoDimensionalArrayCategory** *with* # = any? coerce column  
copy count elt empty empty? eq? every? fill! less? map  
map! maxColIndex maxRowIndex member? members  
minColIndex minRowIndex more? ncols new nrows parts  
qelt qsetelt! row setColumn! setRow! setelt size?

**IndexedVector**{IVECTOR}: **VectorCategory** *with* # \* + - <  
= any? coerce concat construct convert copy copyInto!  
count delete dot elt empty empty? entries entry? eq?  
every? fill! find first index? indices insert less? map map!  
max maxIndex member? members merge min minIndex  
more? new parts position qelt qsetelt! reduce remove  
removeDuplicates reverse reverse! select setelt size? sort  
sort! sorted? swap! zero

**InfiniteTuple**{ITUPLE}: **CoercibleTo** *with* coerce construct  
filterUntil filterWhile generate map select

**InnerFiniteField**{IFF}: **FiniteAlgebraicExtensionField** *with* 0  
1 \* \*\* + - / = Frobenius algebraic? associates? basis  
characteristic charthRoot coerce conditionP coordinates  
createNormalElement createPrimitiveElement  
definingPolynomial degree dimension discreteLog divide  
euclideanSize expressIdealMember exquo  
extendedEuclidean extensionDegree factor  
factorsOfCyclicGroupSize gcd generator inGroundField?  
index init inv lcm lookup minimalPolynomial  
multiEuclidean nextItem norm normal? normalElement  
one? order prime? primeFrobenius primitive?  
primitiveElement principalIdeal quo random recip rem  
representationType represents retract retractIfCan size  
sizeLess? squareFree squareFreePart  
tableForDiscreteLogarithm trace transcendenceDegree  
transcendent? unit? unitCanonical unitNormal zero?

**InnerFreeAbelianMonoid**{IFAMON}:

**FreeAbelianMonoidCategory** *with* 0 \* + - = coefficient coerce  
highCommonTerms mapCoef mapGen nthCoef nthFactor  
retract retractIfCan size terms zero?

**InnerIndexedTwoDimensionalArray**{IIARRAY2}:

**TwoDimensionalArrayCategory** *with* # = any? coerce column  
copy count elt empty empty? eq? every? fill! less? map  
map! maxColIndex maxRowIndex member? members  
minColIndex minRowIndex more? ncols new nrows parts  
qelt qsetelt! row setColumn! setRow! setelt size?

**InnerPAdicInteger**{IPADIC}: **PAdicIntegerCategory** *with* 0 1

\* \*\* + - = approximate associates? characteristic coerce  
complete digits divide euclideanSize expressIdealMember  
exquo extend extendedEuclidean gcd lcm moduloP  
modulus multiEuclidean one? order principalIdeal quo  
quotientByP recip rem sizeLess? sqrt unit? unitCanonical  
unitNormal zero?

**InnerPrimeField**{IPF}: **ConvertibleTo**

**FiniteAlgebraicExtensionField FiniteFieldCategory** *with* 0 1 \*

**\*\* + - / = Frobenius algebraic?** associates? basis  
characteristic charthRoot coerce conditionP convert  
coordinates createNormalElement createPrimitiveElement  
definingPolynomial degree dimension discreteLog divide  
euclideanSize expressIdealMember exquo  
extendedEuclidean extensionDegree factor  
factorsOfCyclicGroupSize gcd generator inGroundField?  
index init inv lcm lookup minimalPolynomial  
multiEuclidean nextItem norm normal? normalElement  
one? order prime? primeFrobenius primitive?  
primitiveElement principalIdeal quo random recip rem  
representationType represents retract retractIfCan size  
sizeLess? squareFree squareFreePart  
tableForDiscreteLogarithm trace transcendenceDegree  
transcendent? unit? unitCanonical unitNormal zero?

**InnerTaylorSeries{ITAYLOR}: IntegralDomain Ring** *with 0 1*  
**\* \*\* + - =** associates? characteristic coefficients coerce  
exquo one? order pole? recip series unit? unitCanonical  
unitNormal zero?

**InputForm{INFORM}: ConvertibleTo SExpressionCategory**  
*with 0 1 # \* \*\* + / =* atom? binary car cdr coerce compile  
convert declare destruct elt eq expr flatten float float?  
function integer integer? interpret lambda list? null? pair?  
string string? symbol symbol? unequal unparse

**IntegerMod{ZMOD}: CommutativeRing ConvertibleTo**  
**Finite StepThrough** *with 0 1 \* \*\* + - =* characteristic coerce  
convert index init lookup nextItem one? random recip size  
zero?

**Integer{INT}: ConvertibleTo IntegerNumberSystem** *with 0 1*  
**\* \*\* + - < =** D abs addmod associates? base binomial bit?  
characteristic coerce convert copy dec differentiate divide  
euclideanSize even? expressIdealMember exquo  
extendedEuclidean factor factorial gcd hash inc init  
invmod lcm length mask max min mulmod multiEuclidean  
negative? nextItem odd? one? patternMatch permutation  
positive? positiveRemainder powmod prime?  
principalIdeal quo random rational rational? rationalIfCan  
recip reducedSystem rem retract retractIfCan shift sign  
sizeLess? squareFree squareFreePart submod  
symmetricRemainder unit? unitCanonical unitNormal  
zero?

**IntegrationResult{IR}: Module RetractableTo** *with 0 \* + -*  
**= D** coerce differentiate elem? integral logpart mkAnswer  
notelem ratpart retract retractIfCan zero?

**Kernel{KERNEL}: CachableSet ConvertibleTo Patternable**  
*with < =* argument coerce convert height is? kernel max  
min name operator position setPosition symbolIfCan

**KeyedAccessFile{KAFILe}: FileCategory TableAggregate**  
*with # = any?* bag close! coerce construct copy count  
dictionary elt empty empty? entries entry? eq? every?  
extract! fill! find first index? indices insert! inspect iomode  
key? keys less? map map! maxIndex member? members  
minIndex more? name open pack! parts qelt qsetelt! read!

reduce remove remove! removeDuplicates reopen! search  
select select! setelt size? swap! table write!

**LaurentPolynomial{LAUPOL}: CharacteristicNonZero**  
**CharacteristicZero ConvertibleTo DifferentialExtension**  
**EuclideanDomain FullyRetractableTo IntegralDomain**  
**RetractableTo** *with 0 1 \* \*\* + - =* D associates?  
characteristic charthRoot coefficient coerce convert degree  
differentiate divide euclideanSize expressIdealMember  
exquo extendedEuclidean gcd lcm leadingCoefficient  
monomial monomial? multiEuclidean one? order  
principalIdeal quo recip reductum rem retract retractIfCan  
separate sizeLess? trailingCoefficient unit? unitCanonical  
unitNormal zero?

**Library{LIB}: TableAggregate** *with # = any?* bag coerce  
construct copy count dictionary elt empty empty? entries  
entry? eq? every? extract! fill! find first index? indices  
insert! inspect key? keys less? library map map! maxIndex  
member? members minIndex more? pack! parts qelt  
qsetelt! reduce remove remove! removeDuplicates search  
select select! setelt size? swap! table

**LieSquareMatrix{LSQM}: CoercibleTo**  
**FramedNonAssociativeAlgebra SquareMatrixCategory** *with 0*  
**1 # \* \*\* + - / =** D JacobiIdentity? JordanAlgebra?  
alternative? antiAssociative? antiCommutative?  
antiCommutator antisymmetric? any? apply associative?  
associator associatorDependence basis characteristic coerce  
column commutative? commutator  
conditionsForIdempotents convert coordinates copy count  
determinant diagonal diagonal? diagonalMatrix  
diagonalProduct differentiate elt empty empty? eq? every?  
exquo flexible? inverse jordanAdmissible? leftAlternative?  
leftCharacteristicPolynomial leftDiscriminant  
leftMinimalPolynomial leftNorm leftPower  
leftRankPolynomial leftRecip leftRegularRepresentation  
leftTrace leftTraceMatrix leftUnit leftUnits less?  
lieAdmissible? lieAlgebra? listOfLists map map! matrix  
maxColIndex maxRowIndex member? members  
minColIndex minRowIndex minordet more? ncols  
noncommutativeJordanAlgebra? nrows nullSpace nullity  
one? parts plenaryPower powerAssociative? qelt rank recip  
reducedSystem represents retract retractIfCan  
rightAlternative? rightCharacteristicPolynomial  
rightDiscriminant rightMinimalPolynomial rightNorm  
rightPower rightRankPolynomial rightRecip  
rightRegularRepresentation rightTrace rightTraceMatrix  
rightUnit rightUnits row rowEchelon scalarMatrix size?  
someBasis square? structuralConstants symmetric? trace  
unit zero?

**LinearOrdinaryDifferentialOperator{LODO}: MonogenicLinearOperator** *with 0 1 \* \*\* + - =* D  
characteristic coefficient coerce degree elt leadingCoefficient  
leftDivide leftExactQuotient leftGcd leftLcm leftQuotient  
leftRemainder minimumDegree monomial one? recip  
reductum rightDivide rightExactQuotient rightGcd



rightLcm rightQuotient rightRemainder zero?

ListMonoidOps{LMOPS}: RetractableTo SetCategory *with*  
= coerce leftMult listOfMonoms makeMulti makeTerm  
makeUnit mapExpon mapGen nthExpon nthFactor  
outputForm plus retract retractIfCan reverse reverse!  
rightMult size

ListMultiDictionary{LMDICT}: MultiDictionary *with* # =  
any? bag coerce construct convert copy count dictionary  
duplicates duplicates? empty empty? eq? every? extract!  
find insert! inspect less? map map! member? members  
more? parts reduce remove remove! removeDuplicates  
removeDuplicates! select select! size? substitute

List{LIST}: ListAggregate *with* # < = any? append child?  
children coerce concat concat! cons construct convert copy  
copyInto! count cycleEntry cycleLength cycleSplit!  
cycleTail cyclic? delete delete! distance elt empty empty?  
entries entry? eq? every? explicitlyFinite? fill! find first  
index? indices insert insert! last leaf? less? list map map!  
max maxIndex member? members merge merge! min  
minIndex more? new nil node? nodes null parts position  
possiblyInfinite? qelt qsetelt! reduce remove remove!  
removeDuplicates removeDuplicates! rest reverse reverse!  
second select select! setDifference setIntersection setUnion  
setchildren! setelt setfirst! setlast! setrest! setvalue! size?  
sort sort! sorted? split! swap! tail third value

LocalAlgebra{LA}: Algebra OrderedRing *with* 0 1 \* \*\* + -  
/ < = abs characteristic coerce denom max min negative?  
numer one? positive? recip sign zero?

Localize{LO}: Module OrderedAbelianGroup *with* 0 \* + - /  
< = coerce denom max min numer zero?

MakeCacheableSet{MKCHSET}: CacheableSet CoercibleTo  
*with* < = coerce max min position setPosition

MakeOrdinaryDifferentialRing{MKODRING}: CoercibleTo  
DifferentialRing *with* 0 1 \* \*\* + - = D characteristic coerce  
differentiate one? recip zero?

Matrix{MATRIX}: MatrixCategory *with* # \* \*\* + - / =  
antisymmetric? any? coerce column copy count  
determinant diagonal? diagonalMatrix elt empty empty?  
eq? every? exquo fill! horizConcat inverse less? listOfLists  
map map! matrix maxColIndex maxRowIndex member?  
members minColIndex minRowIndex minordet more?  
ncols new nrows nullSpace nullity parts qelt qsetelt! rank  
row rowEchelon scalarMatrix setColumn! setRow! setelt  
setsubMatrix! size? square? squareTop subMatrix  
swapColumns! swapRows! symmetric? transpose  
vertConcat zero

ModMonic{MODMON}: Finite

UnivariatePolynomialCategory *with* 0 1 \* \*\* + - / < = An D  
UnVectorise Vectorise associates? characteristic  
charthRoot coefficient coefficients coerce composite  
computePowers conditionP content degree differentiate  
discriminant divide divideExponents elt euclideanSize eval

expressIdealMember exquo extendedEuclidean factor  
factorPolynomial factorSquareFreePolynomial gcd  
gcdPolynomial ground ground? index init integrate isExpt  
isPlus isTimes lcm leadingCoefficient leadingMonomial lift  
lookup mainVariable makeSUP map mapExponents max  
min minimumDegree modulus monicDivide monomial  
monomial? monomials multiEuclidean multiplyExponents  
multivariate nextItem numberOfMonomials one? order  
pow prime? primitiveMonomials primitivePart  
principalIdeal pseudoDivide pseudoQuotient  
pseudoRemainder quo random recip reduce reducedSystem  
reductum rem resultant retract retractIfCan separate  
setPoly size sizeLess? solveLinearPolynomialEquation  
squareFree squareFreePart squareFreePolynomial  
subResultantGcd totalDegree unit? unitCanonical  
unitNormal univariate unmakeSUP variables vectorise  
zero?

ModularField{MODFIELD}: Field *with* 0 1 \* \*\* + - / =  
associates? characteristic coerce divide euclideanSize  
exQuo expressIdealMember exquo extendedEuclidean  
factor gcd inv lcm modulus multiEuclidean one? prime?  
principalIdeal quo recip reduce rem sizeLess? squareFree  
squareFreePart unit? unitCanonical unitNormal zero?

ModularRing{MODRING}: Ring *with* 0 1 \* \*\* + - =  
characteristic coerce exQuo inv modulus one? recip reduce  
zero?

MoebiusTransform{MOEBIUS}: Group *with* 1 \* \*\* / =  
coerce commutator conjugate eval inv moebius one? recip  
scale shift

MonoidRing{MRING}: Algebra CharacteristicNonZero  
CharacteristicZero Finite RetractableTo Ring *with* 0 1 \* \*\* +  
- = characteristic charthRoot coefficient coefficients coerce  
index leadingCoefficient leadingMonomial lookup map  
monomial monomial? monomials numberOfMonomials  
one? random recip reductum retract retractIfCan size  
terms zero?

Multiset{MSET}: MultiSetAggregate *with* # < = any? bag  
brace coerce construct convert copy count dictionary  
difference duplicates empty empty? eq? every? extract!  
find insert! inspect intersect less? map map! member?  
members more? multiset parts reduce remove remove!  
removeDuplicates removeDuplicates! select select! size?  
subset? symmetricDifference union

MultivariatePolynomial{MPOLY}: PolynomialCategory  
*with* 0 1 \* \*\* + - / < = D associates? characteristic  
charthRoot coefficient coefficients coerce conditionP  
content convert degree differentiate discriminant eval  
exquo factor factorPolynomial factorSquareFreePolynomial  
gcd gcdPolynomial ground ground? isExpt isPlus isTimes  
lcm leadingCoefficient leadingMonomial mainVariable map  
mapExponents max min minimumDegree monicDivide  
monomial monomial? monomials multivariate  
numberOfMonomials one? prime? primitiveMonomials

primitivePart recip reducedSystem reductum resultant  
 retract retractIfCan solveLinearPolynomialEquation  
 squareFree squareFreePart squareFreePolynomial  
 totalDegree unit? unitCanonical unitNormal univariate  
 variables zero?

NewDirectProduct{NDP}: DirectProductCategory with 0 1  
 # \* \*\* + - / < = D abs any? characteristic coerce copy  
 count differentiate dimension directProduct dot elt empty  
 empty? entries entry? eq? every? fill! first index index?  
 indices less? lookup map map! max maxIndex member?  
 members min minIndex more? negative? one? parts  
 positive? qelt qsetelt! random recip reducedSystem retract  
 retractIfCan setelt sign size size? sup swap! unitVector  
 zero?

NewDistributedMultivariatePolynomial{NDMP}:  
 PolynomialCategory with 0 1 \* \*\* + - / < = D associates?  
 characteristic charthRoot coefficient coefficients coerce  
 conditionP const content convert degree differentiate  
 discriminant eval exquo factor factorPolynomial  
 factorSquareFreePolynomial gcd gcdPolynomial ground  
 ground? isExpt isPlus isTimes lcm leadingCoefficient  
 leadingMonomial mainVariable map mapExponents max  
 min minimumDegree monicDivide monomial monomial?  
 monomials multivariate numberOfMonomials one? prime?  
 primitiveMonomials primitivePart recip reducedSystem  
 reductum reorder resultant retract retractIfCan  
 solveLinearPolynomialEquation squareFree squareFreePart  
 squareFreePolynomial totalDegree unit? unitCanonical  
 unitNormal univariate variables zero?

None{NONE}: SetCategory with = coerce

NonNegativeInteger{NNI}: Monoid

OrderedAbelianMonoidSup with 0 1 \* \*\* + - < = coerce  
 divide exquo gcd max min one? quo recip rem sup zero?

Octonion{OCT}: FullyRetractableTo OctonionCategory with  
 0 1 \* \*\* + - < = abs characteristic charthRoot coerce  
 conjugate convert elt eval imageE imageI imageJ imageK image  
 imageJ imageK index inv lookup map max min norm octon  
 one? random rational rational? rationalIfCan real recip  
 retract retractIfCan size zero?

OneDimensionalArray{ARRAY1}:

OneDimensionalArrayAggregate with # < = any? coerce  
 concat construct convert copy copyInto! count delete elt  
 empty empty? entries entry? eq? every? fill! find first  
 index? indices insert less? map map! max maxIndex  
 member? members merge min minIndex more? new  
 oneDimensionalArray parts position qelt qsetelt! reduce  
 remove removeDuplicates reverse reverse! select setelt size?  
 sort sort! sorted? swap!

OnePointCompletion{ONECOMP}: AbelianGroup

FullyRetractableTo OrderedRing SetCategory with 0 1 \* \*\* +  
 - < = abs characteristic coerce finite? infinite? infinity max  
 min negative? one? positive? rational rational?  
 rationalIfCan recip retract retractIfCan sign zero?

Operator{OP}: Algebra CharacteristicNonZero  
 CharacteristicZero Eltable RetractableTo Ring with 0 1 \* \*\* +  
 - = characteristic charthRoot coerce elt evaluate one?  
 opeval recip retract retractIfCan zero?

OppositeMonogenicLinearOperator{OMLO}:

DifferentialRing MonogenicLinearOperator with 0 1 \* \*\* + -  
 = D characteristic coefficient coerce degree differentiate  
 leadingCoefficient minimumDegree monomial one? op po  
 recip reductum zero?

OrderedCompletion{ORDCOMP}: AbelianGroup

FullyRetractableTo OrderedRing SetCategory with 0 1 \* \*\* +  
 - < = abs characteristic coerce finite? infinite? max min  
 minusInfinity negative? one? plusInfinity positive?  
 rational rational? rationalIfCan recip retract retractIfCan  
 sign whatInfinity zero?

OrderedDirectProduct{ODP}: DirectProductCategory with  
 0 1 # \* \*\* + - / < = D abs any? characteristic coerce copy  
 count differentiate dimension directProduct dot elt empty  
 empty? entries entry? eq? every? fill! first index index?  
 indices less? lookup map map! max maxIndex member?  
 members min minIndex more? negative? one? parts  
 positive? qelt qsetelt! random recip reducedSystem retract  
 retractIfCan setelt sign size size? sup swap! unitVector  
 zero?

OrderedVariableList{OVAR}: ConvertibleTo OrderedFinite  
 with < = coerce convert index lookup max min random size  
 variable

OrderlyDifferentialPolynomial{ODPOL}:

DifferentialPolynomialCategory RetractableTo with 0 1 \* \*\* +  
 - / < = D associates? characteristic charthRoot coefficient  
 coefficients coerce conditionP content degree  
 differentialVariables differentiate discriminant eval exquo  
 factor factorPolynomial factorSquareFreePolynomial gcd  
 gcdPolynomial ground ground? initial isExpt isPlus  
 isTimes isobaric? lcm leader leadingCoefficient  
 leadingMonomial mainVariable makeVariable map  
 mapExponents max min minimumDegree monicDivide  
 monomial monomial? monomials multivariate  
 numberOfMonomials one? order prime?  
 primitiveMonomials primitivePart recip reducedSystem  
 reductum resultant retract retractIfCan separant  
 solveLinearPolynomialEquation squareFree squareFreePart  
 squareFreePolynomial totalDegree unit? unitCanonical  
 unitNormal univariate variables weight weights zero?

OrderlyDifferentialVariable{ODVAR}:

DifferentialVariableCategory with < = D coerce differentiate  
 makeVariable max min order retract retractIfCan variable  
 weight

OrdinaryDifferentialRing{ODR}: Algebra DifferentialRing

Field with 0 1 \* \*\* + - / = D associates? characteristic  
 coerce differentiate divide euclideanSize  
 expressIdealMember exquo extendedEuclidean factor gcd  
 inv lcm multiEuclidean one? prime? principalIdeal quo

recip rem sizeLess? squareFree squareFreePart unit?  
unitCanonical unitNormal zero?

OrdSetInts{OSI}: OrderedSet *with* < = coerce max min  
value

OutputForm{OUTFORM}: SetCategory *with* \* \*\* + - / <  
<= = > = D SEGMENT ^= and assign blankSeparate box  
brace bracket center coerce commaSeparate differentiate  
div dot elt empty exquo hconcat height hspace infix infix?  
int label left matrix message messagePrint not or  
outputForm over overbar paren pile postfix prefix presub  
presuper prime print prod quo quote rarrow rem right root  
rspace scripts semicolonSeparate slash string sub subHeight  
sum super superHeight supersub vconcat vspace width zag

PAdicInteger{PADIC}: PAdicIntegerCategory *with* 0 1 \* \*\*  
+ - = approximate associates? characteristic coerce  
complete digits divide euclideanSize expressIdealMember  
exquo extend extendedEuclidean gcd lcm moduloP  
modulus multiEuclidean one? order principalIdeal quo  
quotientByP recip rem sizeLess? sqrt unit? unitCanonical  
unitNormal zero?

PAdicRationalConstructor{PAdicRC}:

QuotientFieldCategory *with* 0 1 \* \*\* + - / < = D abs  
approximate associates? ceiling characteristic charthRoot  
coerce conditionP continuedFraction convert denom  
denominator differentiate divide elt euclideanSize eval  
expressIdealMember exquo extendedEuclidean factor  
factorPolynomial factorSquareFreePolynomial floor  
fractionPart gcd gcdPolynomial init inv lcm map max min  
multiEuclidean negative? nextItem numer numerator one?  
patternMatch positive? prime? principalIdeal quo random  
recip reducedSystem rem removeZeroes retract  
retractIfCan sign sizeLess? solveLinearPolynomialEquation  
squareFree squareFreePart squareFreePolynomial unit?  
unitCanonical unitNormal wholePart zero?

PAdicRational{PAdicRAT}: QuotientFieldCategory *with* 0 1  
\* \*\* + - / = D approximate associates? characteristic  
coerce continuedFraction denom denominator differentiate  
divide euclideanSize expressIdealMember exquo  
extendedEuclidean factor fractionPart gcd inv lcm map  
multiEuclidean numer numerator one? prime?  
principalIdeal quo recip reducedSystem rem removeZeroes  
retract retractIfCan sizeLess? squareFree squareFreePart  
unit? unitCanonical unitNormal wholePart zero?

Palette{PALETTE}: SetCategory *with* = bright coerce dark  
dim hue light pastel shade

ParametricPlaneCurve{PARPCURV}: *with* coordinate  
curve

ParametricSpaceCurve{PARSCURV}: *with* coordinate  
curve

ParametricSurface{PARSURF}: *with* coordinate surface

PartialFraction{PFR}: Algebra Field *with* 0 1 \* \*\* + - / =

associates? characteristic coerce compactFraction divide  
euclideanSize expressIdealMember exquo  
extendedEuclidean factor firstDenom firstNumer gcd inv  
lcm multiEuclidean nthFractionalTerm  
numberOfFractionalTerms one? padicFraction  
padicallyExpand partialFraction prime? principalIdeal quo  
recip rem sizeLess? squareFree squareFreePart unit?  
unitCanonical unitNormal wholePart zero?

Partition{PARTITION}: ConvertibleTo  
OrderedCancellationAbelianMonoid *with* 0 \* + - < = coerce  
conjugate convert max min partition pdct powers zero?

PatternMatchListResult{PATLRES}: SetCategory *with* =  
atoms coerce failed failed? lists makeResult new

PatternMatchResult{PATRES}: SetCategory *with* =  
addMatch addMatchRestricted coerce construct destruct  
failed failed? getMatch insertMatch new satisfy? union

Pattern{PATTERN}: RetractableTo SetCategory *with* 0 1 \*  
\*\* + / = addBadValue coerce constant? convert copy depth  
elt generic? getBadValues hasPredicate? hasTopPredicate?  
inR? isExpt isList isOp isPlus isPower isQuotient isTimes  
multiple? optional? optpair patternVariable predicates  
quoted? resetBadValues retract retractIfCan setPredicates  
setTopPredicate symbol? topPredicate variables  
withPredicates

PendantTree{PENDTREE}: BinaryRecursiveAggregate *with*  
# = any? children coerce copy count cyclic? elt empty  
empty? eq? every? leaf? leaves left less? map map!  
member? members more? node? nodes parts ptree right  
setchildren! setelt setleft! setright! setvalue! size? value

PermutationGroup{PERMGRP}: SetCategory *with* < <= =  
base coerce degree elt generators  
initializeGroupForWordProblem member? movedPoints  
orbit orbits order permutationGroup random  
strongGenerators wordInGenerators  
wordInStrongGenerators wordsForStrongGenerators

Permutation{PERM}: PermutationCategory *with* 1 \* \*\* / <  
= coerce coerceImages coerceListOfPairs  
coercePreimagesImages commutator conjugate cycle  
cyclePartition cycles degree elt eval even? fixedPoints inv  
listRepresentation max min movedPoints numberOfCycles  
odd? one? orbit order recip sign sort

Pi{HACKPI}: CharacteristicZero CoercibleTo ConvertibleTo  
Field RealConstant RetractableTo *with* 0 1 \* \*\* + - / =  
associates? characteristic coerce convert divide  
euclideanSize expressIdealMember exquo  
extendedEuclidean factor gcd inv lcm multiEuclidean one?  
pi prime? principalIdeal quo recip rem retract retractIfCan  
sizeLess? squareFree squareFreePart unit? unitCanonical  
unitNormal zero?

PlaneAlgebraicCurvePlot{ACPLOT}:

PlottablePlaneCurveCategory *with* coerce listBranches  
makeSketch refine xRange yRange

Plot3D{PLOT3D}: PlottableSpaceCurveCategory *with* adaptive3D? coerce debug3D listBranches maxPoints3D minPoints3D numFunEvals3D plot pointPlot refine screenResolution3D setAdaptive3D setMaxPoints3D setMinPoints3D setScreenResolution3D tRange tValues xRange yRange zRange zoom

Plot{PLOT}: PlottablePlaneCurveCategory *with* adaptive? coerce debug listBranches maxPoints minPoints numFunEvals parametric? plot plotPolar pointPlot refine screenResolution setAdaptive setMaxPoints setMinPoints setScreenResolution tRange xRange yRange zoom

Point{POINT}: PointCategory *with* # \* + - < = any? coerce concat construct convert copy copyInto! count cross delete dimension dot elt empty empty? entries entry? eq? every? extend fill! find first index? indices insert length less? map map! max maxIndex member? members merge min minIndex more? new parts position qelt qsetelt! reduce remove removeDuplicates reverse reverse! select setelt size? sort sort! sorted? swap! zero

PolynomialIdeals{IDEAL}: SetCategory *with* \* \*\* + = backOldPos coerce contract dimension element? generalPosition generators groebner groebner? groebnerIdeal ideal in? inRadical? intersect leadingIdeal quotient relationsIdeal saturate zeroDim?

PolynomialRing{PR}: FiniteAbelianMonoidRing *with* 0 1 \* \*\* + - / = associates? characteristic charthRoot coefficient coefficients coerce content degree exquo ground ground? leadingCoefficient leadingMonomial map mapExponents minimumDegree monomial monomial? numberOfMonomials one? primitivePart recip reductum retract retractIfCan unit? unitCanonical unitNormal zero?

Polynomial{POLY}: PolynomialCategory *with* 0 1 \* \*\* + - / < = D associates? characteristic charthRoot coefficient coefficients coerce conditionP content convert degree differentiate discriminant eval exquo factor factorPolynomial factorSquareFreePolynomial gcd gcdPolynomial ground ground? integrate isExpt isPlus isTimes lcm leadingCoefficient leadingMonomial mainVariable map mapExponents max min minimumDegree monicDivide monomial monomial? monomials multivariate numberOfMonomials one? patternMatch prime? primitiveMonomials primitivePart recip reducedSystem reductum resultant retract retractIfCan solveLinearPolynomialEquation squareFree squareFreePart squareFreePolynomial totalDegree unit? unitCanonical unitNormal univariate variables zero?

PositiveInteger{PI}: AbelianSemiGroup Monoid OrderedSet *with* 1 \* \*\* + < = coerce gcd max min one? recip

PrimeField{PF}: ConvertibleTo FiniteAlgebraicExtensionField FiniteFieldCategory *with* 0 1 \* \*\* + - / = Frobenius algebraic? associates? basis characteristic charthRoot coerce conditionP convert coordinates createNormalElement createPrimitiveElement

definingPolynomial degree dimension discreteLog divide euclideanSize expressIdealMember exquo extendedEuclidean extensionDegree factor factorsOfCyclicGroupSize gcd generator inGroundField? index init inv lcm lookup minimalPolynomial multiEuclidean nextItem norm normal? normalElement one? order prime? primeFrobenius primitive? primitiveElement principalIdeal quo random recip rem representationType represents retract retractIfCan size sizeLess? squareFree squareFreePart tableForDiscreteLogarithm trace transcendenceDegree transcendent? unit? unitCanonical unitNormal zero?

PrimitiveArray{PRIMARR}: OneDimensionalArrayAggregate *with* # < = any? coerce concat construct convert copy copyInto! count delete elt empty empty? entries entry? eq? every? fill! find first index? indices insert less? map map! max maxIndex member? members merge min minIndex more? new parts position qelt qsetelt! reduce remove removeDuplicates reverse reverse! select setelt size? sort sort! sorted? swap!

Product{PRODUCT}: AbelianGroup AbelianMonoid CancellationAbelianMonoid Finite Group Monoid OrderedAbelianMonoidSup OrderedSet SetCategory *with* 0 1 \* \*\* + - / < = coerce commutator conjugate index inv lookup makeprod max min one? random recip selectfirst selectsecond size sup zero?

QuadraticForm{QFORM}: AbelianGroup *with* 0 \* + - = coerce elt matrix quadraticForm zero?

QuasiAlgebraicSet{QALGSET}: CoercibleTo SetCategory *with* = coerce definingEquations definingInequation empty? idealSimplify quasiAlgebraicSet setStatus simplify

Quaternion{QUAT}: QuaternionCategory *with* 0 1 \* \*\* + - < = D abs characteristic charthRoot coerce conjugate convert differentiate elt eval imagI imagJ imagK inv map max min norm one? quatern rational rational? rationalIfCan real recip reducedSystem retract retractIfCan zero?

QueryEquation{QEQUAT}: *with* equation value variable

Queue{QUEUE}: QueueAggregate *with* # = any? back bag coerce copy count dequeue! empty empty? enqueue! eq? every? extract! front insert! inspect length less? map map! member? members more? parts queue rotate! size?

RadicalFunctionField{RADFF}: FunctionFieldCategory *with* 0 1 \* \*\* + - / = D absolutelyIrreducible? associates? basis branchPoint? branchPointAtInfinity? characteristic characteristicPolynomial charthRoot coerce complementaryBasis convert coordinates definingPolynomial derivationCoordinates differentiate discriminant divide elt euclideanSize expressIdealMember exquo extendedEuclidean factor gcd generator genus integral? integralAtInfinity? integralBasis integralBasisAtInfinity integralCoordinates

integralDerivationMatrix integralMatrix  
 integralMatrixAtInfinity integralRepresents inv  
 inverseIntegralMatrix inverseIntegralMatrixAtInfinity lcm  
 lift minimalPolynomial multiEuclidean nonSingularModel  
 norm normalizeAtInfinity numberOfComponents one?  
 prime? primitivePart principalIdeal quo ramified?  
 ramifiedAtInfinity? rank rationalPoint? rationalPoints  
 recip reduce reduceBasisAtInfinity reducedSystem  
 regularRepresentation rem represents retract retractIfCan  
 singular? singularAtInfinity? sizeLess? squareFree  
 squareFreePart trace traceMatrix unit? unitCanonical  
 unitNormal yCoordinates zero?

RadixExpansion{RADIX}: QuotientFieldCategory with 0 1 \*  
 \*\* + - / <= D abs associates? ceiling characteristic coerce  
 convert cycleRagits denom denominator differentiate divide  
 divide euclideanSize expressIdealMember exquo  
 extendedEuclidean factor floor fractRadix fractRagits  
 fractionPart gcd init inv lcm map max min multiEuclidean  
 negative? nextItem numer numerator one? patternMatch  
 positive? prefixRagits prime? principalIdeal quo random  
 recip reducedSystem rem retract retractIfCan sign  
 sizeLess? squareFree squareFreePart unit? unitCanonical  
 unitNormal wholePart wholeRadix wholeRagits zero?

RectangularMatrix{RMATRIX}: CoercibleTo  
 RectangularMatrixCategory VectorSpace with 0 # \* + - / =  
 antisymmetric? any? coerce column copy count diagonal?  
 dimension elt empty empty? eq? every? exquo less?  
 listOfLists map map! matrix maxColIndex maxRowIndex  
 member? members minColIndex minRowIndex more?  
 ncols nrows nullSpace nullity parts qelt rank  
 rectangularMatrix row rowEchelon size? square?  
 symmetric? zero?

Reference{REF}: Object SetCategory with = coerce deref  
 elt ref setelt setref

RewriteRule{RULE}: Eltable RetractableTo SetCategory  
 with = coerce elt lhs pattern quotedOperators retract  
 retractIfCan rhs rule suchThat

RomanNumeral{ROMAN}: IntegerNumberSystem with 0 1  
 \* \*\* + - <= D abs addmod associates? base binomial bit?  
 characteristic coerce convert copy dec differentiate divide  
 euclideanSize even? expressIdealMember exquo  
 extendedEuclidean factor factorial gcd hash inc init  
 invmod lcm length mask max min mulmod multiEuclidean  
 negative? nextItem odd? one? patternMatch permutation  
 positive? positiveRemainder powmod prime?  
 principalIdeal quo random rational rational? rationalIfCan  
 recip reducedSystem rem retract retractIfCan roman shift  
 sign sizeLess? squareFree squareFreePart submod  
 symmetricRemainder unit? unitCanonical unitNormal  
 zero?

RuleCalled{RULECOLD}: SetCategory with = coerce name

Ruleset{RULESET}: Eltable SetCategory with = coerce elt  
 rules ruleset

ScriptFormulaFormat1{FORMULA1}: Object with coerce

ScriptFormulaFormat{FORMULA}: SetCategory with =  
 coerce convert display epilogue formula new prologue  
 setEpilogue! setFormula! setPrologue!

SegmentBinding{SEGBIND}: SetCategory with = coerce  
 equation segment variable

Segment{SEG}: SegmentCategory

SegmentExpansionCategory with = BY SEGMENT coerce  
 convert expand hi high incr lo low map segment

SemiCancelledFraction{SCFRAC}: ConvertibleTo

QuotientFieldCategory with 0 1 \* \*\* + - / <= D abs  
 associates? ceiling characteristic charthRoot coerce  
 conditionP convert denom denominator differentiate divide  
 elt euclideanSize eval expressIdealMember exquo  
 extendedEuclidean factor factorPolynomial  
 factorSquareFreePolynomial floor fractionPart gcd  
 gcdPolynomial init inv lcm map max min multiEuclidean  
 negative? nextItem normalize numer numerator one?  
 patternMatch positive? prime? principalIdeal quo random  
 recip reducedSystem rem retract retractIfCan sign  
 sizeLess? solveLinearPolynomialEquation squareFree  
 squareFreePart squareFreePolynomial unit? unitCanonical  
 unitNormal wholePart zero?

SequentialDifferentialPolynomial{SDPOL}:

DifferentialPolynomialCategory RetractableTo with 0 1 \* \*\* +  
 - / <= D associates? characteristic charthRoot coefficient  
 coefficients coerce conditionP content degree  
 differentialVariables differentiate discriminant eval exquo  
 factor factorPolynomial factorSquareFreePolynomial gcd  
 gcdPolynomial ground ground? initial isExpt isPlus  
 isTimes isobaric? lcm leader leadingCoefficient  
 leadingMonomial mainVariable makeVariable map  
 mapExponents max min minimumDegree monicDivide  
 monomial monomial? monomials multivariate  
 numberOfMonomials one? order prime?  
 primitiveMonomials primitivePart recip reducedSystem  
 reductum resultant retract retractIfCan separant  
 solveLinearPolynomialEquation squareFree squareFreePart  
 squareFreePolynomial totalDegree unit? unitCanonical  
 unitNormal univariate variables weight weights zero?

SequentialDifferentialVariable{SDVAR}:

DifferentialVariableCategory with <= D coerce differentiate  
 makeVariable max min order retract retractIfCan variable  
 weight

Set{SET}: FiniteSetAggregate with # <= any? bag brace  
 cardinality coerce complement construct convert copy  
 count dictionary difference empty empty? eq? every?  
 extract! find index insert! inspect intersect less? lookup  
 map map! max member? members min more? parts  
 random reduce remove remove! removeDuplicates select  
 select! size size? subset? symmetricDifference union  
 universe

**SExpressionOf{SEXOF}**: **SExpressionCategory** *with* # = atom? car cdr coerce convert destruct elt eq expr float float? integer integer? list? null? pair? string string? symbol symbol? unequal

**SExpression{SEX}**: **SExpressionCategory** *with* # = atom? car cdr coerce convert destruct elt eq expr float float? integer integer? list? null? pair? string string? symbol symbol? unequal

**SimpleAlgebraicExtension{SAE}**: **MonogenicAlgebra** *with* 0 1 \* \*\* + - / = D associates? basis characteristic characteristicPolynomial charthRoot coerce conditionP convert coordinates createPrimitiveElement definingPolynomial derivationCoordinates differentiate discreteLog discriminant divide euclideanSize expressIdealMember exquo extendedEuclidean factor factorsOfCyclicGroupSize gcd generator index init inv lcm lift lookup minimalPolynomial multiEuclidean nextItem norm one? order prime? primeFrobenius primitive? primitiveElement principalIdeal quo random rank recip reduce reducedSystem regularRepresentation rem representationType represents retract retractIfCan size sizeLess? squareFree squareFreePart tableForDiscreteLogarithm trace traceMatrix unit? unitCanonical unitNormal zero?

**SingletonAsOrderedSet{SAOS}**: **OrderedSet** *with* < = coerce create max min

**SingleInteger{SINT}**: **IntegerNumberSystem** *with* 0 1 \* \*\* + - < = And D Not Or ^ abs addmod and associates? base binomial bit? characteristic coerce convert copy dec differentiate divide euclideanSize even? expressIdealMember exquo extendedEuclidean factor factorial gcd hash inc init invmod lcm length mask max min mulmod multiEuclidean negative? nextItem not odd? one? or patternMatch permutation positive? positiveRemainder powmod prime? principalIdeal quo random rational rational? rationalIfCan recip reducedSystem rem retract retractIfCan shift sign sizeLess? squareFree squareFreePart submod symmetricRemainder unit? unitCanonical unitNormal xor zero?

**SparseMultivariatePolynomial{SMP}**: **PolynomialCategory** *with* 0 1 \* \*\* + - / < = D associates? characteristic charthRoot coefficient coefficients coerce conditionP content convert degree differentiate discriminant eval exquo factor factorPolynomial factorSquareFreePolynomial gcd gcdPolynomial ground ground? isExpt isPlus isTimes lcm leadingCoefficient leadingMonomial mainVariable map mapExponents max min minimumDegree monicDivide monomial monomial? monomials multivariate numberOfMonomials one? patternMatch prime? primitiveMonomials primitivePart recip reducedSystem reductum resultant retract retractIfCan solveLinearPolynomialEquation squareFree squareFreePart squareFreePolynomial totalDegree unit? unitCanonical unitNormal univariate variables zero?

**SparseMultivariateTaylorSeries{SMTS}**: **MultivariateTaylorSeriesCategory** *with* 0 1 \* \*\* + - / = D acos acosh acot acoth acsc acsch asec asech asin asinh associates? atan atanh characteristic charthRoot coefficient coerce complete cos cosh cot coth csc csch csbst degree differentiate eval exp exquo extend fintegrate integrate leadingCoefficient leadingMonomial log map monomial monomial? nthRoot one? order pi pole? polynomial recip reductum sec sech sin sinh sqrt tan tanh unit? unitCanonical unitNormal variables zero?

**SparseTable{STBL}**: **TableAggregate** *with* # = any? bag coerce construct copy count dictionary elt empty empty? entries entry? eq? every? extract! fill! find first index? indices insert! inspect key? keys less? map map! maxIndex member? members minIndex more? parts qelt qsetelt! reduce remove remove! removeDuplicates search select select! setelt size? swap! table

**SparseUnivariatePolynomial{SUP}**: **UnivariatePolynomialCategory** *with* 0 1 \* \*\* + - / < = D associates? characteristic charthRoot coefficient coefficients coerce composite conditionP content degree differentiate discriminant divide divideExponents elt euclideanSize eval expressIdealMember exquo extendedEuclidean factor factorPolynomial factorSquareFreePolynomial gcd gcdPolynomial ground ground? init integrate isExpt isPlus isTimes lcm leadingCoefficient leadingMonomial mainVariable makeSUP map mapExponents max min minimumDegree monicDivide monomial monomial? monomials multiEuclidean multiplyExponents multivariate nextItem numberOfMonomials one? order outputForm prime? primitiveMonomials primitivePart principalIdeal pseudoDivide pseudoQuotient pseudoRemainder quo recip reducedSystem reductum rem resultant retract retractIfCan separate sizeLess? solveLinearPolynomialEquation squareFree squareFreePart squareFreePolynomial subResultantGcd totalDegree unit? unitCanonical unitNormal univariate unmakeSUP variables vectorise zero?

**SparseUnivariateTaylorSeries{SUTS}**: **UnivariateTaylorSeriesCategory** *with* 0 1 \* \*\* + - / = D acos acosh acot acoth acsc acsch approximate asec asech asin asinh associates? atan atanh center characteristic charthRoot coefficient coefficients coerce complete cos cosh cot coth csc csch degree differentiate elt eval exp exquo extend integrate leadingCoefficient leadingMonomial log map monomial monomial? multiplyCoefficients multiplyExponents nthRoot one? order pi pole? polynomial quoByVar recip reductum sec sech series sin sinh sqrt tan tanh terms truncate unit? unitCanonical unitNormal variable variables zero?

**SquareMatrix{SQMATRIX}**: **CoercibleTo** **SquareMatrixCategory** *with* 0 1 # \* \*\* + - / = D antisymmetric? any? characteristic coerce column copy count determinant diagonal diagonal? diagonalMatrix

diagonalProduct differentiate elt empty empty? eq? every?  
exquo inverse less? listOfLists map map! matrix  
maxColIndex maxRowIndex member? members  
minColIndex minRowIndex minordet more? ncols nrows  
nullSpace nullity one? parts qelt rank recip reducedSystem  
retract retractIfCan row rowEchelon scalarMatrix size?  
square? squareMatrix symmetric? trace transpose zero?

Stack{STACK}: StackAggregate *with # = any?* bag coerce  
copy count depth empty empty? eq? every? extract!  
insert! inspect less? map map! member? members more?  
parts pop! push! size? stack top

Stream{STREAM}: LazyStreamAggregate *with # = any?*  
child? children coerce complete concat concat! cons  
construct convert copy count cycleEntry cycleLength  
cycleSplit! cycleTail cyclic? delay delete distance elt empty  
empty? entries entry? eq? every? explicitEntries?  
explicitlyEmpty? explicitlyFinite? extend fill! filterUntil  
filterWhile find findCycle first first generate index? indices  
insert last lazy? lazyEvaluate leaf? less? map map!  
maxIndex member? members minIndex more? new node?  
nodes numberOfComputedEntries output parts  
possiblyInfinite? qelt qsetelt! reduce remove  
removeDuplicates repeating repeating? rest rst second  
select setchildren! setelt setfirst! setlast! setrest! setvalue!  
showAll? showAllElements size? split! swap! tail third  
value

StringTable{STRTBL}: TableAggregate *with # = any?* bag  
coerce construct copy count dictionary elt empty empty?  
entries entry? eq? every? extract! fill! find first index?  
indices insert! inspect key? keys less? map map! maxIndex  
member? members minIndex more? parts qelt qsetelt!  
reduce remove remove! removeDuplicates search select  
select! setelt size? swap! table

String{STRING}: StringCategory *with # < = any?* coerce  
concat construct copy copyInto! count delete elt empty  
empty? entries entry? eq? every? fill! find first index?  
indices insert leftTrim less? lowerCase lowerCase! map  
map! match? max maxIndex member? members merge  
min minIndex more? new parts position prefix? qelt  
qsetelt! reduce remove removeDuplicates replace reverse  
reverse! rightTrim select setelt size? sort sort! sorted? split  
string substring? suffix? swap! trim upperCase upperCase!

SubSpaceComponentProperty{COMPPROP}:  
SetCategory *with =* close closed? coerce copy new solid  
solid?

SubSpace{SUBSPACE}: SetCategory *with =* addPoint  
addPoint2 addPointLast birth child children  
closeComponent coerce deepCopy defineProperty  
extractClosed extractIndex extractPoint extractProperty  
internal? leaf? level merge modifyPoint new  
numberOfChildren parent pointData root? separate  
shallowCopy subspace traverse

SuchThat{SUCH}: SetCategory *with =* coerce construct lhs

rhs

Symbol{SYMBOL}: ConvertibleTo OrderedSet  
PatternMatchable *with < =* argscript coerce convert elt list  
max min name new patternMatch resetNew script  
scripted? scripts string subscript superscript

SymmetricPolynomial{SYMPOLY}:  
FiniteAbelianMonoidRing *with 0 1 \* \*\* + - / =* associates?  
characteristic charthRoot coefficient coefficients coerce  
content degree exquo ground ground? leadingCoefficient  
leadingMonomial map mapExponents minimumDegree  
monomial monomial? numberOfMonomials one?  
primitivePart recip reductum retract retractIfCan unit?  
unitCanonical unitNormal zero?

Tableau{TABLEAU}: Object *with* coerce listOfLists tableau

Table{TABLE}: TableAggregate *with # = any?* bag coerce  
construct copy count dictionary elt empty empty? entries  
entry? eq? every? extract! fill! find first index? indices  
insert! inspect key? keys less? map map! maxIndex  
member? members minIndex more? parts qelt qsetelt!  
reduce remove remove! removeDuplicates search select  
select! setelt size? swap! table

TaylorSeries{TS}: MultivariateTaylorSeriesCategory *with 0 1*  
*\* \*\* + - / = D* acos acosh acot acoth acsc acsch asec asech  
asin asinh associates? atan atanh characteristic  
charthRoot coefficient coerce complete cos cosh cot coth  
csc csch degree differentiate eval exp exquo extend  
fintegrate integrate leadingCoefficient leadingMonomial log  
map monomial monomial? nthRoot one? order pi pole?  
polynomial recip reductum sec sech sin sinh sqrt tan tanh  
unit? unitCanonical unitNormal variables zero?

TexFormat1{TEX1}: Object *with* coerce

TexFormat{TEX}: SetCategory *with =* coerce convert  
display epilogue new prologue setEpilogue! setPrologue!  
setTex! tex

TextFile{TEXTFILE}: FileCategory *with =* close! coerce  
endOfFile? iomode name open read! readIfCan! readLine!  
readLineIfCan! reopen! write! writeLine!

ThreeDimensionalViewport{VIEW3D}: SetCategory *with =*  
axes clipSurface close coerce colorDef controlPanel  
diagonals dimensions drawStyle eyeDistance hitherPlane  
intensity key lighting makeViewport3D modifyPointData  
move options outlineRender perspective reset resize rotate  
showClipRegion showRegion subspace title translate  
viewDeltaXDefault viewDeltaYDefault viewPhiDefault  
viewThetaDefault viewZoomDefault viewpoint viewport3D  
write zoom

ThreeSpace{SPACE3}: ThreeSpaceCategory *with =* check  
closedCurve closedCurve? coerce components composite  
composites copy create3Space curve curve? enterPointData  
l1lip l1lp l1prop lp l1prop merge mesh mesh?  
modifyPointData numberOfComponents

numberOfComposites objects point point? polygon  
polygon? subspace

Tree{TREE}: RecursiveAggregate *with* # = any? children  
coerce copy count cyclic? elt empty empty? eq? every?  
leaf? leaves less? map map! member? members more?  
node? nodes parts setchildren! setelt setvalue! size? tree  
value

TubePlot{TUBE}: *with* closed? getCurve listLoops open?  
setClosed tube

Tuple{TUPLE}: CoercibleTo SetCategory *with* = coerce  
length select

TwoDimensionalArray{ARRAY2}:

TwoDimensionalArrayCategory *with* # = any? coerce column  
copy count elt empty empty? eq? every? fill! less? map  
map! maxColIndex maxRowIndex member? members  
minColIndex minRowIndex more? ncols new nrows parts  
qelt qsetelt! row setColumn! setRow! setelt size?

TwoDimensionalViewport{VIEW2D}: SetCategory *with* =  
axes close coerce connect controlPanel dimensions  
getGraph graphState graphStates graphs key  
makeViewport2D move options points putGraph region  
reset resize scale show title translate units viewport2D  
write

UnivariateLaurentSeriesConstructor{ULSCONS}:

UnivariateLaurentSeriesConstructorCategory *with* 0 1 \* \*\* + -  
/ < = D abs acos acosh acot acoth acsc acsch approximate  
asech asin asinh associates? atan atanh ceiling center  
characteristic charthRoot coefficient coerce complete  
conditionP convert cos cosh cot coth csc csch degree denom  
denominator differentiate divide elt euclideanSize eval exp  
expressIdealMember exquo extend extendedEuclidean  
factor factorPolynomial factorSquareFreePolynomial floor  
fractionPart gcd gcdPolynomial init integrate inv laurent  
lcm leadingCoefficient leadingMonomial log map max min  
monomial monomial? multiEuclidean multiplyCoefficients  
multiplyExponents negative? nextItem nthRoot numer  
numerator one? order patternMatch pi pole? positive?  
prime? principalIdeal quo random rationalFunction recip  
reducedSystem reductum rem removeZeroes retract  
retractIfCan sec sech series sign sin sinh sizeLess?  
solveLinearPolynomialEquation sqrt squareFree  
squareFreePart squareFreePolynomial tan tanh taylor  
taylorIfCan taylorRep terms truncate unit? unitCanonical  
unitNormal variable variables wholePart zero?

UnivariateLaurentSeries{ULS}:

UnivariateLaurentSeriesConstructorCategory *with* 0 1 \* \*\* +  
- / = D acos acosh acot acoth acsc acsch approximate asech  
asin asinh associates? atan atanh center  
characteristic charthRoot coefficient coerce complete cos  
cosh cot coth csc csch degree denom denominator  
differentiate divide elt euclideanSize eval exp  
expressIdealMember exquo extend extendedEuclidean  
factor gcd integrate inv laurent lcm leadingCoefficient

leadingMonomial log map monomial monomial?  
multiEuclidean multiplyCoefficients multiplyExponents  
nthRoot numer numerator one? order pi pole? prime?  
principalIdeal quo rationalFunction recip reducedSystem  
reductum rem removeZeroes retract retractIfCan sec sech  
series sin sinh sizeLess? sqrt squareFree squareFreePart  
tan tanh taylor taylorIfCan taylorRep terms truncate unit?  
unitCanonical unitNormal variable variables zero?

UnivariatePolynomial{UP}: UnivariatePolynomialCategory  
*with* 0 1 \* \*\* + - / < = D associates? characteristic  
charthRoot coefficient coefficients coerce composite  
conditionP content degree differentiate discriminant divide  
divideExponents elt euclideanSize eval  
expressIdealMember exquo extendedEuclidean factor  
factorPolynomial factorSquareFreePolynomial gcd  
gcdPolynomial ground ground? init integrate isExpt isPlus  
isTimes lcm leadingCoefficient leadingMonomial  
mainVariable makeSUP map mapExponents max min  
minimumDegree monicDivide monomial monomial?  
monomials multiEuclidean multiplyExponents multivariate  
nextItem numberOfMonomials one? order prime?  
primitiveMonomials primitivePart principalIdeal  
pseudoDivide pseudoQuotient pseudoRemainder quo recip  
reducedSystem reductum rem resultant retract  
retractIfCan separate sizeLess?  
solveLinearPolynomialEquation squareFree squareFreePart  
squareFreePolynomial subResultantGcd totalDegree unit?  
unitCanonical unitNormal univariate unmakeSUP  
variables vectorise zero?

UnivariatePuisseuxSeriesConstructor{UPXSCONS}:

UnivariatePuisseuxSeriesConstructorCategory *with* 0 1 \* \*\* +  
- / = D acos acosh acot acoth acsc acsch approximate asech  
asin asinh associates? atan atanh center  
characteristic charthRoot coefficient coerce complete cos  
cosh cot coth csc csch degree differentiate divide elt  
euclideanSize eval exp expressIdealMember exquo extend  
extendedEuclidean factor gcd integrate inv laurent  
laurentIfCan laurentRep lcm leadingCoefficient  
leadingMonomial log map monomial monomial?  
multiEuclidean multiplyExponents nthRoot one? order pi  
pole? prime? principalIdeal puisseux quo rationalPower  
recip reductum rem retract retractIfCan sec sech series sin  
sinh sizeLess? sqrt squareFree squareFreePart tan tanh  
terms truncate unit? unitCanonical unitNormal variable  
variables zero?

UnivariatePuisseuxSeries{UPXS}:

UnivariatePuisseuxSeriesConstructorCategory *with* 0 1 \* \*\* +  
- / = D acos acosh acot acoth acsc acsch approximate asech  
asin asinh associates? atan atanh center  
characteristic charthRoot coefficient coerce complete cos  
cosh cot coth csc csch degree differentiate divide elt  
euclideanSize eval exp expressIdealMember exquo extend  
extendedEuclidean factor gcd integrate inv laurent  
laurentIfCan laurentRep lcm leadingCoefficient  
leadingMonomial log map monomial monomial?



multiEuclidean multiplyExponents nthRoot one? order pi  
 pole? prime? principalIdeal puseux quo rationalPower  
 recip reductum rem retract retractIfCan sec sech series sin  
 sinh sizeLess? sqrt squareFree squareFreePart tan tanh  
 terms truncate unit? unitCanonical unitNormal variable  
 variables zero?

UnivariateTaylorSeries{UTS}:

UnivariateTaylorSeriesCategory *with* 0 1 \* \*\* + - / = D acos  
 acosh acot acoth acsc acsch approximate asec asech asin  
 asinh associates? atan atanh center characteristic  
 charthRoot coefficient coefficients coerce complete cos cosh  
 cot coth csc csch degree differentiate elt eval evenlambert  
 exp exquo extend generalLambert integrate invmultisect  
 lagrange lambert leadingCoefficient leadingMonomial log  
 map monomial monomial? multiplyCoefficients  
 multiplyExponents multisect nthRoot oddlambert one?  
 order pi pole? polynomial quoByVar recip reductum revert  
 sec sech series sin sinh sqrt tan tanh terms truncate unit?  
 unitCanonical unitNormal univariatePolynomial variable  
 variables zero?

UniversalSegment{UNISEG}: SegmentCategory

SegmentExpansionCategory *with* = BY SEGMENT coerce  
 convert expand hasHi hi high incr lo low map segment

Variable{VARIABLE}: CoercibleTo SetCategory *with* =  
 coerce variable

Vector{VECTOR}: VectorCategory *with* # \* + - < = any?

coerce concat construct convert copy copyInto! count  
 delete dot elt empty empty? entries entry? eq? every? fill!  
 find first index? indices insert less? map map! max  
 maxIndex member? members merge min minIndex more?  
 new parts position qelt qsetelt! reduce remove  
 removeDuplicates reverse reverse! select setelt size? sort  
 sort! sorted? swap! vector zero

Void{VOID}: *with* coerce void



[



# Packages

This is a listing of all packages in the AXIOM library at the time this book was produced. Use the Browse facility (described in Chapter 14) to get more information about these constructors.

This sample entry will help you read the following table:

PackageName{PackageAbbreviation}:	
Category <sub>1</sub> ... Category <sub>N</sub> with operation <sub>1</sub> ... operation <sub>M</sub>	
where	
PackageName	is the full package name, for example, PadeApproximant- Package.
PackageAbbreviation	is the package abbreviation, for example, PADEPAC.
Category <sub>i</sub>	is a category to which the package belongs.
operation <sub>j</sub>	is an operation exported by the package.

AlgebraicFunction{AF}: with \*\* belong?  
 definingPolynomial inrootof iroot minPoly operator rootOf  
 AlgebraicHermiteIntegration{INTHERAL}: with  
 HermiteIntegrate  
 AlgebraicIntegrate{INTALG}: with algintegrate  
 palginfieldint palgintegrate

AlgebraicIntegration{INTAF}: with algint  
 AlgebraicManipulations{ALGMANIP}: with ratDenom  
 ratPoly rootKerSimp rootSimp rootSplit  
 AlgebraicMultFact{ALGMFACT}: with factor  
 AlgebraPackage{ALGPKG}: with basisOfCenter  
 basisOfCentroid basisOfCommutingElements  
 basisOfLeftAnnihilator basisOfLeftNucleus  
 basisOfLeftNucloid basisOfMiddleNucleus basisOfNucleus  
 basisOfRightAnnihilator basisOfRightNucleus  
 basisOfRightNucloid biRank doubleRank leftRank  
 radicalOfLeftTraceForm rightRank weakBiRank  
 AlgFactor{ALGFACT}: with doublyTransitive? factor split  
 AnyFunctions1{ANY1}: with coerce retract retractIfCan  
 retractable?  
 ApplyRules{APPRULE}: with applyRules localUnquote  
 AttachPredicates{PMPRED}: with suchThat  
 BalancedFactorisation{BALFACT}: with  
 balancedFactorisation  
 BasicOperatorFunctions1{BOP1}: with constantOpIfCan  
 constantOperator derivative evaluate  
 BezoutMatrix{BEZOUT}: with bezoutDiscriminant  
 bezoutMatrix bezoutResultant  
 BoundIntegerRoots{BOUNDZRO}: with integerBound  
 CartesianTensorFunctions2{CARTEN2}: with map  
 reshape  
 ChangeOfVariable{CHVAR}: with chvar eval goodPoint  
 mkIntegral radPoly rootPoly  
 CharacteristicPolynomialPackage{CHARPOL}: with  
 characteristicPolynomial

CoerceVectorMatrixPackage{CVMP}: *with* coerce  
coerceP

CombinatorialFunction{COMBF}: *with* \*\* belong?  
binomial factorial factorials iibinom iidprod iidsum iifact  
iiperm iipow ipow operator permutation product  
summation

CommonDenominator{CDEN}: *with* clearDenominator  
commonDenominator splitDenominator

CommonOperators{COMMONOP}: *with* operator

CommuteUnivariatePolynomialCategory{COMMUPC}:  
*with* swap

ComplexFactorization{COMPFAC}: *with* factor

ComplexFunctions2{COMPLEX2}: *with* map

ComplexIntegerSolveLinearPolynomialEquation  
{CINTSLPE}: *with* solveLinearPolynomialEquation

ComplexRootFindingPackage{CRFP}: *with*  
complexZeros divisorCascade factor graeffe norm  
pleskenSplit reciprocalPolynomial rootRadius schwerpunkt  
setErrorBound startPolynomial

ComplexRootPackage{CMPLXRT}: *with* complexZeros

ConstantLODE{ODECONST}: *with* constDsolve

CoordinateSystems{COORDSYS}: *with* bipolar  
bipolarCylindrical cartesian conical cylindrical elliptic  
ellipticCylindrical oblateSpheroidal parabolic  
parabolicCylindrical paraboloidal polar prolateSpheroidal  
spherical toroidal

CRAPackage{CRAPACK}: *with* chineseRemainder  
modTree multiEuclideanTree

CycleIndicators{CYCLES}: *with* SFunction alternating  
cap complete cup cyclic dihedral elementary eval graphs  
powerSum skewSFunction wreath

CyclicStreamTools{CSTTOOLS}: *with*  
computeCycleEntry computeCycleLength cycleElt

CyclotomicPolynomialPackage{CYCLOTOM}: *with*  
cyclotomic cyclotomicDecomposition  
cyclotomicFactorization

DegreeReductionPackage{DEGRED}: *with* expand  
reduce

DiophantineSolutionPackage{DIOSP}: *with* dioSolve

DirectProductFunctions2{DIRPROD2}: *with* map reduce  
scan

DiscreteLogarithmPackage{DLP}: *with*  
shanksDiscLogAlgorithm

DisplayPackage{DISPLAY}: *with* bright center copies  
newLine say sayLength

DistinctDegreeFactorize{DDFACT}: *with* distdfact

exptMod factor irreducible? separateDegrees  
separateFactors tracePowMod

DoubleResultantPackage{DBLRESP}: *with*  
doubleResultant

DrawNumericHack{DRAWHACK}: *with* coerce

DrawOptionFunctions0{DROPT0}: *with* adaptive  
clipBoolean coordinate curveColorPalette  
pointColorPalette ranges space style title toScale  
tubePoints tubeRadius units var1Steps var2Steps

DrawOptionFunctions1{DROPT1}: *with* option

EigenPackage{EP}: *with* characteristicPolynomial  
eigenvalues eigenvector eigenvectors inteigen

ElementaryFunctionODESolver{ODEEF}: *with* solve

ElementaryFunctionSign{SIGNEF}: *with* sign

ElementaryFunctionStructurePackage{EFSTRUC}: *with*  
normalize realElementary rischNormalize validExponential

ElementaryFunctionsUnivariateTaylorSeries{EFUTS}:  
*with* \*\* acos acosh acot acoth acsc acsch asec asech asin  
asinh atan atanh cos cosh cot coth csc csch exp log sec  
sech sin sincos sinh sinhcosh tan tanh

ElementaryFunction{EF}: *with* acos acosh acot acoth  
acsc acsch asec asech asin asinh atan atanh belong? cos  
cosh cot coth csc csch exp iacos iacosh iacot iacoth iacsc  
iiasch iasec iasech iiasin iiasinh iiatan iiatanh iicos iicosh  
iicot iicoth iicsc iicsch iiecp ilog iisec iisech iisin iisinh iitan  
iitanh log operator pi sec sech sin sinh specialTrigs tan  
tanh

ElementaryIntegration{INTEF}: *with* lfextendedint  
lfextlimint lfinfieldint lfintegrate lflimitedint

ElementaryRischDE{RDEEF}: *with* rischDE

EllipticFunctionsUnivariateTaylorSeries{ELFUTS}: *with*  
cn dn sn snendn

EquationFunctions2{EQ2}: *with* map

ErrorFunctions{ERROR}: *with* error

EuclideanGroebnerBasisPackage{GBEUCLID}: *with*  
euclideanGroebner euclideanNormalForm

EvaluateCycleIndicators{EVALCYC}: *with* eval

ExpressionFunctions2{EXPR2}: *with* map

ExpressionSpaceFunctions1{ES1}: *with* map

ExpressionSpaceFunctions2{ES2}: *with* map

ExpressionSpaceODESolver{EXPRODE}: *with* seriesSolve

ExpressionToUnivariatePowerSeries{EXPR2UPS}: *with*  
laurent puisieux series taylor

ExpressionTubePlot{EXPRTUBE}: *with*  
constantToUnaryFunction tubePlot

FactoredFunctions2{FR2}: *with* map

FactoredFunctions{FACTFUNC}: *with* log nthRoot

FactoredFunctionUtilities{FRUTIL}: *with* mergeFactors refine

FactoringUtilities{FACUTIL}: *with* completeEval degree lowerPolynomial normalDeriv raisePolynomial ran variables

FindOrderFinite{FORDER}: *with* order

FiniteDivisorFunctions2{FDIV2}: *with* map

FiniteFieldFunctions{FFF}: *with* createMultiplicationMatrix createMultiplicationTable createZechTable sizeMultiplication

FiniteFieldHomomorphisms{FFHOM}: *with* coerce

FiniteFieldPolynomialPackage2{FFPOLY2}: *with* rootOfIrreduciblePoly

FiniteFieldPolynomialPackage{FFPOLY}: *with* createIrreduciblePoly createNormalPoly createNormalPrimitivePoly createPrimitiveNormalPoly createPrimitivePoly leastAffineMultiple nextIrreduciblePoly nextNormalPoly nextNormalPrimitivePoly nextPrimitiveNormalPoly nextPrimitivePoly normal? numberOfIrreduciblePoly numberOfNormalPoly numberOfPrimitivePoly primitive? random reducedQPowers

FiniteFieldSolveLinearPolynomialEquation{FFSLPE}: *with* solveLinearPolynomialEquation

FiniteLinearAggregateFunctions2{FLAGG2}: *with* map reduce scan

FiniteLinearAggregateSort{FLASORT}: *with* heapSort quickSort shellSort

FiniteSetAggregateFunctions2{FSAGG2}: *with* map reduce scan

FloatingComplexPackage{FLOATCP}: *with* complexRoots complexSolve

FloatingRealPackage{FLOATRP}: *with* realRoots solve

FractionalIdealFunctions2{FRIDEAL2}: *with* map

FractionFunctions2{FRAC2}: *with* map

FunctionalSpecialFunction{FSPECF}: *with* Beta Gamma abs airyAi airyBi belong? bessell bessellJ bessellK bessellY digamma iiGamma iiabs operator polygamma

FunctionFieldCategoryFunctions2{FFCAT2}: *with* map

FunctionFieldIntegralBasis{FFINTBAS}: *with* integralBasis

FunctionSpaceAssertions{PMASFS}: *with* assert constant multiple optional

FunctionSpaceAttachPredicates{PMPREDFS}: *with* suchThat

FunctionSpaceComplexIntegration{FSCINT}: *with* complexIntegrate internalIntegrate

FunctionSpaceFunctions2{FS2}: *with* map

FunctionSpaceIntegration{FSINT}: *with* integrate

FunctionSpacePrimitiveElement{FSPRMELT}: *with* primitiveElement

FunctionSpaceReduce{FSRED}: *with* bringDown newReduce

FunctionSpaceSum{SUMFS}: *with* sum

FunctionSpaceToUnivariatePowerSeries{FS2UPS}: *with* exprToGenUPS exprToUPS

FunctionSpaceUnivariatePolynomialFactor{FSUPFACT}: *with* ffactor qfactor

GaussianFactorizationPackage{GAUSSFAC}: *with* factor prime? sumSquares

GeneralHenselPackage{GHENSEL}: *with* HenselLift completeHensel

GeneralPolynomialGcdPackage{GENPGCD}: *with* gcdPolynomial randomR

GenerateUnivariatePowerSeries{GENUPS}: *with* laurent puisieux series taylor

GenExEuclid{GENEEZ}: *with* compBound reduction solveid tablePow testModulus

GenUFactorize{GENUFACT}: *with* factor

GenusZeroIntegration{INTG0}: *with* palgLODE0 palgRDE0 palgextint0 palgint0 palglimint0

GosperSummationMethod{GOSPER}: *with* GospersMethod

GraphicsDefaults{GRDEF}: *with* adaptive clipPointsDefault drawToScale maxPoints minPoints screenResolution

GrayCode{GRAY}: *with* firstSubsetGray nextSubsetGray

GroebnerFactorizationPackage{GBF}: *with* factorGroebnerBasis groebnerFactorize

GroebnerInternalPackage{GBINTERN}: *with* credPol critB critBonD critM critMTonD1 critMonD1 critT critpOrder fprindINFO gbasis hMonic lepol makeCrit minGbasis prinb prindINFO prinpolINFO prinshINFO redPo redPol sPol updatD updatF virtualDegree

GroebnerPackage{GB}: *with* groebner normalForm

GroebnerSolve{GROEBSOL}: *with* genericPosition groebSolve testDim

HallBasis{HB}: *with* generate inHallBasis? lfunc

HeuGcd{HEUGCD}: *with* content contprim gcd gcdcofact  
 gcdcofactprim gcdprim lintgcd  
 IdealDecompositionPackage{IDECOMP}: *with*  
 primaryDecomp prime? radical zeroDimPrimary?  
 zeroDimPrime?  
 IncrementingMaps{INCRMAPS}: *with* increment  
 incrementBy  
 InfiniteTupleFunctions2{ITFUN2}: *with* map  
 InfiniteTupleFunctions3{ITFUN3}: *with* map  
 Infinity{INFINITY}: *with* infinity minusInfinity  
 plusInfinity  
 InnerAlgFactor{IALGFACT}: *with* factor  
 InnerCommonDenominator{ICDEN}: *with*  
 clearDenominator commonDenominator splitDenominator  
 InnerMatrixLinearAlgebraFunctions{IMATLIN}: *with*  
 determinant inverse nullSpace nullity rank rowEchelon  
 InnerMatrixQuotientFieldFunctions{IMATQF}: *with*  
 inverse nullSpace nullity rank rowEchelon  
 InnerModularGcd{INMODGCD}: *with* modularGcd  
 reduction  
 InnerMultFact{INNMFAC}: *with* factor  
 InnerNormalBasisFieldFunctions{INBFF}: *with* \* \*\* /  
 basis dAndcExp expPot index inv lookup  
 minimalPolynomial norm normal? normalElement pol  
 qPot random repSq setFieldInfo trace xn  
 InnerNumericEigenPackage{INEP}: *with* charpol  
 innerEigenvectors  
 InnerNumericFloatSolvePackage{INFSP}: *with*  
 innerSolve innerSolve1 makeEq  
 InnerPolySign{INPSIGN}: *with* signAround  
 InnerPolySum{ISUMP}: *with* sum  
 InnerTrigonometricManipulations{ITRIGMNP}: *with*  
 F2FG FG2F GF2FG explogs2trigs trigs2explogs  
 InputFormFunctions1{INFORM1}: *with* interpret  
 packageCall  
 IntegerCombinatoricFunctions{COMBINAT}: *with*  
 binomial factorial multinomial partition permutation  
 stirling1 stirling2  
 IntegerFactorizationPackage{INTFACT}: *with*  
 BasicMethod PollardSmallFactor factor squareFree  
 IntegerLinearDependence{ZLINDEP}: *with*  
 linearDependenceOverZ linearlyDependentOverZ?  
 solveLinearlyOverQ  
 IntegerNumberTheoryFunctions{INTHEORY}: *with*  
 bernoulli chineseRemainder divisors euler eulerPhi  
 fibonacci harmonic jacobi legendre moebiusMu  
 numberOfDivisors sumOfDivisors sumOfKthPowerDivisors  
 IntegerPrimesPackage{PRIMES}: *with* nextPrime  
 prevPrime prime? primes  
 IntegerRetractions{INTRET}: *with* integer integer?  
 integerIfCan  
 IntegerRoots{IROOT}: *with* approxNthRoot approxSqrt  
 perfectNthPower? perfectNthRoot perfectSqrt  
 perfectSquare?  
 IntegralBasisTools{IBATool}: *with* diagonalProduct  
 idealiser leastPower  
 IntegrationResultFunctions2{IR2}: *with* map  
 IntegrationResultRFToFunction{IRRF2F}: *with*  
 complexExpand complexIntegrate expand integrate split  
 IntegrationResultToFunction{IR2F}: *with*  
 complexExpand expand split  
 IntegrationTools{INTTOOLS}: *with* kmax ksec mkPrim  
 union vark varselect  
 InverseLaplaceTransform{INVLAPLA}: *with*  
 inverseLaplace  
 IrredPolyOverFiniteField{IRREDFFX}: *with*  
 generateIrredPoly  
 IrrRepSymNatPackage{IRSN}: *with*  
 dimensionOfIrreducibleRepresentation  
 irreducibleRepresentation  
 KernelFunctions2{KERNEL2}: *with* constantIfCan  
 constantKernel  
 Kovacic{KOVACIC}: *with* kovacic  
 LaplaceTransform{LAPLACE}: *with* laplace  
 LeadingCoefDetermination{LEADCDET}: *with* distFact  
 polCase  
 LinearDependence{LINDEP}: *with* linearDependence  
 linearlyDependent? solveLinear  
 LinearPolynomialEquationByFractions{LPEFRAC}: *with*  
 solveLinearPolynomialEquationByFractions  
 LinearSystemMatrixPackage{LSMP}: *with* aSolution  
 hasSolution? rank solve  
 LinearSystemPolynomialPackage{LSPP}: *with* linSolve  
 LinGrobnerPackage{LGROBP}: *with* anticoord  
 chooseMon computeBasis coordinate groebgen  
 intcompBasis linGenPos minPol tototex transform  
 LiouvillianFunction{LF}: *with* Ci Ei Si belong? dilog erf  
 integral li operator  
 ListFunctions2{LIST2}: *with* map reduce scan  
 ListFunctions3{LIST3}: *with* map



ListToMap{LIST2MAP}: *with match*  
 MakeBinaryCompiledFunction{MKBCFUNC}: *with binaryFunction compiledFunction*  
 MakeFloatCompiledFunction{MKFLCFN}: *with makeFloatFunction*  
 MakeFunction{MKFUNC}: *with function*  
 MakeRecord{MKRECORD}: *with makeRecord*  
 MakeUnaryCompiledFunction{MKUCFUNC}: *with compiledFunction unaryFunction*  
 MappingPackage1{MAPPKG1}: *with \*\* coerce fixedPoint id nullary recur*  
 MappingPackage2{MAPPKG2}: *with const constant curry diag*  
 MappingPackage3{MAPPKG3}: *with \* constantLeft constantRight curryLeft curryRight twist*  
 MappingPackageInternalHacks1{MAPHACK1}: *with iter recur*  
 MappingPackageInternalHacks2{MAPHACK2}: *with arg1 arg2*  
 MappingPackageInternalHacks3{MAPHACK3}: *with comp*  
 MatrixCategoryFunctions2{MATCAT2}: *with map reduce*  
 MatrixCommonDenominator{MCDEN}: *with clearDenominator commonDenominator splitDenominator*  
 MatrixLinearAlgebraFunctions{MATLIN}: *with determinant inverse minordet nullSpace nullity rank rowEchelon*  
 MergeThing{MTHING}: *with mergeDifference*  
 MeshCreationRoutinesForThreeDimensions{MESH}: *with meshFun2Var meshPar1Var meshPar2Var ptFunc*  
 ModularDistinctDegreeFactorizer{MDDFACT}: *with ddFact exptMod factor gcd separateFactors*  
 ModularHermitianRowReduction{MHROWRED}: *with rowEch rowEchelon*  
 MonoidRingFunctions2{MRF2}: *with map*  
 MoreSystemCommands{MSYSCMD}: *with systemCommand*  
 MPolyCatFunctions2{MPC2}: *with map reshape*  
 MPolyCatFunctions3{MPC3}: *with map*  
 MPolyCatRationalFunctionFactorizer{MPRFF}: *with factor pushdown pushdterm pushcoef pushconst pushup totalfract*  
 MRationalFactorize{MRATFAC}: *with factor*  
 MultFiniteFactorize{MFINFACT}: *with factor*  
 MultipleMap{MMAP}: *with map*  
 MultivariateFactorize{MULTFACT}: *with factor*  
 MultivariateLifting{MLIFT}: *with corrPoly lifting lifting1*  
 MultivariateSquareFree{MULTSQFR}: *with squareFree squareFreePrim*  
 NonCommutativeOperatorDivision{NCODIV}: *with leftDivide leftExactQuotient leftGcd leftLcm leftQuotient leftRemainder*  
 NoneFunctions1{NONE1}: *with coerce*  
 NonLinearFirstOrderODESolver{NODE1}: *with solve*  
 NonLinearSolvePackage{NLINSOL}: *with solve solveInField*  
 NPCoef{NPCOEF}: *with listexp npcoef*  
 NumberFieldIntegralBasis{NFINTBAS}: *with discriminant integralBasis*  
 NumberFormats{NUMFMT}: *with FormatArabic FormatRoman ScanArabic ScanRoman*  
 NumberTheoreticPolynomialFunctions{NTPOLFN}: *with bernoulliB cyclotomic eulerE*  
 NumericalOrdinaryDifferentialEquations{NUMODE}: *with rk4 rk4a rk4f rk4qc*  
 NumericalQuadrature{NUMQUAD}: *with aromberg asimpson atrapezoidal romberg romberg simpson simpsono trapezoidal trapezoidal*  
 NumericComplexEigenPackage{NCEP}: *with characteristicPolynomial complexEigenvalues complexEigenvectors*  
 NumericContinuedFraction{NCNTFRAC}: *with continuedFraction*  
 NumericRealEigenPackage{NREP}: *with characteristicPolynomial realEigenvalues realEigenvectors*  
 NumericTubePlot{NUMTUBE}: *with tube*  
 Numeric{NUMERIC}: *with complexNumeric numeric*  
 OctonionCategoryFunctions2{OCTCT2}: *with map*  
 ODEIntegration{ODEINT}: *with expint int*  
 ODETools{ODETOOLS}: *with particularSolution variationOfParameters wronskianMatrix*  
 OneDimensionalArrayFunctions2{ARRAY12}: *with map reduce scan*  
 OnePointCompletionFunctions2{ONECOMP2}: *with map*  
 OperationsQuery{OPQUERY}: *with getDatabase*

OrderedCompletionFunctions2{ORDCOMP2}: *with* map  
 OrderingFunctions{ORDFUNS}: *with* pureLex reverseLex totalLex  
 OrthogonalPolynomialFunctions{ORTHPOL}: *with* ChebyshevU chebyshevT hermiteH laguerreL legendreP  
 OutputPackage{OUT}: *with* output  
 PadeApproximantPackage{PADEPAC}: *with* pade  
 PadeApproximants{PADE}: *with* pade padeCf  
 ParadoxicalCombinatorsForStreams{YSTREAM}: *with* Y  
 PartitionsAndPermutations{PARTPERM}: *with* conjugate conjugates partitions permutations sequences shuffle shuffleIn  
 PatternFunctions1{PATTERN1}: *with* addBadValue badValues predicate satisfy? suchThat  
 PatternFunctions2{PATTERN2}: *with* map  
 PatternMatchAssertions{PMAS}: *with* assert constant multiple optional  
 PatternMatchFunctionSpace{PMFS}: *with* patternMatch  
 PatternMatchIntegerNumberSystem{PMINS}: *with* patternMatch  
 PatternMatchKernel{PMKERNEL}: *with* patternMatch  
 PatternMatchListAggregate{PMLSAGG}: *with* patternMatch  
 PatternMatchPolynomialCategory{PMPLCAT}: *with* patternMatch  
 PatternMatchPushDown{PMDOWN}: *with* fixPredicate patternMatch  
 PatternMatchQuotientFieldCategory{PMQFCAT}: *with* patternMatch  
 PatternMatchResultFunctions2{PATRES2}: *with* map  
 PatternMatchSymbol{PMSYM}: *with* patternMatch  
 PatternMatchTools{PMTTOOLS}: *with* patternMatch patternMatchTimes  
 PatternMatch{PATMATCH}: *with* Is is?  
 Permanent{PERMAN}: *with* permanent  
 PermutationGroupExamples{PGE}: *with* abelianGroup alternatingGroup cyclicGroup dihedralGroup janko2 mathieu11 mathieu12 mathieu22 mathieu23 mathieu24 rubiksGroup symmetricGroup youngGroup  
 PiCoercions{PICOERCE}: *with* coerce  
 PlotFunctions1{PLOT1}: *with* plot plotPolar  
 PlotTools{PLOTTOOL}: *with* calcRanges  
 PointFunctions2{PTFUNC2}: *with* map  
 PointPackage{PTPACK}: *with* color hue phiCoord rCoord shade thetaCoord xCoord yCoord zCoord  
 PointsOfFiniteOrderRational{PFOQ}: *with* order torsion? torsionIfCan  
 PointsOfFiniteOrderTools{PFOTOOLS}: *with* badNum doubleDisc getGoodPrime mix polyred  
 PointsOfFiniteOrder{PFO}: *with* order torsion? torsionIfCan  
 PolToPol{POLTOPOL}: *with* dmpToNdmp dmpToP ndmpToDmp ndmpToP pToDmp pToNdmp  
 PolyGroebner{PGROEB}: *with* lexGroebner totalGroebner  
 PolynomialAN2Expression{PAN2EXPR}: *with* coerce  
 PolynomialCategoryLifting{POLYLIFT}: *with* map  
 PolynomialCategoryQuotientFunctions{POLYCATQ}: *with* isExpt isPlus isPower isTimes mainVariable multivariate univariate variables  
 PolynomialFactorizationByRecursionUnivariate{PFBRU}: *with* bivariateSLPEBR factorByRecursion factorSFBRLcUnit factorSquareFreeByRecursion randomR solveLinearPolynomialEquationByRecursion  
 PolynomialFactorizationByRecursion{PFBR}: *with* bivariateSLPEBR factorByRecursion factorSFBRLcUnit factorSquareFreeByRecursion randomR solveLinearPolynomialEquationByRecursion  
 PolynomialFunctions2{POLY2}: *with* map  
 PolynomialGcdPackage{PGCD}: *with* gcd gcdPrimitive  
 PolynomialInterpolationAlgorithms{PINTERPA}: *with* LagrangeInterpolation  
 PolynomialInterpolation{PINTERP}: *with* interpolate  
 PolynomialNumberTheoryFunctions{PNTHEORY}: *with* bernoulli chebyshevT chebyshevU cyclotomic euler fixedDivisor hermite laguerre legendre  
 PolynomialRoots{POLYROOT}: *with* froot qroot rroot  
 PolynomialSolveByFormulas{SOLVEFOR}: *with* aCubic aLinear aQuadratic aQuartic aSolution cubic linear mapSolve quadratic quartic solve  
 PolynomialSquareFree{PSQFR}: *with* squareFree  
 PolynomialToUnivariatePolynomial{POLY2UP}: *with* univariate  
 PowerSeriesLimitPackage{LIMITPS}: *with* complexLimit limit  
 PrimitiveArrayFunctions2{PRIMARR2}: *with* map reduce scan  
 PrimitiveElement{PRIMELT}: *with* primitiveElement

PrimitiveRatDE{ODEPRIM}: *with* denomLODE  
 PrimitiveRatRicDE{ODEPRIC}: *with* changevar  
 constantCoefficientRicDE denomRicDE  
 leadingCoefficientRicDE polyRicDE singRicDE  
 PrintPackage{PRINT}: *with* print  
 PureAlgebraicIntegration{INTPAF}: *with* palgLODE  
 palgRDE palgextint palgint palglimit  
 PureAlgebraicLODE{ODEPAL}: *with* algDsolve  
 QuasiAlgebraicSet2{QALGSET2}: *with* radicalSimplify  
 QuaternionCategoryFunctions2{QUATCT2}: *with* map  
 QuotientFieldCategoryFunctions2{QFCAT2}: *with* map  
 RadicalEigenPackage{REP}: *with* eigenMatrix  
 gramSchmidt normalise orthonormalBasis  
 radicalEigenvalues radicalEigenvector radicalEigenvectors  
 RadicalSolvePackage{SOLVERAD}: *with* contractSolve  
 radicalRoots radicalSolve  
 RadixUtilities{RADUTIL}: *with* radix  
 RandomNumberSource{RANDSRC}: *with* randnum  
 reseed size  
 RationalFactorize{RATFACT}: *with* factor  
 RationalFunctionDefiniteIntegration{DEFINTRF}: *with*  
 integrate  
 RationalFunctionFactorizer{RFFACTOR}: *with*  
 factorFraction  
 RationalFunctionFactor{RFFACT}: *with* factor  
 RationalFunctionIntegration{INTRF}: *with*  
 extendedIntegrate infieldIntegrate internalIntegrate  
 limitedIntegrate  
 RationalFunctionLimitPackage{LIMITRF}: *with*  
 complexLimit limit  
 RationalFunctionSign{SIGNRF}: *with* sign  
 RationalFunctionSum{SUMRF}: *with* sum  
 RationalFunction{RF}: *with* coerce eval mainVariable  
 multivariate univariate variables  
 RationalIntegration{INTRAT}: *with* extendedint infieldint  
 integrate limitedint  
 RationalLODE{ODERAT}: *with* ratDsolve  
 RationalRetractions{RATRET}: *with* rational rational?  
 rationalIfCan  
 RationalRicDE{ODERTRIC}: *with* changevar  
 constantCoefficientRicDE polyRicDE ricDsolve singRicDE  
 RatODETools{RTODETLS}: *with* genericPolynomial  
 RealSolvePackage{REALSOLV}: *with* realSolve solve  
 RealZeroPackageQ{REAL0Q}: *with* realZeros refine  
 RealZeroPackage{REAL0}: *with* midpoint midpoints  
 realZeros refine  
 RectangularMatrixCategoryFunctions2{RMCAT2}: *with*  
 map reduce  
 ReducedDivisor{RDIV}: *with* order  
 ReduceLODE{ODERED}: *with* reduceLODE  
 ReductionOfOrder{REDORDER}: *with* ReduceOrder  
 RepeatedDoubling{REPDB}: *with* double  
 RepeatedSquaring{REPSQ}: *with* expt  
 RepresentationPackage1{REP1}: *with*  
 antisymmetricTensors createGenericMatrix  
 permutationRepresentation symmetricTensors  
 tensorProduct  
 RepresentationPackage2{REP2}: *with* areEquivalent?  
 completeEchelonBasis createRandomElement  
 cyclicSubmodule isAbsolutelyIrreducible? meatAxe  
 scanOneDimSubspaces split  
 standardBasisOfCyclicSubmodule  
 ResolveLatticeCompletion{RESLATC}: *with* coerce  
 RetractSolvePackage{RETSOL}: *with* solveRetract  
 SAERationalFunctionAlgFactor{SAERFFC}: *with* factor  
 SegmentBindingFunctions2{SEGBIND2}: *with* map  
 SegmentFunctions2{SEG2}: *with* map  
 SimpleAlgebraicExtensionAlgFactor{SAEFACT}: *with*  
 factor  
 DoubleFloatSpecialFunctions{DFLOATSFUN}: *with* Beta  
 Gamma airyAi airyBi besselI besselJ besselK besselY  
 digamma hypergeometric0F1 logGamma polygamma  
 SortedCache{SCACHE}: *with* cache clearCache  
 enterInCache  
 SparseUnivariatePolynomialFunctions2{SUP2}: *with*  
 map  
 SpecialOutputPackage{SPECOUT}: *with*  
 outputAsFortran outputAsScript outputAsTex  
 StorageEfficientMatrixOperations{MATSTOR}: *with* \*\*  
 copy! leftScalarTimes! minus! plus! power!  
 rightScalarTimes! times!  
 StreamFunctions1{STREAM1}: *with* concat  
 StreamFunctions2{STREAM2}: *with* map reduce scan  
 StreamFunctions3{STREAM3}: *with* map  
 StreamTaylorSeriesOperations{STTAYLOR}: *with* \* + - /  
 adddiag coerce compose deriv eval evenlambert gderiv  
 generalLambert int integers integrate invmultisect lagrange

lambert lazyGintegrate lazyIntegrate mapdiv mapmult  
monom multisection nld oddintegers oddlambert power  
powern recip revert

StreamTranscendentalFunctions{STTF}: *with* \*\* acos  
acosh acot acoth acsc acsch asec asech asin asinh atan  
atanh cos cosh cot coth csc csch exp log sec sech sin sincos  
sinh sinhcos tan tanh

SubResultantPackage{SUBRESP}: *with* primitivePart  
subresultantVector

SymmetricFunctions{SYMFUNC}: *with* symFunc

SymmetricGroupCombinatoricFunctions{SGCF}: *with*  
coleman inverseColeman listYoungTableaus  
makeYoungTableau nextColeman nextLatticePermutation  
nextPartition numberOfImproperPartitions subSet  
unrankImproperPartitions0 unrankImproperPartitions1

SystemODESolver{ODESYS}: *with* solveInField  
triangulate

SystemSolvePackage{SYSSOLP}: *with* solve  
triangularSystems

TableauxBumpers{TABLBUMP}: *with* bat bat1 bumprow  
bumpstab bumpstab1 inverse lex maxrow mr slex tab tab1  
untab

TangentExpansions{TANEXP}: *with* tanAn tanNa  
tanSum

ToolsForSign{TOOLSIGN}: *with* direction nonQsign sign

TopLevelDrawFunctionsForAlgebraicCurves  
{DRAWCURV}: *with* draw

TopLevelDrawFunctionsForCompiledFunctions  
{DRAWCFUN}: *with* draw makeObject recolor

TopLevelDrawFunctions{DRAW}: *with* draw makeObject

TopLevelThreeSpace{TOPSP}: *with* createThreeSpace

TranscendentalHermiteIntegration{INTHERTR}: *with*  
HermiteIntegrate

TranscendentalIntegration{INTTR}: *with* expextendedint  
expintegrate expintfldpoly explimitedint primextendedint  
primextintfrac primintegrate primintegratefrac  
primintfldpoly primlimintfrac primlimitedint

TranscendentalManipulations{TRMANIP}: *with* cos2sec  
cosh2sech cot2tan cot2trig coth2tanh coth2trigh csc2sin  
csch2sinh expand expandLog expandPower htrigs  
removeCosSq removeCoshSq removeSinSq removeSinhSq  
sec2cos sech2cosh simplify simplifyExp sin2csc sinh2csch  
tan2cot tan2trig tanh2coth tanh2trigh

TranscendentalRischDE{RDETR}: *with* DSPDE SPDE  
baseRDE expRDE primRDE

TransSolvePackageService{SOLVESER}: *with*  
decomposeFunc unvectorise

TransSolvePackage{SOLVETRA}: *with* solve

TriangularMatrixOperations{TRIMAT}: *with*  
LowTriBddDenomInv UpTriBddDenomInv

TrigonometricManipulations{TRIGMNIP}: *with*  
complexElementary complexNormalize imag real real?  
trigs

TubePlotTools{TUBETOOL}: *with* \* + - cosSinInfo cross  
dot loopPoints point unitVector

TwoDimensionalPlotClipping{CLIP}: *with* clip  
clipParametric clipWithRanges

TwoFactorize{TWOFACT}: *with* generalSqFr  
generalTwoFactor twoFactor

UnivariateFactorize{UNIFACT}: *with* factor  
factorSquareFree genFact henselFact henselfact quadratic  
sqroot trueFactors

UnivariateLaurentSeriesFunctions2{ULS2}: *with* map

UnivariatePolynomialCategoryFunctions2{UPOLYC2}:  
*with* map

UnivariatePolynomialCommonDenominator{UPCDEN}:  
*with* clearDenominator commonDenominator  
splitDenominator

UnivariatePolynomialFunctions2{UP2}: *with* map

UnivariatePolynomialSquareFree{UPSQFREE}: *with*  
BumInSepFFE squareFree squareFreePart

UnivariatePuisseuxSeriesFunctions2{UPXS2}: *with* map

UnivariateTaylorSeriesFunctions2{UTS2}: *with* map

UnivariateTaylorSeriesODESolver{UTSODE}: *with*  
mpsode ode ode1 ode2 stFunc1 stFunc2 stFuncN

UniversalSegmentFunctions2{UNISEG2}: *with* map

UserDefinedPartialOrdering{UDPO}: *with* getOrder  
largest less? more? setOrder userOrdered?

UserDefinedVariableOrdering{UDVO}: *with*  
getVariableOrder resetVariableOrder setVariableOrder

VectorFunctions2{VECTOR2}: *with* map reduce scan

ViewDefaultsPackage{VIEWDEF}: *with*  
axesColorDefault lineColorDefault pointColorDefault  
pointSizeDefault tubePointsDefault tubeRadiusDefault  
unitsColorDefault var1StepsDefault var2StepsDefault  
viewDefaults viewPosDefault viewSizeDefault  
viewWriteAvailable viewWriteDefault

ViewportPackage{VIEW}: *with* coerce drawCurves  
graphCurves

WeierstrassPreparation{WEIER}: *with* cfirst clikeUniv  
crest qqg sts2stst weierstrass

WildFunctionFieldIntegralBasis{WFFINTBS}: *with*

integralBasis listSquaredFactors



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# Operations

This appendix contains a partial list of AXIOM operations with brief descriptions. For more details, use the Browse facility of HyperDoc: enter the name of the operation for which you want more information in the input area on the main Browse menu and then click on **Operations**.

]

**#aggregate**

*#a* returns the number of items in *a*.

**x\*\*y**

*x\*\*y* returns *x* to the power *y*. Also, this operation returns, if *x* is:

an equation: a new equation by raising both sides of *x* to the power *y*.

a float or small float: **sign**(*x*)**exp**(*y* log(|*x*|)).

See also `InputForm` and `OutputForm`.

**x\*y**

The binary operator `*` denotes multiplication. Its meaning depends on the type of its arguments:

if *x* and *y* are members of a ring (more generally, a domain of category `SemiGroup`), *x\*y* returns the product of *x* and *y*.

if *r* is an integer and *x* is an element of a ring, or if *r* is a scalar and *x* is a vector, matrix, or direct product: *r\*x* returns the left multiplication of *r* by *x*. More generally, if *r* is an integer and *x* is a member of a domain of category `AbelianMonoid`, or *r* is a member of domain *R* and *x* is a domain of category `Module(R)`, `GradedModule`, or `GradedAlgebra` defined over *R*, *r\*x*

returns the left multiplication of *r* by *x*. Here *x* can be a vector, a matrix, or a direct product. Similarly, *x\*n* returns the right integer multiple of *x*.

if *a* and *b* are monad elements, the product of *a* and *b* (see `Monad`).

if *A* and *B* are matrices, returns the product of *A* and *B*. If *v* is a row vector, *v\*A* returns the product of *v* and *A*. If *v* is column vector, *A\*v* returns the product of *A* with column vector *v*. In each case, the operation calls **error** if the dimensions are incompatible.

if *s* is an integer or float and *c* is a color, *s\*c* returns the weighted shade scaled by *s*.

if *s* and *t* are Cartesian tensors, *s\*t* is the inner product of the tensors *s* and *t*. This contracts the last index of *s* with the first index of *t*, that is,

$t * s = \text{contract}(t, \text{rank } t, s, 1),$

$t * s = \sum_{k=1}^N t([i_1, \dots, i_N, k] * s[k, j_1, \dots, j_M]).$

if *eq* is an equation, *r\*eq* multiplies both sides of *eq* by *r*.

if *I* and *J* are ideals, the product of ideals.

See also `OutputForm`, `Monad`, `LeftModule`, `RightModule`, and `FreeAbelianMonoidCategory`,

See also `InputForm` and `OutputForm`.

**x+y**

The binary operator `+` denotes addition. Its meaning depends on the type of its arguments. If *x* and *y* are:

members of a ring (more generally, of a domain of category `AbelianSemiGroup`): the sum of *x* and *y*.

matrices: the matrix sum if *x* and *y* have the same dimensions, and **error** otherwise.

vectors: the component-wise sum if *x* and *y* have the same length, and **error** otherwise.

colors: a color which additively mixes colors *x* and *y*.

equations: an equation created by adding the respective left- and right-hand sides of  $x$  and  $y$ .

elements of graded module or algebra: the sum of  $x$  and  $y$  in the module of elements of the same degree as  $x$  and  $y$ .

ideals: the ideal generated by the union of  $x$  and  $y$ .

See also `FreeAbelianMonoidCategory`, `InputForm` and `OutputForm`.

**[x]−y**

$-x$  returns the negative (additive inverse) of  $x$ , where  $x$  is a member of a ring (more generally, a domain of category `AbelianGroup`). Also,  $x$  may be a matrix, a vector, or a member of a graded module.

$x - y$  returns  $x + (-y)$ .

See also `CancellationAbelianMonoid` and `OutputForm`.

**x/y**

The binary operator `/` generally denotes binary division. Its precise meaning, however, depends on the type of its arguments:

$x$  and  $y$  are elements of a group: multiplies  $x$  by the inverse `inv`( $y$ ) of  $y$ .

$x$  and  $y$  are elements of a field: divides  $x$  by  $y$ , calling **error** if  $y = 0$ .

$x$  is a matrix or a vector and  $y$  is a scalar: divides each element of  $x$  by  $y$ .

$x$  and  $y$  are floats or small floats: divides  $x$  by  $y$ .

$x$  and  $y$  are fractions: returns the quotient as another fraction.

$x$  and  $y$  are polynomials: returns the quotient as a fraction of polynomials.

See also `AbelianMonoidRing`, `InputForm` and `OutputForm`.

**0**

The additive identity element for a ring (more generally, for an `AbelianMonoid`). Also, for a graded module or algebra, the zero of degree 0 (see `GradedModule`). See also `InputForm`.

**1**

The multiplicative identity element for a ring (more generally, for a `Monoid` and `MonadWithUnit`). or a graded algebra. See also `InputForm`.

**x<y**

The binary operator `<` denotes the boolean-valued “less than” function. Its meaning depends on the type of its arguments. The operation  $x < y$  for  $x$  and  $y$ :

elements of a totally ordered set (such as integer and floating point numbers): tests if  $x$  is less than  $y$ .

sets: tests if all the elements of  $x$  are also elements of  $y$ .

permutations: tests if  $x$  is less than  $y$ ; see `Permutation` for details. Note: this order relation is total if and only if the underlying domain is of category `Finite` or `OrderedSet`.

permutation groups: tests if  $x$  is a proper subgroup of  $y$ . See also `OutputForm`.

**x=y**

The meaning of binary operator  $x = y$  depends on the value expected of the operation. If the value is expected to be:

a boolean:  $x = y$  tests that  $x$  and  $y$  are equal.

an equation:  $x = y$  creates an equation.

See also `OutputForm`.

**abelianGroup** (*listOfPositiveIntegers*)

**abelianGroup** ( $[p_1, \dots, p_k]$ ) constructs the abelian group that is the direct product of cyclic groups with order  $p_i$ .

**absolutelyIrreducible?** ()

**absolutelyIrreducible?** ( $()F$ ) tests if the algebraic function field  $F$  remains irreducible over the algebraic closure of the ground field. See `FunctionFieldCategory` using `Browse`.

**abs** (*element*)

**abs** ( $x$ ) returns the absolute value of  $x$ , an element of an `OrderedRing` or a `Complex`, `Quaternion`, or `Octonion` value.

**acos** (*expression*)

**acosIfCan** (*expression*)

Argument  $x$  can be a `Complex`, `Float`, `DoubleFloat`, or `Expression` value or a series.

**acos** ( $x$ ) returns the arccosine of  $x$ .

**acosIfCan** ( $x$ ) returns **acos** ( $x$ ) if possible, and **"failed"** otherwise.

**acosh** (*expression*)

**acoshIfCan** (*expression*)

Argument  $x$  can be a `Complex`, `Float`, `DoubleFloat`, or `Expression` value or a series.

**acosh** ( $x$ ) returns the hyperbolic arccosine of  $x$ .

**acoshIfCan** ( $x$ ) returns **acosh** ( $x$ ) if possible, and **"failed"** otherwise.

**acoth** (*expression*)

**acothIfCan** (*expression*)

Argument  $x$  can be a `Complex`, `Float`, `DoubleFloat`, or `Expression` value or a series.

**acoth** ( $x$ ) returns the hyperbolic arccotangent of  $x$ .

**acothIfCan** ( $x$ ) returns **acoth** ( $x$ ) if possible, and **"failed"** otherwise.

**acot** (*expression*)

**acotIfCan** (*expression*)

Argument  $x$  can be a Complex, Float, DoubleFloat, or Expression value or a series.

**acot** ( $x$ ) returns the arccotangent of  $x$ .

**acotIfCan** ( $x$ ) returns **acot** ( $x$ ) if possible, and "failed" otherwise.

**acsch** (*expression*)

**acschIfCan** (*expression*)

Argument  $x$  can be a Complex, Float, DoubleFloat, or Expression value or a series.

**acsch** ( $x$ ) returns the hyperbolic arccosecant of  $x$ .

**acschIfCan** ( $x$ ) returns **acsch** ( $x$ ) if possible, and "failed" otherwise.

**acsc** (*expression*)

**acscIfCan** (*expression*)

Argument  $x$  can be a Complex, Float, DoubleFloat, or Expression value or a series.

**acsc** ( $x$ ) returns the arccosecant of  $x$ .

**acscIfCan** ( $x$ ) returns **acsc** ( $x$ ) if possible, and "failed" otherwise.

**adaptive** ([*boolean*])

**adaptive** () tests whether plotting will be done adaptively.

**adaptive** (*true*) turns adaptive plotting on;

**adaptive** (*false*) turns it off. Note: this command can be expressed by the draw option *adaptive* ==  $b$ .

**addmod** (*integer, integer, integer*)

**addmod** ( $a, b, p$ ),  $0 \leq a, b < p > 1$ , means  $a + b \bmod p$ .

**airyAi** (*complexDoubleFloat*)

**airyBi** (*complexDoubleFloat*)

**airyAi** ( $x$ ) is the Airy function  $\text{Ai}(x)$  satisfying the differential equation  $\text{Ai}''(x) - x\text{Ai}(x) = 0$ .

**airyBi** ( $x$ ) is the Airy function  $\text{Bi}(x)$  satisfying the differential equation  $\text{Bi}''(x) - x\text{Bi}(x) = 0$ .

**Aleph** (*nonNegativeInteger*)

**Aleph** ( $n$ ) provides the named (infinite) cardinal number.

**algebraic?** ()

**algebraic?** ( $a$ ) tests whether an element  $a$  is algebraic with respect to the ground field  $F$ .

**alphabetic** ()

**alphabetic?** (*character*)

**alphabetic** () returns the class of all characters  $ch$  for which **alphabetic?** ( $ch$ ) is *true*.

**alphabetic?** ( $ch$ ) tests if  $ch$  is an alphabetic character a...z, A...B.

**alphanumeric** ()

**alphanumeric?** (*character*)

**alphanumeric** () returns the class of all characters  $ch$  for which **alphanumeric?** ( $ch$ ) is *true*.

**alphanumeric?** ( $ch$ ) tests if  $ch$  is either an alphabetic character a...z, A...B or digit 0...9.

**alternating** (*integer*)

**alternating** ( $n$ ) is the cycle index of the alternating group of degree  $n$ . See CycleIndicators for details.

**alternatingGroup** (*listOfIntegers*)

**alternatingGroup** ( $li$ ) constructs the alternating group acting on the integers in the list  $li$ . If  $n$  is odd, the generators are in general the  $(n-2)$ -cycle  $(li.3, \dots, li.n)$  and the 3-cycle  $(li.1, li.2, li.3)$ . If  $n$  is even, the generators are the product of the 2-cycle  $(li.1, li.2)$  with  $(n-2)$ -cycle  $(li.3, \dots, li.n)$  and the 3-cycle  $(li.1, li.2, li.3)$ . Duplicates in the list will be removed.

**alternatingGroup** ( $n$ ) constructs the alternating group  $A_n$  acting on the integers  $1, \dots, n$ . If  $n$  is odd, the generators are in general the  $(n-2)$ -cycle  $(3, \dots, n)$  and the 3-cycle  $(1, 2, 3)$ . If  $n$  is even, the generators are the product of the 2-cycle  $(1, 2)$  with  $(n-2)$ -cycle  $(3, \dots, n)$  and the 3-cycle  $(1, 2, 3)$  if  $n$  is even.

**alternative?** ()

**alternative?** ()\$ $F$  tests if

$2\text{associator}(a, a, b) = 0 = 2\text{associator}(a, b, b)$  for all  $a, b$  in the algebra  $F$ . Note: in general,  $2a = 0$  does not necessarily imply  $a = 0$ .

**and** (*boolean, boolean*)

$x$  **and**  $y$  returns the logical *and* of two BitAggregates  $x$  and  $y$ .

$b_1$  **and**  $b_2$  returns the logical *and* of Boolean  $b_1$  and  $b_2$ .

$si_1$  **and**  $si_2$  returns the bit-by-bit logical *and* of the small integers  $si_1$  and  $si_2$ .

See also OutputForm.

**approximants** (*continuedFraction*)

**approximants** ( $cf$ ) returns the stream of approximants of the continued fraction  $cf$ . If the continued fraction is finite, then the stream will be infinite and periodic with period 1.

**approximate** (*series, integer*)

**approximate** ( $s, r$ ) returns a truncated power series as an expression in the coefficient domain of the power series.

For example, if  $R$  is Fraction Polynomial Integer and  $s$  is a series over  $R$ , then **approximate**( $s, r$ ) returns the power series  $s$  truncated after the exponent  $r$  term.

**approximate** ( $p\text{AdicInteger}, integer$ )

**approximate** ( $x, n$ ),  $x$  a  $p$ -adic integer, returns an integer  $y$  such that  $y = x \bmod p^n$  when  $n$  is positive, and 0 otherwise.

**approxNthRoot** (*integer, nonNegativeInteger*)

**approxNthRoot** ( $n, p$ ) returns an integer approximation  $i$  to  $n^{1/p}$  such that  $-1 < i - n^{1/p} < 1$ .

**approxSqrt** (*integer*)

**approxSqrt** ( $n$ ) returns an integer approximation  $i$  to  $\sqrt{n}$  such that  $-1 < i - \sqrt{n} < 1$ . A variable precision Newton iteration is used with running time  $O(\log(n)^2)$ .

**areEquivalent?** (*listOfMatrices, listOfMatrices* [, *randomElements?*, *numberOfTries*])

**areEquivalent?** ( $lM, lM', b, numberOfTries$ ) tests whether the two lists of matrices, assumed of the same square shape, can be simultaneously conjugated by a non-singular matrix. If these matrices represent the same group generators, the representations are equivalent. The algorithm tries *numberOfTries* times to create elements in the generated algebras in the same fashion. For details, consult Browse.

**areEquivalent?** ( $aG0, aG1, numberOfTries$ ) calls

**areEquivalent?** ( $aG0, aG1, true, 25$ ).

**areEquivalent?** ( $aG0, aG1$ ) calls **areEquivalent?** ( $aG0, aG1, true, 25$ ).

**argscript** (*symbol, listOfOutputForms*)

**argscript** ( $f, [o_1, \dots, o_n]$ ) returns a new symbol with  $f$  with scripts  $o_1, \dots, o_n$ .

**argument** (*complexExpression*)

**argument** ( $c$ ) returns the angle made by complex expression  $c$  with the positive real axis.

**arity** (*basicOperator*)

**arity** ( $op$ ) returns  $n$  if  $op$  is  $n$ -ary, and "failed" if  $op$  has arbitrary arity.

**asec** (*expression*)

**asecIfCan** (*expression*)

Argument  $x$  can be a Complex, Float, DoubleFloat, or Expression value or a series.

**asec** ( $x$ ) returns the arcsecant of  $x$ .

**asecIfCan** ( $x$ ) returns **asec** ( $x$ ) if possible, and "failed" otherwise.

**asech** (*expression*)

**asechIfCan** (*expression*)

Argument  $x$  can be a Complex, Float, DoubleFloat, or Expression value or a series.

**asech** ( $x$ ) returns the hyperbolic arcsecant of  $x$ .

**asechIfCan** ( $x$ ) returns **asech** ( $x$ ) if possible, and "failed" otherwise.

**asin** (*expression*)

**asinIfCan** (*expression*)

Argument  $x$  can be a Complex, Float, DoubleFloat, or

Expression value or a series.

**asin** ( $x$ ) returns the arcsine of  $x$ .

**asinIfCan** ( $x$ ) returns **asin** ( $x$ ) if possible, and "failed" otherwise.

**asinh** (*expression*)

**asinhIfCan** (*expression*)

Argument  $x$  can be a Complex, Float, DoubleFloat, or Expression value or a series.

**asinh** ( $x$ ) returns the hyperbolic arcsine of  $x$ .

**asinhIfCan** ( $x$ ) returns **asinh** ( $x$ ) if possible, and "failed" otherwise.

**assign** (*outputForm, outputForm*)

**assign** ( $f, g$ ) creates an OutputForm object for the assignment  $f:=g$ .

**associates?** (*element, element*)

**associates?** ( $x, y$ ) tests whether  $x$  and  $y$  are associates, that is, that  $x$  and  $y$  differ by a unit factor.

**associative?** ()

**associative?** ()\$ $F$  tests if multiplication in  $F$  is associative, where  $F$  is a FiniteRankNonAssociativeAlgebra.

**associatorDependence** ()

**associatorDependence** ()\$ $F$  computes associator identities for  $F$ . Consult FiniteRankNonAssociativeAlgebra using Browse for details..

**associator** (*element, element, element*)

**associator** ( $a, b, c$ ) returns  $(ab)c - a(bc)$ , where  $a, b$ , and  $c$  are all members of a domain of category NonAssociateRng.

**assoc** (*element, associationList*)

**assoc** ( $k, al$ ) returns the element  $x$  in the AssociationList  $al$  stored under key  $k$ , or "failed" if no such element exists.

**atan** (*expression* [, *phase*])

**atanIfCan** (*expression*)

Argument  $x$  can be a Complex, Float, DoubleFloat, or Expression value or a series.

**atan** ( $x$ ) returns the arctangent of  $x$ .

**atan** ( $x, y$ ) computes the arc tangent from  $x$  with phase  $y$ .

**atanIfCan** ( $x$ ) returns the **atan** ( $x$ ) if possible, and "failed" otherwise.

**atanh** (*expression*)

**atanhIfCan** (*expression*)

Argument  $x$  can be a Complex, Float, DoubleFloat, or Expression value or a series.

**atanh** ( $x$ ) returns the hyperbolic arctangent of  $x$ .

**atanhIfCan** ( $x$ ) returns **atanh** ( $x$ ) if possible, and "failed" otherwise.

**atom?** (*sExpression*)

**atom?** (*s*) tests if *x* is atomic, where *x* is an SExpression or OutputForm.

**antiCommutator** (*element*, *element*)

**antiCommutator** (*x*, *y*) returns  $xy + yx$ , where *x* and *y* are elements of a non-associative ring, possibly without identity. See NonAssociativeRng using Browse.

**antisymmetric?** (*matrix*)

**antisymmetric?** (*m*) tests if the matrix *m* is square and antisymmetric, that is,  $m_{i,j} = -m_{j,i}$  for all *i* and *j*.

**antisymmetricTensors** (*matrices*, *positiveInteger*)

**antisymmetricTensors** (*A*, *n*), where *A* is an *m* by *m* matrix, returns a matrix obtained by applying to *A* the irreducible, polynomial representation of the general linear group  $GL_m$  corresponding to the partition  $(1, 1, \dots, 1, 0, 0, \dots, 0)$  of *n*. A call to **error** occurs if *n* is greater than *m*. Note: this corresponds to the symmetrization of the representation with the sign representation of the symmetric group  $S_n$ . The carrier spaces of the representation are the antisymmetric tensors of the *n*-fold tensor product.

**antisymmetricTensors** (*lA*, *n*), where *lA* is a list of *m* by *m* matrices, similarly applies the representation of  $GL_m$  to each matrix *A* of *lA*, returning a list of matrices.

**any?** (*predicate*, *aggregate*)

**any?** (*pred*, *a*) tests if predicate **pred** (*x*) is *true* for any element *x* of aggregate *a*. Note: for collections, **any?**(*p*, *u*) = **reduce**(**or**, **map**(*p*, *u*), **false**, **true**).

**any** (*type*, *object*)

**any** (*type*, *object*) is a technical function for creating an *object* of Any. Argument *type* is a LISP form for the *type* of *object*.

**append** (*list*, *list*)

**append** (*l*<sub>1</sub>, *l*<sub>2</sub>) appends the elements of list *l*<sub>1</sub> onto the front of list *l*<sub>2</sub>. See also **concat**.

**axesColorDefault** ([*palette*])

**axesColorDefault** (*p*) sets the default color of the axes in a two-dimensional viewport to the palette *p*.

**axesColorDefault** () returns the default color of the axes in a two-dimensional viewport.

**back** (*queue*)

**back** (*q*) returns the element at the back of the queue, or calls **error** if *q* is empty.

**bag** ([*bag*])

**bag** ([*x*, *y*, ..., *z*]) creates a bag with elements *x*, *y*, ..., *z*.

**balancedBinaryTree** (*nonNegativeInteger*, *element*)

**balancedBinaryTree** (*n*, *s*) creates a balanced binary tree with *n* nodes, each with value *s*.

**base** (*group*)

**base** (*gp*) returns a base for the group *gp*. Consult PermutationGroup using Browse for details.

**basis** ()

**basis** ()\$*R* returns a fixed basis of *R* or a subspace of *R*. See FiniteAlgebraicExtensionField, FramedAlgebra, FramedNonAssociativeAlgebra using Browse for details.

**basisOfCenter** ()

**basisOfCenter** ()\$*R* returns a basis of the space of all *x* in *R* satisfying **commutator** (*x*, *a*) = 0 and **associator** (*x*, *a*, *b*) = **associator** (*a*, *x*, *b*) = **associator** (*a*, *b*, *x*) = 0 for all *a*, *b* in *R*. Domain *R* is a domain of category FramedNonAssociativeAlgebra.

**basisOfCentroid** ()

**basisOfCentroid** ()\$*R* returns a basis of the centroid of *R*, that is, the endomorphism ring of *R* considered as (*R*, *R*)-bimodule. Domain *R* is a domain of category FramedNonAssociativeAlgebra.

**basisOfCommutingElements** ()

**basisOfCommutingElements** ()\$*R* returns a basis of the space of all *x* of *R* satisfying **commutator** (*x*, *a*) = 0 for all *a* in *R*. Domain *R* is a domain of category FramedNonAssociativeAlgebra.

**basisOfLeftAnnihilator** (*element*)

**basisOfRightAnnihilator** (*element*)

These operations return a basis of the space of all *x* in *R* of category FramedNonAssociativeAlgebra, satisfying

**basisOfLeftAnnihilator** (*a*): 0 = *x**a*.

**basisOfRightAnnihilator** (*a*): 0 = *a**x*.

**basisOfNucleus** ()

**basisOfLeftNucleus** ()

**basisOfMiddleNucleus** ()

**basisOfRightNucleus** ()

Each operation returns a basis of the space of all *x* of *R*, a domain of category FramedNonAssociativeAlgebra, satisfying for all *a* and *b*:

**basisOfNucleus** ()\$*R*: **associator** (*x*, *a*, *b*) = **associator** (*a*, *x*, *b*) = **associator** (*a*, *b*, *x*) = 0;

**basisOfLeftNucleus** ()\$*R*: **associator** (*x*, *a*, *b*) = 0;

**basisOfMiddleNucleus** ()\$*R*: **associator** (*a*, *x*, *b*) = 0;

**basisOfRightNucleus** ()\$*R*: **associator** (*a*, *b*, *x*) = 0.

**basisOfLeftNucloid** ()

### **basisOfRightNucloid** ()

Each operation returns a basis of the space of endomorphisms of  $R$ , a domain of category `FramedNonAssociativeAlgebra`, considered as:

**basisOfLeftNucloid** (): a right module.

**basisOfRightNucloid** (): a left module.

Note: if  $R$  has a unit, the left and right nucloid coincide with the left and right nucleus.

### **belong?** (*operator*)

**belong?** ( $op$ )\$ $R$  tests if  $op$  is known as an operator to  $R$ . For example,  $R$  is an Expression domain or AlgebraicNumber.

### **bernoulli** (*integer*)

**bernoulli** ( $n$ ) returns the  $n^{\text{th}}$  Bernoulli number, that is,  $B(n, 0)$  where  $B(n, x)$  is the  $n^{\text{th}}$  Bernoulli polynomial.

**besselI** (*complexDoubleFloat*, *complexDoubleFloat*)

**besselJ** (*complexDoubleFloat*, *complexDoubleFloat*)

**besselK** (*complexDoubleFloat*, *complexDoubleFloat*)

**besselY** (*complexDoubleFloat*, *complexDoubleFloat*)

**besselI** ( $v, x$ ) is the modified Bessel function of the first kind,  $I(v, x)$ , satisfying the differential equation  $x^2 w''(x) + xw'(x) - (x^2 + v^2)w(x) = 0$ .

**besselJ** ( $v, x$ ) is the Bessel function of the second kind,  $J(v, x)$ , satisfying the differential equation  $x^2 w''(x) + xw'(x) + (x^2 - v^2)w(x) = 0$ .

**besselK** ( $v, x$ ) is the modified Bessel function of the first kind,  $K(v, x)$ , satisfying the differential equation  $x^2 w''(x) + xw'(x) - (x^2 + v^2)w(x) = 0$ . Note: The default implementation uses the relation  $K(v, x) = \pi/2(I(-v, x) - I(v, x))/\sin(v\pi)$  so is not valid for integer values of  $v$ .

**besselY** ( $v, x$ ) is the Bessel function of the second kind,  $Y(v, x)$ , satisfying the differential equation  $x^2 w''(x) + xw'(x) + (x^2 - v^2)w(x) = 0$ . Note: The default implementation uses the relation  $Y(v, x) = (J(v, x)\cos(v\pi) - J(-v, x))/\sin(v\pi)$  so is not valid for integer values of  $v$ .

**Beta** (*complexDoubleFloat*, *complexDoubleFloat*)

**Beta** ( $x, y$ ) is the Euler beta function,  $B(x, y)$ , defined by

**Beta** ( $x, y$ )  $\int_0^1 t^{x-1}(1-t)^{y-1}dt$ . Note: this function is defined by **Beta** ( $x, y$ ) =  $\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ .

### **binaryTournament** (*listOfElements*)

**binaryTournament** ( $ls$ ) creates a BinaryTournament tree with the elements of  $ls$  as values at the nodes.

### **binaryTree** (*value*)

**binaryTree** ( $x$ ) creates a binary tree consisting of one

node for which the **value** is  $x$  and the **left** and **right** subtrees are empty.

### **binary** (*various*)

**binary** ( $rn$ ) converts rational number  $rn$  to a binary expansion.

**binary** ( $op, [a_1, \dots, a_n]$ ) returns the input form corresponding to  $a_1 op \dots op a_n$ , where  $op$  and the  $a_i$ 's are of type InputForm.

### **binomial** (*integerNumber*, *integerNumber*)

**binomial** ( $x, y$ ) returns the binomial coefficient  $C(x, y) = x!/(y!(x-y)!)$ , where  $x \geq y \geq 0$ , the number of combinations of  $x$  objects taken  $y$  at a time. Arguments  $x$  and  $y$  can come from any Expression or IntegerNumberSystem domain.

### **bipolar** ( $x$ )

#### **bipolarCylindrical** ( $x$ )

**bipolar** ( $a$ ) returns a function for transforming bipolar coordinates to Cartesian coordinates; this function maps the point  $(u, v)$  to  $(x = a \sinh(v)/(\cosh(v) - \cos(u)), y = a \sin(u)/(\cosh(v) - \cos(u)))$ .

**bipolarCylindrical** ( $a$ ) returns a function for transforming bipolar cylindrical coordinates to Cartesian coordinates; this function maps the point  $(u, v, z)$  to  $(x = a \sinh(v)/(\cosh(v) - \cos(u)), y = a \sin(u)/(\cosh(v) - \cos(u)), z)$ .

### **biRank** (*element*)

**biRank** ( $x$ )\$ $R$ , where  $R$  is a domain of category `FramedNonAssociativeAlgebra`, returns the number of linearly independent elements among  $x, xb_i, b_i x, b_i x b_j, i, j = 1, \dots, n$ , where  $b = [b_1, \dots, b_n]$  is the fixed basis for  $R$ . Note: if  $R$  has a unit, then **doubleRank**, **weakBiRank** and **biRank** coincide.

### **bit?** (*integer*, *integer*)

**bit?** ( $i, n$ ) tests if the  $n^{\text{th}}$  bit of  $i$  is a 1.

### **bits** ()

**bits** () returns the precision of floats in bits. Also see **precision**.

### **blankSeparate** (*listOfOutputForms*)

**blankSeparate** ( $lo$ ), where  $lo$  is a list of objects of type OutputForm (normally unexposed), returns a single output form consisting of the elements of  $lo$  separated by blanks.

### **blue** ()

**blue** () returns the position of the blue hue from total hues.

### **bottom!** (*dequeue*)

**bottom!** ( $q$ ) removes then returns the element at the bottom (back) of the dequeue  $q$ .

**box** (*expression*)

**box** (*e*), where *e* is an expression, returns *e* with a box around it that prevents *e* from being evaluated when operators are applied to it. For example, **log** (1) returns 0, but **log** (**box**(1)) returns the formal kernel **log** (1).

**box** ( $f_1, \dots, f_n$ ), where the  $f_i$  are expressions, returns  $(f_1, \dots, f_n)$  with a box around them that prevents the  $f_i$  from being evaluated when operators are applied to them, and makes them applicable to a unary operator. For example, **atan** (**box**[*x*, 2]) returns the formal kernel **atan** (*x*, 2).

**box** (*o*), where *o* is an object of type `OutputForm` (normally unexposed), returns an output form enclosing *o* in a box.

**brace** (*outputForm*)

**brace** (*o*), where *o* is an object of type `OutputForm` (normally unexposed), returns an output form enclosing *o* in braces.

**bracket** (*outputForm*)

**bracket** (*o*), where *o* is an object of type `OutputForm` (normally unexposed), returns an output form enclosing *o* in brackets.

**branchPoint** (*element*)

**branchPointAtInfinity?** ()

**branchPoint?** (*a*)\$*F* tests if  $x = a$  is a branch point of the algebraic function field *F*.

**branchPointAtInfinity?** ()\$*F* tests if the algebraic function field *F* has a branch point at infinity.

**bright** (*color*)

**bright** (*c*) sets the shade of a hue, *c*, above dim but below pastel.

**bright** (*ls*) sets the font property of a list of strings *ls* to bold-face type.

**cap** (*symmetricPolynomial*, *symmetricPolynomial*)

**cap** ( $s_1, s_2$ ), introduced by Redfield, is the scalar product of two cycle indices, where the  $s_i$  are `SymmetricPolynomials` with rational number coefficients. See also **cup**. See `CycleIndicators` for details.

**cardinality** (*finiteSetAggregate*)

**cardinality** (*u*) returns the number of elements of *u*. Note: **cardinality**(*u*) = #*u*.

**car** (*sExpression*)

**car** (*se*) returns  $a_1$  when *se* is the `SExpression` object  $(a_1, \dots, a_n)$ .

**cdr** (*sExpression*)

**cdr** (*se*) returns  $(a_2, \dots, a_n)$  when *se* is the `SExpression` object  $(a_1, \dots, a_n)$ .

**ceiling** (*floatOrRationalNumber*)

Argument *x* is a floating point number or fraction of numbers.

**ceiling** (*x*) returns the smallest integral element above *x*.

**center** (*stringsOrSeries*)

**center** (*s*) returns the point about which the series *s* is expanded.

**center** (*ls*, *n*, *s*) takes a list of strings *ls*, and centers them within a list of strings which is *n* characters long. The remaining spaces are filled with strings composed of as many repetitions as possible of the last string parameter *s*.

**center** ( $s_1, n, s_2$ ) is equivalent to **center** ( $[s_1], n, s_2$ ).

**char** (*character*)

**char** (*i*) returns a `Character` object with integer code *i*.

Note: **ord**(**char**(*i*)) = *i*.

**char** (*s*) returns the unique character of a string *s* of length one.

**characteristic** ()

**characteristic** ()\$*R* returns the characteristic of ring *R*: the smallest positive integer *n* such that  $nx = 0$  for all *x* in the ring, or zero if no such *n* exists.

**characteristicPolynomial** (*matrix* [, *symbol*])

**characteristicPolynomial** (*a*) returns the characteristic polynomial of the regular representation of *a* with respect to any basis.

**characteristicPolynomial** (*m*) returns the characteristic polynomial of the matrix *m* expressed as polynomial with a new symbol as variable.

**characteristicPolynomial** (*m*, *sy*) is similar except that the resulting polynomial has variable *sy*.

**characteristicPolynomial** (*m*, *r*), where *r* is a member of the coefficient domain of matrix *m*, evaluates the characteristic polynomial at *r*. In particular, if *r* is the polynomial '*x*', then it returns the characteristic polynomial expressed as a polynomial in '*x*'.

**charClass** (*strings*)

**charClass** (*s*) creates a character class containing exactly the characters given in the string *s*.

**charClass** (*ls*) creates a character class which contains exactly the characters given in the list *ls* of strings.

**charthRoot** (*element*)

**charthRoot** (*r*), where *r* is an element of domain with

**characteristic**  $p \neq 0$ , returns the  $p^{\text{th}}$  root of *r*, or "failed" if none exists in the domain.

**charthRoot** (*f*)\$*R* takes the  $p^{\text{th}}$  root of finite field element *f*, where *p* is the characteristic of the finite field *R*. Note: such a root is always defined in finite fields.

**chebyshevT** (*positiveInteger*, *element*)

**chebyshevT** ( $n, x$ ) returns the  $n^{\text{th}}$  Chebyshev polynomial of the first kind,  $T_n(x)$ , defined by  $(1 - tx)/(1 - 2tx + t^2) = \sum_{n=0}^{\infty} T_n(x) t^n$ .

**children** (*recursiveAggregate*)

**children** ( $u$ ) returns a list of the children of aggregate  $u$ .

**chineseRemainder** (*listOfElements*, *listOfModuli*)

**chineseRemainder** (*integer*, *modulus*, *integer*, *modulus*)

**chineseRemainder** ( $lv, lm$ ) where  $lv$  is a list of values  $[v_1, \dots, v_n]$  and  $lm$  is a list of moduli  $[m_1, \dots, m_n]$ , returns  $m$  such that  $m = n_i \bmod p_i$ ; the  $p_i$  must be relatively prime.

**chineseRemainder** ( $n_1, p_1, n_2, p_2$ ) is equivalent to **chineseRemainder** ( $[n_1, n_2], [p_1, p_2]$ ), where all arguments are integers.

**clearDenominator** (*fraction*)

**clearDenominator** ( $[q_1, \dots, ]$ ) returns  $[p_1, \dots, ]$  such that  $q_i = p_i/d$  where  $d$  is a common denominator for the  $q_i$ 's.

**clearDenominator** ( $A$ ), where  $A$  is a matrix of fractions, returns matrix  $B$  such that  $A = B/d$  where  $d$  is a common denominator for the elements of  $A$ .

**clearDenominator** ( $p$ ) returns polynomial  $q$  such that  $p = q/d$  where  $d$  is a common denominator for the coefficients of polynomial  $p$ .

**clip** (*rangeOrBoolean*)

**clip** ( $b$ ) turns two-dimensional clipping on if  $b$  is *true*, and off if  $b$  is *false*. This command may be given as a draw option: **clip** == **b**.

**clip** ( $[a..b]$ ) defines the range for user-defined clipping. This command may be given as a draw option: **range** ==  $[a..b]$ .

**clipPointsDefault** (*[boolean]*)

**clipPointsDefault** () tests if automatic clipping is to be done.

**clipPointsDefault** ( $b$ ) turns on automatic clipping for  $b = \text{true}$ , and off if  $b = \text{false}$ . This command may be given as a draw option: **clip** == **b**.

**close** (*filename*)

**close** ( $v$ ) closes the viewport window of the given two-dimensional or three-dimensional viewport  $v$  and terminates the corresponding **Unix** process. Argument  $v$  is a member of domain **TwoDimensionalViewport** or **ThreeDimensionalViewport**.

**close!** (*filename*)

**close!** ( $fn$ ) returns the file  $fn$  closed to input and output.

**closedCurve?** (*threeSpace*)

**closedCurve?** ( $sp$ ) tests if the **ThreeSpace** object  $sp$  contains a single closed curve component.

**closedCurve** (*listsOfPoints* [, *listOfPoints*])

**closedCurve** ( $lpt$ ) returns a **ThreeSpace** object containing a single closed curve described by the list of points  $lpt$  of the form  $[p_0, p_1, \dots, p_n, p_0]$ .

**closedCurve** ( $sp$ ) returns a closed curve as a list of points, where  $sp$  must be a **ThreeSpace** object containing a single closed curve.

**closedCurve** ( $sp, lpt$ ) returns **ThreeSpace** object with the closed curve denoted by  $lpt$  added. Argument  $lpt$  is a list of points of the form  $[p_0, p_1, \dots, p_n, p_0]$ .

**coefficient** (*polynomialOrSeries*, *nonNegativeInteger*)

**coefficient** ( $p, n$ ) extracts the coefficient of the monomial with exponent  $n$  from polynomial  $p$ , or returns zero if exponent is not present.

**coefficient** ( $u, x, n$ ) returns the coefficient of variable  $x$  to the power  $n$  in  $u$ , a multivariate polynomial or series.

**coefficient** ( $u, [x_1, \dots, ], [n_1, \dots, ]$ ) returns the coefficient of  $x_1^{n_1} \dots x_k^{n_k}$  in  $u$ , a multivariate series or polynomial.

Also defined for domain **CliffordAlgebra** and categories **AbelianMonoidRing**, **FreeAbelianCategory**, and **MonogenicLinearOperator**.

**coefficient** ( $s, n$ ) returns the terms of total degree  $n$  of series  $s$  as a polynomial.

**coefficients** (*polynomialOrStream*)

**coefficients** ( $p$ ) returns the list of non-zero coefficients of polynomial  $p$  starting with the coefficient of the maximum degree.

**coefficients** ( $s$ ) returns a stream of coefficients  $[a_0, a_1, a_2, \dots]$  for the stream  $s$ :  $a_0 + a_1x + a_2x^2 + \dots$ .

Note: the entries of the stream may be zero.

**coerceImages** (*listOfElements*)

**coerceImages** ( $ls$ ) coerces the list  $ls$  to a permutation whose image is given by  $ls$  and whose preimage is fixed to be  $[1, \dots, n]$ . Note: **coerceImages** ( $ls$ ) = **coercePreimagesImages** ( $[1, \dots, n], ls$ ).

**coerceListOfPairs** (*listOfPairsOfElements*)

**coerceListOfPairs** ( $lls$ ) coerces a list of pairs  $lls$  to a permutation, or calls *error* if not consistent, that is, the set of the first elements coincides with the set of second elements.

**coercePreimagesImages** (*listOfListOfElements*)

**coercePreimagesImages** ( $lls$ ) coerces the representation  $lls$  of a permutation as a list of preimages and images to a permutation.

**coleman** (*listOfIntegers*, *listOfIntegers*, *listOfIntegers*)

**coleman** ( $\alpha, \beta, \pi$ ) generates the Coleman-matrix of a certain double coset of the symmetric group given by an representing element  $\pi$  and  $\alpha$  and  $\beta$ . The matrix has nonnegative entries, row sums  $\alpha$  and column sums  $\beta$ . Consult **SymmetricGroupCombinatoricFunctions** using



Browse for details.

**color** (*integer*)

**color** (*i*) returns a color of the indicated hue *i*.

**colorDef** (*viewPort, color, color*)

**colorDef** (*v, c<sub>1</sub>, c<sub>2</sub>*) sets the range of colors along the colormap so that the lower end of the colormap is defined by *c<sub>1</sub>* and the top end of the colormap is defined by *c<sub>2</sub>* for the given three-dimensional viewport *v*.

**colorFunction** (*smallFloatFunction*)

**colorFunction** (*fn*) specifies the color for three-dimensional plots. Function *fn* can take one to three DoubleFloat arguments and always returns a DoubleFloat value. If one argument, the color is based upon the *z*-component of plot. If two arguments, the color is based on two parameter values. If three arguments, the color is based on the *x*, *y*, and *z* components. This command may be given as a draw option: **colorFunction** == *fn*.

**column** (*matrix, positiveInteger*)

**column** (*M, j*) returns the *j*<sup>th</sup> column of the matrix or TwoDimensionalArrayCategory object *M*, or calls **error** if the index is outside the proper range.

**commaSeparate** (*listOfOutputForms*)

**commaSeparate** (*lo*), where *lo* is a list of objects of type OutputForm (normally unexposed), returns an output form which separates the elements of *lo* by commas.

**commonDenominator** (*fraction*)

**commonDenominator** (*[q<sub>1</sub>, ..., ]*) returns a common denominator for the *q<sub>i</sub>*'s.

**commonDenominator** (*A*), where *A* is a matrix of fractions, returns a common denominator for the elements of *A*.

**commonDenominator** (*p*) returns a common denominator for the coefficients of polynomial *p*.

**commutative?** ()

**commutative?** ()\$*R* tests if multiplication in the algebra *R* is commutative.

**commutator** (*groupElement, groupElement*)

**commutator** (*p, q*) computes **inv** (*p*) \* **inv** (*q*) \* *p* \* *q* where *p* and *q* are members of a Group domain.

**commutator** (*a, b*) returns *ab* - *ba* where *a* and *b* are members of a NonAssociativeRing domain.

**compactFraction** (*partialFraction*)

**compactFraction** (*u*) normalizes the partial fraction *u* to a compact representation where it has only one fractional term per prime in the denominator.

**comparison** (*basicOperator, property*)

**comparison** (*op, p*) attaches *p* as the "%less?" property to *op*. If *op1* and *op2* have the same name, and one of them has a "%less?" property *p*, then *p*(*op1, op2*) is called to decide whether *op1* < *op2*.

**compile** (*symbol, listOfTypes*)

**compile** (*f, [T<sub>1</sub>, ..., T<sub>n</sub>]*) forces the interpreter to compile the function with name *f* with signature (*T<sub>1</sub>, ..., T<sub>n</sub>*) -> *T*, where *T* is a type determined by type analysis of the function body of *f*. If the compilation is successful, the operation returns the name *f*. The operation calls **error** if *f* is not defined beforehand in the interpreter, or if the *T<sub>i</sub>*'s are not valid types, or if the compiler fails. See also **function**, **interpret**, **lambda**, and **compiledFunction**.

**compiledFunction** (*expression, symbol [ , symbol]*)

Argument *expression* may be of any type that is coercible to type InputForm (most commonly used types). These functions must be package called to define the type of the function produced.

**compiledFunction** (*expr, x*)\$*P*, where *P* is MakeUnaryCompiledFunction(*E, S, T*), returns an anonymous function of type *ST* defined by defined by *x* ↦ *expr*. The anonymous function is compiled and directly applicable to objects of type *S*.

**compiledFunction** (*expr, x, y*)\$*P*, where *P* is MakeBinaryCompiledFunction(*E, A, B, T*) returns an anonymous function of type (*A, B*) → *T* defined by (*x, y*) ↦ *expr*. The anonymous function is compiled and is then directly applicable to objects of type (*A, B*). See also **compile**, **function**, and **lambda**.

**complement** (*finiteSetElement*)

**complement** (*u*) returns the complement of the finite set *u*, that is, the set of all values not in *u*.

**complementaryBasis** (*vector*)

**complementaryBasis** (*b<sub>1</sub>, ..., b<sub>n</sub>*) returns the complementary basis (*b'<sub>1</sub>, ..., b'<sub>n</sub>*) of (*b<sub>1</sub>, ..., b<sub>n</sub>*) for a domain of category FunctionFieldCategory.

**complete** (*streamOrInteger*)

**complete** (*u*) causes all terms of a stream or continued fraction *u* to be computed. If not called on a finite stream or continued fraction, this function will compute until interrupted.

**complete** (*n*) is the *n*<sup>th</sup> complete homogeneous symmetric function expressed in terms of power sums. Alternatively, it is the cycle index of the symmetric group of degree *n*. See CycleIndicators for details.

**completeEchelonBasis** (*vectorOfVectors*)

**completeEchelonBasis** (*vv*) returns a completed basis

from *vv*, a vector of vectors of domain elements. Consult RepresentationPackage2 using Browse for details.

**complex** (*element*, *element*)

**complex** (*x*, *y*) creates the complex expression  $x + \%i*y$ .

**complexEigenvalues** (*matrix*, *precision*)

**complexEigenvalues** (*m*, *eps*) computes the eigenvalues of the matrix *m* to precision *eps*, chosen as a float or a rational number so as to agree with the type of the coefficients of the matrix *m*.

**complexEigenvectors** (*matrix*, *precision*)

**complexEigenvectors** (*m*, *eps*) (*m*, a matrix) returns a list of records, each containing a complex eigenvalue, its algebraic multiplicity, and a list of associated eigenvectors. All results are expressed as complex floats or rationals with precision *eps*.

**complexElementary** (*expression* [, *symbol*])

**complexElementary** (*e*) rewrites *e* in terms of the two fundamental complex transcendental elementary functions: *log*, *exp*.

**complexElementary** (*e*, *x*) does the same but only rewrites kernels of *e* involving *x*.

**complexExpand** (*integrationResult*)

**complexExpand** (*ir*), where *ir* is an IntegrationResult, returns the expanded complex function corresponding to *ir*.

**complexIntegrate** (*expression*, *variable*)

**complexIntegrate** (*f*, *x*) returns  $\int f(x)dx$  where *x* is viewed as a complex variable.

**complexLimit** (*expression*, *equation*)

**complexLimit** (*f*(*x*), *x* = *a*) computes the complex limit of *f* as its argument *x* approaches *a*.

**complexNormalize** (*expression* [, *symbol*])

**complexNormalize** (*e*) rewrites *e* using the least possible number of complex independent kernels.

**complexNormalize** (*e*, *x*) rewrites *e* using the least possible number of complex independent kernels involving *x*.

**complexNumeric** (*expression* [, *positiveInteger*])

**complexNumeric** (*u*) returns a complex approximation of *u*, where *u* is a polynomial or an expression.

**complexNumeric** (*u*, *n*) does the same but requires accuracy to be up to *n* decimal places.

**complexRoots** (*rationalFunctions* [, *options*])

**complexRoots** (*rf*, *eps*) finds all the complex solutions of a univariate rational function with rational number coefficients with precision given by *eps*. The complex

solutions are returned either as rational numbers or floats depending on whether *eps* is a rational number or a float.

**complexRoots** (*lrf*, *lv*, *eps*) similarly finds all the complex solutions of a list of rational functions with rational number coefficients with respect the variables appearing in *lv*. Solutions are computed to precision *eps* and returned as a list of values corresponding to the order of variables in *lv*.

**complexSolve** (*eq*, *x*)

See **solve** (*u*, *v*).

**complexZeros** (*polynomial*, *floatOrRationalNumber*)

**complexZeros** (*poly*, *eps*) finds the complex zeros of the univariate polynomial *poly* to precision *eps*. Solutions are returned either as complex floats or rationals depending on the type of *eps*.

**components** (*threeSpace*)

**components** (*sp*) takes the ThreeSpace object *sp*, and returns a list of ThreeSpace objects, each having a single component.

**composite** (*polynomial*, *polynomial*)

**composite** (*p*, *q*), for polynomials *p* and *q*, returns *f* if  $p = f(q)$ , and "failed" if no such *f* exists.

**composite** (*lsp*), where *lsp* is a list [*sp*<sub>1</sub>, *sp*<sub>2</sub>, ..., *sp*<sub>*n*</sub>] of ThreeSpace objects, returns a single ThreeSpace object containing the union of all objects in the parameter list grouped as a single composite.

**composites** (*threeSpace*)

**composites** (*sp*) takes the ThreeSpace object *sp* and returns a list of ThreeSpace objects, one for each single composite of *sp*. If *sp* has no defined composites (composites need to be explicitly created), the list returned is empty. Note that not all the components need to be part of a composite.

**concat** (*aggregate*, *aggregate*)

**concat!** (*aggregate*, *aggregate*)

**concat** (*u*, *x*) returns list *u* with additional element *x* at the end. Note: equivalent to **concat** (*u*, [*x*]).

**concat** (*u*, *v*) returns an aggregate consisting of the elements of *u* followed by the elements of *v*.

**concat** (*u*), where *u* is a list of aggregates [*a*, *b*, ..., *c*], returns a single aggregate consisting of the elements of *a* followed by those of *b* followed ... by the elements of *c*.

**concat!** (*u*, *x*), where *u* is extensible, destructively adds element *x* to the end of aggregate *u*; if *u* is a stream, it must be finite.

**concat!** (*u*, *v*) destructively appends *v* to the end of *u*; if *u* is a stream, it must be finite.

**conditionP** (*matrix*)

**conditionP** (*M*), given a matrix *M* representing a homogeneous system of equations over a field *F* with

**characteristic**  $p$ , returns a non-zero vector whose  $p^{\text{th}}$  power is a non-trivial solution to these equations, or "failed" if no such vector exists.

**conditionsForIdempotents** ()

**conditionsForIdempotents** () determines a complete list of polynomial equations for the coefficients of idempotents with respect to the  $R$ -module basis. See also **FramedNonAssociativeAlgebra** for an alternate definition.

**conical** (*smallFloat*, *smallFloat*)

**conical** ( $a, b$ ) returns a function of two parameters for mapping conical coordinates to Cartesian coordinates. The function maps the point  $(\lambda, \mu, \nu)$  to  $x = \lambda\mu\nu/(ab)$ ,  
 $y = \lambda/a\sqrt{((\mu^2 - a^2)(\nu^2 - a^2)/(a^2 - b^2))}$ ,  
 $z = \lambda/b\sqrt{((\mu^2 - b^2)(\nu^2 - b^2)/(b^2 - a^2))}$ .

**conjugate** (*element* [, *element*])

**conjugate** ( $u$ ) returns the conjugate of a complex, quaternion, or octonian expression  $u$ . For example, if  $u$  is the complex expression  $x + \%iy$ , **conjugate** ( $u$ ) returns  $x - \%iy$ .

**conjugate** ( $pt$ ) returns the conjugate of a partition  $pt$ . See **PartitionsAndPermutations** using **Browse**.

**conjugate** ( $p, q$ ) returns **inv** ( $q$ )  $* p * q$  for elements  $p$  and  $q$  of a group. Note: this operation is called *right action by conjugation*.

**conjugates** (*streamOfPartitions*)

**conjugates** ( $lp$ ) is the stream of conjugates of a stream of partitions  $lp$ .

**connect** (*twoDimensionalViewport*, *positiveInteger*, *string*)

**connect** ( $v, n, s$ ) displays the lines connecting the graph points in field  $n$  of the two-dimensional viewport  $v$  if  $s = "on"$ , and does not display the lines if  $s = "off"$ .

**constant** (*variableOrfunction*)

**constantLeft** (*function*, *element*)

**constantRight** (*function*, *element*)

These operations add an argument to a function and must be package-called from package  $P$  as indicated. See also **curry**, **curryLeft**, and **curryRight**.

**constant** ( $f$ )\$ $P$  returns the function  $g$  such that  $g(a) = f()$ , where function  $f$  has type  $\rightarrow C$  and  $a$  has type  $A$ . The function must be package-called from  $P = \text{MappingPackage2}(A, C)$ .

**constantRight** ( $f$ )\$ $P$  returns the function  $g$  such that  $g(a, b) = f(a)$ , where function  $f$  has type  $A \rightarrow C$  and  $b$  has type  $B$ . This function must be package-called from  $P = \text{MappingPackage3}(A, B, C)$ .

**constantLeft** ( $f$ )\$ $P$  returns the function  $g$  such that  $g(a, b) = f(b)$ , where function  $f$  has type  $B \rightarrow C$  and  $a$  has type  $A$ . The function must be package-called from  $P =$

**MappingPackage3**( $A, B, C$ ).

**constant** ( $x$ ) tells the pattern matcher that  $x$  should match the symbol ' $x$ ' and no other quantity, or calls **error** if  $x$  is not a symbol.

**constantOperator** (*property*)

**constantOpIfCan** ( $f$ )

**constantOperator** ( $f$ ) returns a nullary operator  $op$  such that  $op()$  always evaluate to  $f$ .

**constantOpIfCan** ( $op$ ) returns  $f$  if  $op$  is the constant nullary operator always returning  $f$ , and "failed" otherwise.

**construct** (*element*, ...)

**construct** ( $x, y, \dots, z$ )\$ $R$  returns the collection of elements  $x, y, \dots, z$  from domain  $R$  ordered as given. This is equivalently written as  $[x, y, \dots, z]$ . The qualification  $R$  may be omitted for domains of type **List**. Infinite tuples such as  $[x_i \text{ for } i \text{ in } 1..]$  are converted to a **Stream** object.

**cons** (*element*, *listOrStream*)

**cons** ( $x, u$ ), where  $u$  is a list or stream, creates a new list or stream whose **first** element is  $x$  and whose **rest** is  $u$ . Equivalent to **concat** ( $x, u$ ).

**content** (*polynomial* [, *symbol*])

**content** ( $p$ ) returns the greatest common divisor (**gcd**) of the coefficients of polynomial  $p$ .

**content** ( $p, v$ ), where  $p$  is a multivariate polynomial type, returns the *gcd* of the coefficients of the polynomial  $p$  viewed as a univariate polynomial with respect to the variable  $v$ . For example, if  $p = 7x^2y + 14xy^2$ , the *gcd* of the coefficients with respect to  $x$  is  $7y$ .

**continuedFraction** (*fractionOrFloat* [, *options*])

**continuedFraction** ( $f$ ) converts the floating point number  $f$  to a reduced continued fraction.

**continuedFraction** ( $r$ ) converts the fraction  $r$  with components of type  $R$  to a continued fraction over  $R$ .

**continuedFraction** ( $r, s, s'$ ), where  $s$  and  $s'$  are streams over a domain  $R$ , constructs a continued fraction in the following way: if  $s = [a1, a2, \dots]$  and  $s' = [b1, b2, \dots]$  then the result is the continued fraction  $r + a1/(b1 + a2/(b2 + \dots))$ .

**contract** (*idealOrTensors* [, *options*])

**contract** ( $I, lvar$ ) contracts the ideal  $I$  to the polynomial ring  $F[lvar]$ .

**contract** ( $t, i, j$ ) is the contraction of tensor  $t$  which sums along the  $i^{\text{th}}$  and  $j^{\text{th}}$  indices. For example, if  $r = \text{contract}(t, 1, 3)$  for a rank 4 tensor  $t$ , then  $r$  is the rank 2 ( $= 4 - 2$ ) tensor given by  $r(i, j) = \sum_{h=1}^{\text{dim}} t(h, i, h, j)$ .

**contract** ( $t, i, s, j$ ) is the inner product of tensors  $s$  and  $t$  which sums along the  $k_1$ st index of  $t$  and the  $k_2$ st index of  $s$ . For example, if  $r = \text{contract}(s, 2, t, 1)$  for rank 3 tensors

$s$  and  $t$ , then  $r$  is the rank 4 ( $= 3 + 3 - 2$ ) tensor given by  $r(i, j, k, l) = \sum_{h=1}^{\dim} s(i, h, j) t(h, k, l)$ .

**contractSolve** (*equation, symbol*)

**contractSolve** ( $eq, x$ ) finds the solutions expressed in terms of radicals of the equation of rational functions  $eq$  with respect to the symbol  $x$ . The result contains new symbols for common subexpressions in order to reduce the size of the output. Alternatively, an expression  $u$  may be given for  $eq$  in which case the equation  $eq$  is defined as  $u = 0$

**controlPanel** (*viewport, string*)

**controlPanel** ( $v, s$ ) displays the control panel of the given two-dimensional or three-dimensional viewport  $v$  if  $s = "on"$ , or hides the control panel if  $s = "off"$ .

**convergents** (*continuedFraction*)

**convergents** ( $cf$ ) returns the stream of the convergents of the continued fraction  $cf$ . If the continued fraction is finite, then the stream will be finite.

**coordinate** (*curveOrSurface, nonNegativeInteger*)

**coordinate** ( $u, n$ ) returns the  $n^{\text{th}}$  coordinate function for the curve or surface  $u$ . See `ParametericPlaneCurve`, `ParametericSpaceCurve`, and `ParametericSurface`, using `Browse`.

**coordinates** (*pointOrvector[, basis]*)

**coordinates** ( $pt$ ) specifies a change of coordinate systems of point  $pt$ . This option is expressed in the form `coordinates == pt`.

The following operations return a matrix representation of the coordinates of an argument vector  $v$  of the form  $[v_1 \dots v_n]$  with respect to the basis  $a$  domain  $R$ . The coordinates of  $v_i$  are contained in the  $i^{\text{th}}$  row of the matrix returned.

**coordinates** ( $v, b$ ) returns the matrix representation with respect to the basis  $b$  for vector  $v$  of elements from domain  $R$  of category `FiniteRankNonAssociativeAlgebra` or `FiniteRankAlgebra`. If a second argument is not given, the basis is taken to be the fixed basis of  $R$ .

**coordinates** ( $v$ )\$ $R$ , returns a matrix representation for  $v$  with respect to a fixed basis for domain  $R$  of category `FiniteAlgebraicExtensionField`, `FramedNonAssociativeAlgebra`, or `FramedAlgebra`.

**copies** (*integer, string*)

**copies** ( $n, s$ ) returns a string composed of  $n$  copies of string  $s$ .

**copy** (*aggregate*)

**copy** ( $u$ ) returns a top-level (non-recursive) copy of an aggregate  $u$ . Note: for lists, `copy(u) == [x for x in u]`.

**copyInto!** (*aggregate, aggregate, integer*)

**copyInto!** ( $u, v, p$ ) returns linear aggregate  $u$  with elements of  $u$  replaced by the successive elements of  $v$  starting at index  $p$ . Arguments  $u$  and  $v$  can be elements of any `FiniteLinearAggregate`.

**cos** (*expression*)

**cosIfCan** (*expression*)

Argument  $x$  can be a Complex, Float, DoubleFloat, or Expression value or a series.

**cos** ( $x$ ) returns the cosine of  $x$ .

**cosIfCan** ( $x$ ) returns **cos** ( $x$ ) if possible, and "failed" otherwise.

**cos2sec** (*expression*)

**cos2sec** ( $e$ ) converts every **cos** ( $u$ ) appearing in  $e$  into  $1/\sec(u)$ .

**cosh2sech** (*expression*)

**cosh2sech** ( $e$ ) converts every **cosh** ( $u$ ) appearing in  $e$  into  $1/\sech(u)$ .

**cosh** (*expression*)

**coshIfCan** (*expression*)

Argument  $x$  can be a Complex, Float, DoubleFloat, or Expression value or a series.

**cosh** ( $x$ ) returns the hyperbolic cosine of  $x$ .

**coshIfCan** ( $x$ ) returns **cosh** ( $x$ ) if possible, and "failed" otherwise.

**cot** (*expression*)

Argument  $x$  can be a Complex, Float, DoubleFloat, or Expression value or a series.

**cot** ( $x$ ) returns the cotangent of  $x$ .

**cotIfCan** ( $x$ ) returns **cot** ( $x$ ) if possible, and "failed" otherwise.

**cot2tan** (*expression*)

**cot2tan** ( $e$ ) converts every **cot** ( $u$ ) appearing in  $e$  into  $1/\tan(u)$ .

**cot2trig** (*expression*)

**cot2trig** ( $e$ ) converts every **cot** ( $u$ ) appearing in  $e$  into  $\cos(u)/\sin(u)$ .

**coth** (*expression*)

**cothIfCan** (*expression*)

Argument  $x$  can be a Complex, Float, DoubleFloat, or Expression value or a series.

**coth** ( $x$ ) returns the hyperbolic cotangent of  $x$ .

**cothIfCan** ( $x$ ) returns **coth** ( $x$ ) if possible, and "failed" otherwise.

**coth2tanh** (*expression*)

**coth2tanh** ( $e$ ) converts every **coth** ( $u$ ) appearing in  $e$  into  $1/\tanh(u)$ .

**coth2trigh** (*expression*)

**coth2trigh** (*expression*) converts every  $\coth(u)$  appearing in  $e$  into  $\cosh(u)/\sinh(u)$ .

**count** (*predicate, aggregate*)

**count** (*pred, u*) returns the number of elements  $x$  in  $u$  such that **pred** ( $x$ ) is *true*. For collections, **count**(*p, u*) = **reduce**(+, [1 for  $x$  in  $u$  | **p**( $x$ )], 0).

**count** ( $x, u$ ) returns the number of occurrences of  $x$  in  $u$ . For collections, **count**( $x, u$ ) = **reduce**(+, [ $x=y$  for  $y$  in  $u$ ], 0).

**countable?** (*cardinal*)

**countable?** ( $u$ ) tests if the cardinal number  $u$  is countable, that is, if  $u \leq \aleph_0$ .

**createThreeSpace** ()

**createThreeSpace** ()\$ThreeSpace( $R$ ) creates a ThreeSpace object capable of holding point, curve, mesh components or any combination of the three. The ring  $R$  is usually DoubleFloat. If you do not package call this function, DoubleFloat is assumed.

**createThreeSpace** ( $s$ ) creates a ThreeSpace object containing objects pre-defined within some SubSpace  $s$ .

**createGenericMatrix** (*nonNegativeInteger*)

**createGenericMatrix** ( $n$ ) creates a square matrix of dimension  $n$  whose entry at the  $i$ -th row and  $j$ -th column is the indeterminate  $x_{i,j}$  (double subscripted). See RepresentationPackage1 using Browse.

**createIrreduciblePoly** (*nonNegativeInteger*)

**createIrreduciblePoly** ( $n$ )\$FFPOLY( $GF$ ) generates a monic irreducible polynomial of degree  $n$  over the finite field  $GF$ .

**createNormalElement** ()

**createNormalElement** ()\$ $F$  computes a normal element over the ground field of a finite algebraic extension field  $F$ , that is, an element  $a$  such that

$a^q, 0 \leq i < \text{extensionDegree}()$ \$ $F$  is an  $F$ -basis, where  $q$  is the size of the ground field.

**createNormalPrimitivePoly** (*element*)

**createNormalPrimitivePoly** ( $n$ )\$FFPOLY( $GF$ ) generates a normal and primitive polynomial of degree  $n$  over the field  $GF$ .

**createPrimitiveElement** ()

**createPrimitiveElement** ()\$ $F$  computes a generator of the (cyclic) multiplicative group of a finite field  $F$ .

**createRandomElement** (*listOfMatrices, matrix*)

**createRandomElement** ( $lm, m$ ) creates a random element of the group algebra generated by  $lm$ , where  $lm$  is a list of matrices and  $m$  is a matrix. See RepresentationPackage2 using Browse.

**csc2sin** (*expression*)

**csc2sin** (*expression*) converts every **csc** ( $u$ ) appearing in  $f$  into  $1/\sin(u)$ .

**csch2sinh** (*expression*)

**csch2sinh** (*expression*) converts every **csch** ( $u$ ) appearing in  $f$  into  $1/\sinh(u)$ .

**csch** (*expression*)

**cschIfCan** (*expression*)

Argument  $x$  can be a Complex, Float, DoubleFloat, or Expression value or a series.

**csch** ( $x$ ) returns the hyperbolic cosecant of  $x$ .

**cschIfCan** ( $x$ ) returns **csch** ( $x$ ) if possible, and "failed" otherwise.

**cscIfCan** (*expression*)

Argument  $x$  can be a Complex, Float, DoubleFloat, or Expression value or a series.

**csc** ( $x$ ) returns the cosecant of  $x$ .

**cscIfCan** ( $x$ ) returns **csc** ( $x$ ) if possible, and "failed" otherwise.

**cup** (*symmetricPolynomial, symmetricPolynomial*)

**cup** ( $s_1, s_2$ ), introduced by Redfield, is the scalar product of two cycle indices, where the  $s_i$  are of type SymmetricPolynomial with rational number coefficients. See also **cap**. See CycleIndicators for details.

**curry** (*function*)

**curryLeft** (*function, element*)

**curryRight** (*function, element*)

These functions drop an argument from a function.

**curry** ( $f, a$ ) returns the function  $g$  such that  $g() = f(a)$ , where function  $f$  has type  $A \rightarrow C$  and element  $a$  has type  $A$ .

**curryRight** ( $f, b$ ) returns the function  $g$  such that  $g(a) = f(a, b)$ , where function  $f$  has type  $(A, B) \rightarrow C$  and element  $b$  has type  $B$ .

**curryLeft** ( $f, a$ ) is the function  $g$  such that  $g(b) = f(a, b)$ , where function  $f$  has type  $(A, B) \rightarrow C$  and element  $a$  has type  $A$ .

See also **constant**, **constantLeft**, and **constantRight**.

**curve** (*listOfPoints[, options]*)

**curve** ( $[p_0, p_1, \dots, p_n]$ ) creates a space curve defined by the list of points  $p_0$  through  $p_n$  and returns a ThreeSpace object whose component is the curve.

**curve** ( $sp$ ) checks to see if the ThreeSpace object  $sp$  is composed of a single curve defined by a list of points; if so, the list of points defining the curve is returned. Otherwise, the operation calls **error**.

**curve** ( $c_1, c_2$ ) creates a plane curve from two component functions  $c_1$  and  $c_2$ . See ComponentFunction using Browse. **curve**( $sp, [[p_0], [p_1], \dots, [p_n]]$ ) adds a space curve defined by a list of points  $p_0$  through  $p_n$  to a ThreeSpace object  $sp$ .

Each  $p_i$  is from a domain **PointDomain** ( $m, R$ ), where  $R$  is the Ring over which the point elements are defined and  $m$  is the dimension of the points.

**curve** ( $s, [p_0, p_1, \dots, p_n]$ ) adds the space curve component designated by the list of points  $p_0$  through  $p_n$  to the ThreeSpace object  $sp$ .

**curve** ( $c_1, c_2, c_3$ ) creates a space curve from three component functions  $c_1$ ,  $c_2$ , and  $c_3$ .

**curve?** ( $threeSpace$ )

**curve?** ( $sp$ ) tests if the ThreeSpace object  $sp$  contains a single curve object.

**curveColor** ( $float$ )

**curveColor** ( $p$ ) specifies a color index for two-dimensional graph curves from the palette  $p$ . This option is expressed in the form `curveColor == p`.

**cycle** ( $listOfPermutations$ )

**cycle** ( $ls$ ) converts a cycle  $ls$ , a list with no repetitions, to the permutation, which maps  $ls.i$  to  $ls.(i + 1)$  (index modulo the length of the list).

**cycleEntry** ( $aggregate$ )

**cycleEntry** ( $u$ ) returns the head of a top-level cycle contained in aggregate  $u$ , or **empty** () if none exists.

**cycleLength** ( $aggregate$ )

**cycleLength** ( $u$ ) returns the length of a top-level cycle contained in aggregate  $u$ , or 0 if  $u$  has no such cycle.

**cyclePartition** ( $permutation$ )

**cyclePartition** ( $p$ ) returns the cycle structure of a permutation  $p$  including cycles of length 1. The permutation is assumed to be a member of **Permutation**( $S$ ) where  $S$  is a finite set.

**cycleRagits** ( $radixExpansion$ )

**cycleRagits** ( $rx$ ) returns the cyclic part of the ragits of the fractional part of a radix expansion. For example, if  $x = 3/28 = 0.10714285714285\dots$ , then **cycleRagits**( $x$ ) = [7, 1, 4, 2, 8, 5].

**cycleSplit!** ( $aggregate$ )

**cycleSplit!** ( $u$ ) splits the recursive aggregate (for example, a list)  $u$  into two aggregates by dropping off the cycle. The value returned is the cycle entry, or *nil* if none exists. For example, if  $w = \text{concat}(u, v)$  is the cyclic list where  $v$  is the head of the cycle, **cycleSplit!** ( $w$ ) will drop  $v$  off  $w$ . Thus  $w$  is destructively changed to  $u$ , and  $v$  is returned.

**cycles** ( $listOfListOfElements$ )

**cycles** ( $lls$ ) coerces a list of list of cycles  $lls$  to a permutation. Each cycle, represented as a list  $ls$  with no repetitions, is coerced to the permutation, which maps  $ls.i$  to  $ls.(i + 1)$  (index modulo the length of the list). These

permutations are then multiplied.

**cycleTail** ( $aggregate$ )

**cycleTail** ( $u$ ) returns the last node in the cycle of a recursive aggregate (for example, a list)  $u$ , or empty if none exists.

**cyclic** ( $integer$ )

**cyclic** ( $n$ ) returns the cycle index of the cyclic group of degree  $n$ . **CycleIndicators** for details.

**cyclic?** ( $aggregate$ )

**cyclic?** ( $u$ ) tests if recursive aggregate (for example, a list)  $u$  has a cycle.

**cyclicGroup** ( $listOfIntegers$ )

**cyclicGroup** ( $[i_1, \dots, i_k]$ ) constructs the cyclic group of order  $k$  acting on the list of integers  $i_1, \dots, i_k$ . Note: duplicates in the list will be removed.

**cyclicGroup** ( $positiveInteger$ )

**cyclicGroup** ( $n$ ) constructs the cyclic group of order  $n$  acting on the integers  $1, \dots, n$ ,  $n > 0$ .

**cyclicSubmodule** ( $listOfMatrices, vector$ )

**cyclicSubmodule** ( $lm, v$ ), where  $lm$  is a list of  $n$  by  $n$  square matrices and  $v$  is a vector of size  $n$ , generates a basis in echelon form. Consult **RepresentationPackage2** using **Browse** for details.

**cylindrical** ( $point$ )

**cylindrical** ( $pt$ ) transforms  $pt$  from polar coordinates to Cartesian coordinates, by mapping the point  $(r, theta, z)$  to  $x = r \cos(theta)$ ,  $y = r \sin(theta)$ ,  $z$ .

**D** ( $expression$  [,  $options$ ])

**D** ( $x$ ) returns the derivative of  $x$ . This function is a simple differential operator where no variable needs to be specified.

**D** ( $x, [s_1, \dots, s_n]$ ) computes successive partial derivatives, that is,  $D(\dots D(x, s_1) \dots, s_n)$ .

**D** ( $u, x$ ) computes the partial derivative of  $u$  with respect to  $x$ .

**D** ( $u, deriv$  [,  $n$ ]) differentiates  $u$   $n$  times using a derivation which extends  $deriv$  on  $R$ . Argument  $n$  defaults to 1.

**D** ( $p, d, x'$ ) extends the  $R$ -derivation  $d$  to an extension  $R$  in  $R[x]$  where  $Dx$  is given by  $x'$ , and returns  $Dp$ .

**D** ( $x, [s_1, \dots, s_n], [n_1, \dots, n_m]$ ) computes multiple partial derivatives, that is,  $D(\dots D(x, s_1, n_1) \dots, s_n, n_m)$ .

**D** ( $u, x, n$ ) computes multiple partial derivatives, that is,  $n^{\text{th}}$  derivative of  $u$  with respect to  $x$ .

**D** ( $of$  [,  $n$ ]), where  $of$  is an object of type **OutputForm** (normally unexposed), returns an output form for the  $n^{\text{th}}$  derivative of  $f$ , for example,  $f'$ ,  $f''$ ,  $f'''$ ,  $f^{iv}$ , and so on.

**D()** $\$A$  provides the operator corresponding to the derivation in the differential ring  $A$ .

**dark** (*color*)

**dark** (*color*) returns the shade of the indicated hue of *color* to its lowest value.

**ddFact** (*polynomial*, *primeInteger*)

**ddFact** ( $q, p$ ) computes a distinct degree factorization of the polynomial  $q$  modulo the prime  $p$ , that is, such that each factor is a product of irreducibles of the same degrees.

**decimal** (*rationalNumber*)

**decimal** ( $rn$ ) converts a rational number  $rn$  to a decimal expansion.

**declare** (*listOfInputForms*)

**declare** ( $t$ ) returns a name  $f$  such that  $f$  has been declared to the interpreter to be of type  $t$ , but has not been assigned a value yet.

**decreasePrecision** (*integer*)

**decreasePrecision** ( $n$ ) $\$R$  decreases the current **precision** by  $n$  decimal digits.

**definingPolynomial** ()

**definingPolynomial** () $\$R$  returns the minimal polynomial for a MonogenicAlgebra domain  $R$ , that is, one which **generator** () $\$R$  satisfies.

**definingPolynomial** ( $x$ ) returns an expression  $p$  such that  $p(x) = 0$ , where  $x$  is an AlgebraicNumber or an object of type Expression.

**degree** (*polynomial* [, *symbol*])

The meaning of **degree**( $u$ ,  $s$ ) depends on the type of  $u$ .

if  $u$  is a polynomial: **degree** ( $u, x$ ) returns the degree of polynomial  $u$  with respect to the variable  $x$ . Similarly, **degree** ( $u, lv$ ), where  $lv$  is a list of variables, returns a list of degrees of polynomial  $u$  with respect to each of the variables in  $lv$ .

if  $u$  is an element of an AbelianMonoidRing or GradedModule domain: **degree** ( $u$ ) returns the maximum of the exponents of the terms of  $u$ .

if  $u$  is a series: **degree** ( $u$ ) returns the degree of the leading term of  $u$ .

if  $u$  is an element of a domain of category ExtensionField: **degree** ( $u$ ) returns the degree of the minimal polynomial of  $u$  if  $u$  is algebraic with respect to the ground field  $F$ , and **%infinity** otherwise.

if  $u$  is a permutation: **degree** ( $u$ ) returns the number of points moved by the permutation.

if  $u$  is a permutation group: **degree** ( $u$ ) returns the number of points moved by all permutations of the group  $u$ . For additional information on **degree**, consult Browse.

**delete** (*aggregate*, *integerOrSegment*)

**delete** ( $u, i$ ) returns a copy of linear aggregate  $u$  with the  $i^{\text{th}}$  element deleted. Note: for lists, **delete**( $a, i$ ) == **concat**( $a(0..i-1)$ ,  $a(i+1..)$ ).

**delete** ( $u, i..j$ ) returns a copy of  $u$  with the  $i^{\text{th}}$  through  $j^{\text{th}}$  element deleted. Note: for lists, **delete**( $a, i..j$ ) = **concat**( $a(0..i-1)$ ,  $a(j+1..)$ ).

**delete!** ( $u, i$ ) destructively deletes the  $i^{\text{th}}$  element of  $u$ . **delete!** ( $u, i..j$ ) destructively deletes elements  $u.i$  through  $u.j$  of  $u$ .

**deleteProperty** (*basicOperator*, *string*)

**deleteProperty** ( $op, s$ ) destructively removes property  $s$  from  $op$ .

**denom** (*expression*)

**denominator** (*expression*)

Argument  $x$  can be from domain Fraction( $R$ ) for some domain  $R$ , or of type Expression if the result is of type  $R$ .

**denom** ( $x$ ) returns the denominator of  $x$  as an object of domain  $R$ ; if  $x$  is of type Expression, it returns an object of domain SMP( $R$ , Kernel(Expression  $R$ )).

**denominator** ( $x$ ) returns the denominator of  $x$  as an element of Fraction( $R$ ); if  $x$  is of type Expression, it returns an object of domain Expression( $R$ ).

**denominators** (*fractionOrContinuedFraction*)

**denominator** (*frac*) is the denominator of the fraction *frac*.

**denominators** (*cf*) returns the stream of denominators of the approximants of the continued fraction  $x$ . If the continued fraction is finite, then the stream will be finite.

**depth** (*stack*)

**depth** ( $st$ ) returns the number of elements of stack  $st$ .

**dequeue** (*queue*)

**dequeue!** (*queue*)

**dequeue** ( $[x, y, \dots, z]$ ) creates a dequeue with first (top or front) element  $x$ , second element  $y$ , ..., and last (bottom or back) element  $z$ .

**dequeue!** ( $q$ ) destructively extracts the first (top) element from queue  $q$ . The element previously second in the queue becomes the first element. A call to **error** occurs if  $q$  is empty.

**derivationCoordinates** (*vectorOfElements*, *derivationFunction*)

**derivationCoordinates** ( $v, '$ ) returns a matrix  $M$  such that  $v' = Mv$ . Argument  $v$  is a vector of elements from  $R$ , a domain of category MonogenicAlgebra over a ring  $R$ . Argument  $'$  is a derivation function defined on  $R$ .

**derivative** (*basicOperator* [, *property*])

**derivative** ( $op$ ) returns the value of the "%diff" property

of  $op$  if it has one, and "failed" otherwise.

**derivative** ( $op, dprop$ ) attaches  $dprop$  as the "%diff" property of  $op$ . Note: if  $op$  has a "%diff" property  $f$ , then applying a derivation  $D$  to  $op(a)$  returns  $f(a)D(a)$ .

Argument  $op$  must be unary.

**derivative** ( $op, [f_1, \dots, f_n]$ ) attaches  $[f_1, \dots, f_n]$  as the "%diff" property of  $op$ . Note: if  $op$  has such a "%diff" property, then applying a derivation  $D$  to  $op(a_1, \dots, a_n)$  returns  $f_1(a_1, \dots, a_n)D(a_1) + \dots + f_n(a_1, \dots, a_n)D(a_n)$ . See also  $D$ .

**destruct** ( $sExpression$ )

**destruct** ( $se$ ), where  $se$  is the  $SExpression$   $(a_1, \dots, a_n)$ , returns the list  $[a_1, \dots, a_n]$ .

**determinant** ( $matrix$ )

**determinant** ( $m$ ) returns the determinant of the matrix  $m$ , or calls **error** if the matrix is not square. Note: the underlying coefficient domain of  $m$  is assumed to have a commutative "\*".

**diagonal** ( $matrix$ )

**diagonal** ( $m$ ), where  $m$  is a square matrix, returns a vector consisting of the diagonal elements of  $m$ .

**diagonal** ( $f$ ), where  $f$  is a function of type  $(A, A) \rightarrow T$  is the function  $g$  such that  $g(a) = f(a, a)$ . See MappingPackage for related functions.

**diagonal?** ( $matrix$ )

**diagonal?** ( $m$ ) tests if the matrix  $m$  is square and diagonal.

**diagonalMatrix** ( $listOfElements$ )

**diagonalMatrix** ( $l$ ), where  $l$  is a list or vector of elements, returns a (square) diagonal matrix with those elements of  $l$  on the diagonal.

**diagonalMatrix** ( $[m_1, \dots, m_k]$ ) creates a block diagonal matrix  $M$  with block matrices  $m_1, \dots, m_k$  down the diagonal, with 0 block matrices elsewhere.

**diagonalProduct** ( $matrix$ )

**diagonalProduct** ( $m$ ) returns the product of the elements on the diagonal of the matrix  $m$ .

**dictionary** ()

**dictionary** ()\$ $R$  creates an empty dictionary of type  $R$ .

**dictionary** ( $[x, y, \dots, z]$ ) creates a dictionary consisting of entries  $x, y, \dots, z$ .

**difference** ( $setAggregate, element$ )

**difference** ( $u, x$ ) returns the set aggregate  $u$  with element  $x$  removed.

**difference** ( $u, v$ ) returns the set aggregate  $w$  consisting of elements in set aggregate  $u$  but not in set aggregate  $v$ .

**differentialVariables** ( $differentialPolynomial$ )

**differentialVariables** ( $p$ ) returns a list of differential indeterminates occurring in a differential polynomial  $p$ .

**differentiate** ( $expression[, options]$ )

See **D**.

**digamma** ( $complexDoubleFloat$ )

**digamma** ( $x$ ) is the function,  $\psi(x)$ , defined by  $\psi(x) = \Gamma'(x)/\Gamma(x)$ . Argument  $x$  is either a small float or a complex small float.

**digit** ()

**digit** () returns the class of all characters for which **digit?** is *true*.

**digit?** ( $character$ )

**digit?** ( $ch$ ) tests if character  $c$  is a digit character, that is, one of 0..9.

**digits** ( $[positiveInteger]$ )

**digits** () returns the current precision of floats in numbers of digits.

**digits** ( $n$ ) set the **precision** of floats to  $n$  digits.

**digits** ( $x$ ) returns a stream of  $p$ -adic digits of  $p$ -adic integer  $n$ . See PAdicInteger using Browse.

**dihedral** ( $integer$ )

**dihedral** ( $n$ ) is the cycle index of the dihedral group of degree  $n$ .

**dihedralGroup** ( $listOfIntegers$ )

**dihedralGroup** ( $[i_1, \dots, i_k]$ ) constructs the dihedral group of order  $2k$  acting on the integers  $i_1, \dots, i_k$ . Note: duplicates in the list will be removed.

**dihedralGroup** ( $n$ ) constructs the dihedral group of order  $2n$  acting on integers  $1, \dots, n$ .

**dilog** ( $expression$ )

**dilog** ( $x$ ) returns the dilogarithm of  $x$ , that is,  $\int \log(x)/(1-x)dx$ .

**dim** ( $color$ )

**dim** ( $c$ ) sets the shade of a hue  $c$ , above dark but below bright.

**dimension** ( $[various]$ )

**dimension** ()\$ $R$  returns the dimensionality of the vector space or rank of Lie algebra  $R$ .

**dimension** ( $I$ ) gives the dimension of the ideal  $I$ .

**dimension** ( $s$ ) returns the dimension of the point category  $s$ .

**dioSolve** ( $equation$ )

**dioSolve** ( $eq$ ) computes a basis of all minimal solutions for a linear homomogeneous Diophantine equation  $eq$ , then all



minimal solutions of the inhomogeneous equation. Alternatively, an expression  $u$  may be given for  $eq$  in which case the equation  $eq$  is defined as  $u = 0$ .

**directory** (*filename*)

**directory** ( $f$ ) returns the directory part of the file name.

**directProduct** (*vector*)

**directProduct** ( $v$ ) converts the vector  $v$  to become a direct product

**discreteLog** (*finiteFieldElement*)

**discreteLog** ( $a$ )\$ $F$  computes the discrete logarithm of  $a$  with respect to **primitiveElement** ()\$ $F$  of the field  $F$ .

**discreteLog** (*finiteFieldElement*, *finiteFieldElement*)

**discreteLog** ( $b, a$ ) computes  $s$  such that  $b^s = a$  if such an  $s$  exists.

**discriminant** (*polynomial* [, *symbol*])

**discriminant** ( $p$  [,  $x$ ]) returns the discriminant of the polynomial  $p$  with respect to the variable  $x$ . If  $x$  is univariate, the second argument may be omitted.

**discriminant** ()\$ $R$  returns

**determinant** (**traceMatrix**()\$ $R$ ) of a FramedAlgebra domain  $R$ .

**discriminant** ( $[v_1, \dots, v_n]$ ) returns

**determinant** (*traceMatrix* ( $[v_1, \dots, v_n]$ )) where the  $v_i$  each have  $n$  elements.

**display** (*text* [, *width*])

**display** ( $t$  [,  $w$ ]), where  $t$  is either IBM SCRIPT Formula Format or  $\text{\TeX}$  text, outputs  $t$  so that each line has length  $\leq w$ . The default value of  $w$  is that length set by the system command )set output length.

**display** ( $op, f$ ) attaches  $f$  as the "%display" property of  $op$ .

**display** ( $op$ ) returns the "%display" property of  $op$  if it has one attached, and "failed" otherwise.

Value  $f$  either has type  $\text{OutputForm} \rightarrow \text{OutputForm}$  or else  $\text{List}(\text{OutputForm}) \rightarrow \text{OutputForm}$ . Argument  $op$  must be unary. Note: if  $op$  has a "%display" property  $f$  of the former type, then  $op(a)$  gets converted to  $\text{OutputForm}$  as  $f(a)$ . If  $f$  has the latter type, then  $op(a_1, \dots, a_n)$  gets converted to  $\text{OutputForm}$  as  $f(a_1, \dots, a_n)$ .

**distance** (*aggregate*, *aggregate*)

**distance** ( $u, v$ ), where  $u$  and  $v$  are recursive aggregates (for example, lists) returns the path length (an integer) from node  $u$  to  $v$ .

**distdfact** (*polynomial*, *boolean*)

**distdfact** ( $p$ , *squareFreeFlag*) produces the complete factorization of the polynomial  $p$  returning an internal data structure. If argument *squareFreeFlag* is true, the polynomial is assumed square free.

**distribute** (*expression* [,  $f$ ])

**distribute** ( $f$  [,  $g$ ]) expands all the kernels in  $f$  that contain  $g$  in their arguments and that are formally enclosed by a **box** or a **paren** expression. By default,  $g$  is the list of all kernels in  $f$ .

**divide** (*element*, *element*)

**divide** ( $x, y$ ) divides  $x$  by  $y$  producing a record containing a *quotient* and *remainder*, where the remainder is smaller (see **sizeLess?**) than the divisor  $y$ .

**divideExponents** (*polynomial*, *nonNegativeInteger*)

**divideExponents** ( $p, n$ ) returns a new polynomial resulting from dividing all exponents of the polynomial  $p$  by the non negative integer  $n$ , or "failed" if no exponent is exactly divisible by  $n$ .

**divisors** (*integer*)

**divisors** ( $i$ ) returns a list of the divisors of integer  $i$ .

**domain** (*typeAnyObject*)

**domain** ( $a$ ) returns the type of the original object that was converted to Any as object of type  $\text{SExpression}$

**domainOf** (*typeAnyObject*)

**domainOf** ( $a$ ) returns a printable form of the type of the original type of  $a$ , an object of type Any.

**dot** (*vector*, *vector*)

**dot** ( $v_1, v_2$ ) computes the inner product of the vectors  $v_1$  and  $v_2$ , or calls **error** if  $x$  and  $y$  are not of the same length.

**dot** ( $of$ ), where  $of$  is an object of type  $\text{OutputForm}$  (normally unexposed), returns an output form with one dot overhead ( $\dot{x}$ ).

**doubleRank** (*element*)

**doubleRank** ( $x$ ), where  $x$  is an element of a domain  $R$  of category  $\text{FramedNonAssociativeAlgebra}$ , determines the number of linearly independent elements in  $b_1x, \dots, b_nx$ , where  $b = [b_1, \dots, b_n]$  is the fixed basis for  $R$ .

**doublyTransitive?** ()

**doublyTransitive?** ( $p$ ) tests if polynomial  $p$ , is irreducible over the field  $K$  generated by its coefficients, and if  $p(X)/(X - a)$  is irreducible over  $K(a)$  where  $p(a) = 0$ .

**draw** (*functionOrExpression*, *range* [, *options*])

$f$ ,  $g$ , and  $h$  below denote user-defined functions which map one or more  $\text{DoubleFloat}$  values to a  $\text{DoubleFloat}$  value.

**draw** ( $f, a..b$ ) draws the two-dimensional graph of  $y = f(x)$  as  $x$  ranges from **min** ( $a, b$ ) to **max** ( $a, b$ ).

**draw** (*curve* ( $f, g$ ),  $a..b$ ) draws the two-dimensional graph of the parametric curve  $x = f(t), y = g(t)$  as  $t$  ranges from **min** ( $a, b$ ) to **max** ( $a, b$ ).

**draw** ( $f, a..b, c..d$ ) draws the three-dimensional graph of  $z = f(x, y)$  as  $x$  ranges from **min** ( $a, b$ ) to **max** ( $a, b$ ) and  $y$  ranges from **min** ( $c, d$ ) to **max** ( $c, d$ ).

**draw** ( $curve(f, g, h), a..b$ ) draws a three-dimensional graph of the parametric curve  $x = f(t), y = g(t), z = h(t)$  as  $t$  ranges from **min** ( $a, b$ ) to **max** ( $a, b$ ).

**draw** ( $surface(f, g, h), a..b, c..d$ ) draws the three-dimensional graph of the parametric surface  $x = f(u, v), y = g(u, v), z = h(u, v)$  as  $u$  ranges from **min** ( $a, b$ ) to **max** ( $a, b$ ) and  $v$  ranges from **min** ( $c, d$ ) to **max** ( $c, d$ ).

Arguments  $f, g$ , and  $h$  below denote an Expression involving the variables indicated as arguments. For example,  $f(x, y)$  denotes an expression involving the variables  $x$  and  $y$ .

**draw** ( $f(x), x = a..b$ ) draws the two-dimensional graph of  $y = f(x)$  as  $x$  ranges from **min** ( $a, b$ ) to **max** ( $a, b$ ).

**draw** ( $curve(f(t), g(t)), t = a..b$ ) draws the two-dimensional graph of the parametric curve  $x = f(t), y = g(t)$  as  $t$  ranges from **min** ( $a, b$ ) to **max** ( $a, b$ ).

**draw** ( $f(x, y), x = a..b, y = c..d$ ) draws the three-dimensional graph of  $z = f(x, y)$  as  $x$  ranges from **min** ( $a, b$ ) to **max** ( $a, b$ ) and  $y$  ranges from **min** ( $c, d$ ) to **max** ( $c, d$ ).

**draw** ( $curve(f(t), g(t), h(t)), t = a..b$ ) draws the three-dimensional graph of the parametric curve  $x = f(t), y = g(t), z = h(t)$  as  $t$  ranges from **min** ( $a, b$ ) to **max** ( $a, b$ ).

**draw** ( $surface(f(u, v), g(u, v), h(u, v)), u = a..b, v = c..d$ ) draws the three-dimensional graph of the parametric surface  $x = f(u, v), y = g(u, v), z = h(u, v)$  as  $u$  ranges from **min** ( $a, b$ ) to **max** ( $a, b$ ) and  $v$  ranges from **min** ( $c, d$ ) to **max** ( $c, d$ ).

Each of the **draw** operations optionally take options given as extra arguments.

**adaptive**== **true** turns on adaptive plotting.

**clip**== **true** turns on two-dimensional clipping.

**colorFunction**==  $f$  specifies the color based on a function.

**coordinates**==  $p$  specifies a change of coordinate systems of point  $p$ : *bipolar*, *bipolarCylindrical*, *conical*, *elliptic*, *ellipticCylindrical*, *oblateSpheroidal*, *parabolic*, *parabolicCylindrical*, *paraboloidal*, *prolateSpheroidal*, *spherical*, and *toroidal*.

**curveColor**==  $p$  specifies a color index for two-dimensional graph curves from the palette  $p$ .

**pointColor**==  $p$  specifies a color index for two-dimensional graph points from the palette  $p$ .

**range**== [ $a..b$ ] provides a user-specified range for implicit curve plots.

**space**==  $sp$  adds the current graph to ThreeSpace object  $sp$ .

**style**==  $s$  specifies the drawing style in which the graph will be plotted: *wire*, *solid*, *shade*, *smooth*.

**title**==  $s$  titles the graph with string  $s$ .

**toScale**== **true** causes the graph to be drawn to scale.

**tubePoints**==  $n$  specifies the number of points  $n$  defining the circle which creates the tube around a three-dimensional curve. The default value is 6.

**tubeRadius**==  $r$  specifies a Float radius  $r$  for a tube plot around a three-dimensional curve.

**unit**== [ $f_1, f_2$ ] marks off the units of a two-dimensional graph in increments  $f_1$  along the x-axis,  $f_2$  along the y-axis.

**var1Steps**==  $n$  indicates the number of subdivisions  $n$  of the first range variable.

**var2Steps**==  $n$  indicates the number of subdivisions  $n$  of the second range variable.

**drawToScale** ( $[boolean]$ )

**drawToScale** () tests if plots are currently to be drawn to scale.

**drawToScale** (*true*) causes plots to be drawn to scale.

**drawToScale** (*false*) causes plots to be drawn to fill up the viewport window. The default setting is *false*.

**duplicates** (*dictionary*)

**duplicates** ( $d$ ) returns a list of values which have duplicates in  $d$

**Ei** (*variable*)

**Ei** ( $x$ ) returns the exponential integral of  $x$ :  $\int exp(x)/xdx$ .

**eigenMatrix** (*matrix*)

**eigenMatrix** ( $A$ ) returns the matrix  $B$  such that  $BA(inverse\ B)$  is diagonal, or "failed" if no such  $B$  exists.

**eigenvalues** (*matrix*)

**eigenvalues** ( $A$ ), where  $A$  is a matrix with rational function coefficients, returns the eigenvalues of the matrix  $A$  which are expressible as rational functions over the rational numbers.

**eigenvector** (*eigenvalue, matrix*)

**eigenvector** (*eigval, A*) returns the eigenvectors belonging to the eigenvalue *eigval* for the matrix  $A$ .

**eigenvectors** (*matrix*)

**eigenvectors** ( $A$ ) returns the eigenvalues and eigenvectors for the matrix  $A$ . The rational eigenvalues and the corresponding eigenvectors are explicitly computed. The non-rational eigenvalues are defined via their minimal polynomial. Their corresponding eigenvectors are expressed in terms of a "generic" root of this polynomial.

**element?** (*polynomial, ideal*)

**element?** ( $f, I$ ) tests if the polynomial  $f$  belongs to the ideal  $I$ .

**elementary** (*integer*)

**elementary** ( $n$ ) is the  $n^{\text{th}}$  elementary symmetric function expressed in terms of power sums. See `CycleIndicators` for details.

**elliptic** (*scaleFactor*)

**elliptic** ( $r$ ) returns a function for transforming elliptic coordinates to Cartesian coordinates. The function returned will map the point  $(u, v)$  to  $x = r \cosh(u) \cos(v)$ ,  $y = r \sinh(u) \sin(v)$ .

**ellipticCylindrical** (*scaleFactor*)

**ellipticCylindrical** ( $r$ ) returns a function for transforming elliptic cylindrical coordinates to Cartesian coordinates as a function of the scale factor  $r$ . The function returned will map the point  $(u, v, z)$  to  $x = r \cosh(u) \cos(v)$ ,  $y = r \sinh(u) \sin(v)$ ,  $z$ .

**elt** (*structure, various* [ , ...])

**elt** ( $u, v$ ), usually written as  $u.v$  or  $u(v)$ , regards the structure  $u$  as a function and applies structure  $u$  to argument  $v$ . Many types export **elt** with multiple arguments; **elt** ( $u, v, w, \dots$ ) is generally written  $u(v, w, \dots)$ . The interpretation of  $u$  depends on its type. If  $u$  is:

an indexed aggregate such as a list, stream, vector, or string:  $u.i$ ,  $1 \leq i \leq \text{maxIndex}(u)$ , is equivalently written  $u(i)$  and returns the  $i^{\text{th}}$  element of  $u$ . Also,  $u(i, y)$  returns  $u(i)$  if  $i$  is an appropriate index for  $u$ , and  $y$  otherwise.

a linear aggregate:  $u(i..j)$  returns the aggregate of elements of  $u(k)$  for  $k = i, i+1, \dots, j$  in that order.

a basic operator:  $u(x)$  applies the unary operator  $u$  to  $x$ ; similarly,  $u.x_1, \dots, x_n$  applies the  $n$ -ary operator  $u$  to  $x_1, \dots, x_n$ . Also,  $u(x, y)$ ,  $u(x, y, z)$ , and  $u(x, y, z, w)$  respectively apply the binary, ternary, or 4-ary operator  $u$  to arguments.

a univariate polynomial or rational function:  $u(y)$  evaluates the rational function or polynomial with the distinguished variable replaced by the value of  $y$ ; this value may either be another rational function or polynomial or a member of the underlying coefficient domain.

a list:  $u.first$  is equivalent to **first** ( $u$ ) and returns the first element of list  $u$ . Also,  $u.last$  is equivalent to **last** ( $u$ ) and returns the last element of list  $u$ . Both of these call **error** if  $u$  is the empty list. Similarly,  $u.rest$  is equivalent to **rest** ( $u$ ) and returns the list  $u$  beginning at its second element, or calls **error** if  $u$  has less than two elements.

a library:  $u(\text{name})$  returns the entry in the library

stored under the key *name*.

a linear ordinary differential operator:  $u(x)$  applies the differential operator  $u$  to the value  $x$ .

a matrix or two-dimensional array:  $u(i, j[x])$ ,  $1 \leq i \leq \text{nrows}(u)$ ,  $1 \leq j \leq \text{ncols}(m)$ , returns the element in the  $i^{\text{th}}$  row and  $j^{\text{th}}$  column of the matrix  $m$ . If the indices are out of range and an extra argument  $x$  is provided, then  $x$  is returned; otherwise, **error** is called. Also,  $u([i_1, \dots, i_m], [j_1, \dots, j_m])$  returns the  $m$ -by- $n$  matrix consisting of elements  $u(i_k, j_l)$  of  $u$ . a permutation group:  $u(i)$  returns the  $i$ -th generator of the group  $u$ .

a point:  $u.i$  returns the  $i^{\text{th}}$  component of the point  $u$ .

a rewrite rule:  $u(f[n])$  applies rewrite rule  $u$  to expression  $f$  at most  $n$  times, where  $n = \infty$  by default. When the left-hand side of  $u$  matches a subexpression of  $f$ , the subexpression is replaced by the right-hand side of  $u$  producing a new  $f$ . After  $n$  iterations or when no further match occurs, the transformed  $f$  is returned.

a ruleset:  $u(f[n])$  applies ruleset  $u$  to expression  $f$  at most  $n$  times, where  $n = \infty$  by default. Similar to last case, except that on each iteration, each rule in the ruleset is applied in turn in attempt to find a match.

an SExpression ( $a_1, \dots, a_n \ . \ b$ ) (where  $b$  denotes the **cdr** of the last node):  $u.i$  returns  $a_i$ ; similarly  $u.[i_1, \dots, i_m]$  returns  $(a_{i_1}, \dots, a_{i_m})$ .

a univariate series:  $u(r)$  returns the coefficient of the term of degree  $r$  in  $u$ .

a symbol:  $u[a_1, \dots, a_n]$  returns  $u$  subscripted by  $a_1, \dots, a_n$ .

a cartesian tensor:  $u(r)$  gives a component of a rank 1 tensor;  $u([i_1, \dots, i_n])$  gives a component of a rank  $n$  tensor;  $u()$  gives the component of a rank 0 tensor.

Also:  $u(i, j)$ ,  $u(i, j, k)$ , and  $u(i, j, k, l)$  gives a component of a rank 2, 3, and 4 tensors respectively.

See also `QuadraticForm`, `FramedNonAssociativeAlgebra`, and `FunctionFieldCategory`.

**empty** ()

**empty** ()\$ $R$  creates an aggregate of type  $R$  with 0 elements.

**empty?** (*aggregate*)

**empty?** ( $u$ ) tests if aggregate  $u$  has 0 elements.

**endOfFile?** (*file*)

**endOfFile?** ( $f$ ) tests whether the file  $f$  is positioned after the end of all text. If the file is open for output, then this test always returns *true*.

**enqueue!** (*value, queue*)

**enqueue!** ( $x, q$ ) inserts  $x$  into the queue  $q$  at the back end.

**enterPointData** (*space*, *listOfPoints*)

**enterPointData** (*s*, [*p*<sub>0</sub>, *p*<sub>1</sub>, ..., *p*<sub>*n*</sub>]) adds a list of points from *p*<sub>0</sub> through *p*<sub>*n*</sub> to the ThreeSpace *s*, and returns the index of the start of the list.

**entry?** (*value*, *aggregate*)

**entry?** (*x*, *u*), where *u* is an indexed aggregate (such as a list, vector, or string), tests if *x* equals *u*.*i* for some index *i*.

**epilogue** (*formattedObject*)

**epilogue** (*t*) extracts the epilogue section of an IBM SCRIPT Formula Format or T<sub>E</sub>X formatted object *t*.

**eq** (*sExpression*, *sExpression*)

**eq** (*s*, *t*), for SExpressions *s* and *t* returns *true* if EQ(*s*, *t*) is *true* in Common LISP.

**eq?** (*aggregate*, *aggregate*)

**eq?** (*u*, *v*) tests if two aggregates *u* and *v* are same objects in the AXIOM store.

**equality** (*operator*, *function*)

**equality** (*op*, *f*) attaches *f* as the "%equal?" property to *op*. Argument *f* must be a boolean-valued "equality" function defined on BasicOperator objects. If *op*<sub>1</sub> and *op*<sub>2</sub> have the same name, and one of them has an "%equal?" property *f*, then *f*(*op*<sub>1</sub>, *op*<sub>2</sub>) is called to decide whether *op*<sub>1</sub> and *op*<sub>2</sub> should be considered equal.

**equation** (*expression*, *expression*)

**equation** (*a*, *b*) creates the equation *a* = *b*.

**equation** (*v*, *a..b*), also written: *v* = *a..b*, creates a segment binding value with variable *v* and segment *a..b*.

**erf** (*variable*)

**erf** (*x*) returns the error function of *x*:  $\frac{2}{\sqrt{\pi}} \int \exp^{-x^2} dx$ .

**error** (*string* [, *string*])

**error** (*msg*) displays error message *msg* and terminates. Argument *msg* is either a string or a list of strings.

**error** (*name*, *msg*) is similar except that the error message is preceded by a message saying that the error occurred in a function named *name*.

**euclideanGroebner** (*ideal* [, *string*, *string*])

**euclideanGroebner** (*lp* [, "info", "redcrit"]) computes a Gröbner basis for a polynomial ideal over a Euclidean domain generated by the list of polynomials *lp*. If the string "info" is given as a second argument, a summary is given of the critical pairs. If the string "redcrit" is given as a third argument, the critical pairs are printed.

**euclideanNormalForm** (*polynomial*, *groebnerBasis*)

**euclideanNormalForm** (*poly*, *gb*) reduces the polynomial *poly* modulo the precomputed Gröbner basis *gb* giving a

canonical representative of the residue class.

**euclideanSize** (*element*)

**euclideanSize** (*x*) returns the Euclidean size of the element *x*, or calls **error** if *x* is zero.

**eulerPhi** (*positiveInteger*)

**eulerPhi** (*n*) returns the number of integers between 1 and *n* (including 1) which are relatively prime to *n*. This is the Euler phi function  $\phi(n)$ , also called the totient function.

**euler** (*positiveInteger*)

**euler** (*n*) returns the *n*<sup>th</sup> Euler number. This is  $2^n E(n, 1/2)$ , where  $E(n, x)$  is the *n*<sup>th</sup> Euler polynomial.

**eval** (*expression* [, *options*])

Many domains have forms of the **eval** defined. Here are some the most common forms.

**eval** (*f*) unquotes all the quoted operators in *f*.

**eval** (*f*, *x* = *v*) replaces symbol or expression *x* by *v* in *f*; if *x* is an expression, it must be retractable to a single Kernel.

**eval** (*f*, [*x*<sub>1</sub> = *v*<sub>1</sub>, ..., *x*<sub>*n*</sub> = *v*<sub>*n*</sub>]) returns *f* with symbols or expressions *x*<sub>*i*</sub> replaced by *v*<sub>*i*</sub> in parallel; if *x*<sub>*i*</sub> is an expression, it must be retractable to a single Kernel.

**eval** (*f*, [*x*<sub>1</sub>, ..., *x*<sub>*n*</sub>]) unquotes all the quoted operations in *f* whose name is one of the *x*<sub>*i*</sub>'s.

**eval** (*f*, *x*) unquotes all quoted operators in *f* whose name is *x*.

**eval** (*e*, *s*, *f*) replaces every subexpression of *e* of the form *s*(*a*<sub>1</sub>, ..., *a*<sub>*n*</sub>) by *f*(*a*<sub>1</sub>, ..., *a*<sub>*n*</sub>). The function *f* can have type Expression → Expression if *s* is a unary operator; otherwise *f* must have signature List(Expression) → Expression.

**eval** (*e*, [*s*<sub>1</sub>, ..., *s*<sub>*n*</sub>], [*f*<sub>1</sub>, ..., *f*<sub>*n*</sub>]), replaces every subexpression of *e* of the form *s*<sub>*i*</sub>(*a*<sub>1</sub>, ..., *a*<sub>*n*<sub>*i*</sub></sub>) by *f*<sub>*i*</sub>(*a*<sub>1</sub>, ..., *a*<sub>*n*<sub>*i*</sub></sub>). If all the *s*<sub>*i*</sub>'s are unary operators, the functions *f*<sub>*i*</sub> can have signature Expression → Expression; otherwise, the *f*<sub>*i*</sub> must have signature List(Expression) → Expression.

**eval** (*p*, *el*), where *p* is a permutation, returns the image of element *el* under *p*.

**eval** (*s*), where *s* is of type SymmetricPolynomial with rational number coefficients, returns the sum of the coefficients of a cycle index. See CycleIndicators for details.

**eval** (*f*, *s*), where *s* is of type SymmetricPolynomial with rational number coefficients and *f* is a function of type Integer → Algebra Fraction Integer, evaluates the cycle index *s* by applying the function *f* to each integer in a monomial partition, forms their product and sums the results over all monomials. See EvaluateCycleIndicators for details.

**evaluate** (*operator*, *function*)

**evaluate** (*op*) returns the value of the "%eval" property of BasicOperator object *op* if it has one, and "failed"

otherwise.

**evaluate** (*op*, *f*) attaches *f* as the "%eval" property of *op*. If *op* has an "%eval" property *f*, then applying *op* to *a* returns the result of *f*(*a*). If *f* takes a single argument, then applying *op* to a value *a* returns the result *f*(*a*). If *f* takes a list of arguments, then applying *op* to *a*<sub>1</sub>, ..., *a*<sub>*n*</sub> returns the result of *f*(*a*<sub>1</sub>, ..., *a*<sub>*n*</sub>).

Argument *f* may also be an anonymous function of the form *u* + - > *g*(*u*). In this case, *g* must be additive, that is, *g*(*a* + *b*) = *g*(*a*) + *g*(*b*) for any *a* and *b* in *R*. This implies that *g*(*na*) = *ng*(*a*) for any *a* in *R* and integer *n* > 0.

**even?** (*integerNumber*)

**even?** (*n*) tests if integer *n* is even.

**even?** (*p*) tests if permutation *p* is an even permutation, that is, that the **sign** (*p*) = 1.

**every?** (*predicate*, *aggregate*)

**every?** (*pred*, *u*) tests if *pred*(*x*) is true for all elements *x* of *u*.

**exists?** (*file*)

**exists?** (*f*) tests if the file *f* exists in the file system.

**exp** (*expression*)

**expIfCan** (*x*)

**exp** (*x*) returns %e to the power *x*.

**expIfCan** (*z*) returns exp(*z*) if possible, and "failed" otherwise.

**exp1** ()

**exp1** ()\$*R* returns exp 1: 2.7182818284... either a float or a small float according to whether *R* = Float or *R* = DoubleFloat.

**expand** (*expression*)

**expand** (*f*) performs the following expansions on Expression *f* :

Logs of products are expanded into sums of logs.

Trigonometric and hyperbolic trigonometric functions of sums are expanded into sums of products of trigonometric and hyperbolic trigonometric functions.

Formal powers of the form  $(a/b)^c$  are expanded into  $a^c b^{(-c)}$ .

**expand** (*ir*), where *ir* is an IntegrationResult, returns the list of possible real functions corresponding to *ir*.

**expand** (*lseg*), where *lseg* is a list of segments, returns a list with all segments expanded. For example, **expand** [1..4, 7..9] = [1, 2, 3, 4, 7, 8, 9].

**expand** (*l..h* by *k*) returns a list of explicit elements. For example, **expand**(1..5 by 2) = [1, 3, 5].

**expand** (*f*) returns an unfactored form of factored object *f*.

**expandLog** (*expression*)

**expandLog** (*f*) converts every **log** (*a/b*) appearing in Expression *f* into log(*a*) - log(*b*).

**expandPower** (*expression*)

**expandPower** (*f*) converts every power  $(a/b)^c$  appearing in Expression *f* into  $a^c b^{-c}$ .

**explicitEntries?** (*stream*)

**explicitEntries?** (*s*) tests if the stream *s* has explicitly computed entries.

**explicitlyEmpty?** (*stream*)

**explicitlyEmpty?** (*s*) tests if the stream is an (explicitly) empty stream. Note: this is a null test which will not cause lazy evaluation.

**explicitlyFinite?** (*stream*)

**explicitlyFinite?** (*s*) tests if the stream *s* has a finite number of elements. Note: for many datatypes, **explicitlyFinite?**(*s*) = **not possiblyInfinite?**(*s*).

**exponent** (*floatOrFactored*)

**exponent** (*fl*) returns the **exponent** part of a float or small float *fl*.

**exponent** (*u*), where *u* is a factored object, returns the exponent of the first factor of *u*, or 0 if the factored object consists solely of a unit.

**expressIdealMember** (*listOfIdeals*, *ideal*)

**expressIdealMember** ([*f*<sub>1</sub>, ..., *f*<sub>*n*</sub>], *h*) returns a representation of ideal *h* as a linear combination of the ideals *f*<sub>*i*</sub> or "failed" if *h* is not in the ideal generated by the *f*<sub>*i*</sub>.

**exptMod** (*polynomial*, *nonNegativeInteger*, *polynomial* [ , *prime*])

**exptMod** (*u*, *k*, *v* [, *p*]) raises the polynomial *u* to the *k*<sup>th</sup> power modulo the polynomial *v*. If a prime *p* is given, the power is also computed modulo that prime.

**exquo** (*element*, *element*)

**exquo** (*a*, *b*) either returns an element *c* such that *cb* = *a* or "failed" if no such element can be found. Values *a* and *b* are members of a domain of category IntegralDomain.

**exquo** (*A*, *r*) returns the exact quotient of the elements of matrix *A* by coefficient *r*, or calls **error** if this is not possible.

**extend** (*stream*, *integer*)

**extend** (*ps*, *n*), where *ps* is a power series, causes all terms of *ps* of degree ≤ *n* to be computed.

**extend** (*st*, *n*), where *st* is a stream, causes entries to be computed so that *st* has at least *n* explicit entries, or so that all entries of *st* are finite with length ≤ *n*.

**extendedEuclidean** (*element*, *element* [, *element*])

Arguments *x*, *y*, and *z* are members of a domain of category EuclideanDomain.

**extendedEuclidean** (*x*, *y*) returns a record *rec* containing three fields: *coef1*, *coef2*, and *generator* where  $rec.coef1 * x + rec.coef2 * y = rec.generator$  and *rec.generator* is a gcd of *x* and *y*. The gcd is unique only up to associates if **canonicalUnitNormal** is not asserted.

Note: See **principalIdeal** for a version of this operation which accepts an arbitrary length list of arguments.

**extendedEuclidean** (*x*, *y*, *z*) either returns a record *rec* of two fields *coef1* and *coef2* where  $rec.coef1 * x + rec.coef2 * y = z$ , and "failed" if *z* cannot be expressed as such a linear combination of *x* and *y*.

**extendedIntegrate** (*rationalFunct*, *symbol*, *rationalFunct*)

**extendedIntegrate** (*f*, *x*, *g*) returns fractions [*h*, *c*] such that  $dc/dx = 0$  and  $dh/dx = f - cg$  if (*h*, *c*) exist, and "failed" otherwise.

**extensionDegree** ()

**extensionDegree** ()\$*F* returns the degree of the field extension *F* if the extension is algebraic, and **infinity** if it is not.

**extension** (*filename*)

**extension** (*fn*) returns the type part of the file name *fn* as a string.

**extract!** (*bag*)

**extract!** (*bg*) destructively removes a (random) item from bag *bg*.

**extractBottom!** (*dequeue*)

**extractBottom!** (*d*) destructively extracts the bottom (back) element from the dequeue *d*, or calls **error** if *d* is empty.

**extractTop!** (*dequeue*)

**extractTop!** (*d*) destructively extracts the top (front) element from the dequeue *d*, or calls **error** if *d* is empty.

**e** (*positiveInteger*)

**e** (*n*) produces the appropriate unit element of a CliffordAlgebra.

**factor** (*polynomial* [, *numbers*])

**factor** (*x*) returns the factorization of *x* into irreducibles, where *x* is a member of any domain of category UniqueFactorizationDomain.

**factor** (*p*, *lan*), where *p* is a polynomial and *lan* is a list of algebraic numbers, factors *p* over the extension generated by the algebraic numbers given by the list *lan*.

**factor** (*upoly*, *prime*), where *upoly* is a univariate polynomial and *prime* is a prime integer, returns the list of factors of *upoly* modulo the integer prime *p*, or calls **error**

if *upoly* is not square-free modulo *p*.

**factorFraction** (*fraction*)

**factorFraction** (*r*) factors the numerator and the denominator of the polynomial fraction *r*.

**factorGroebnerBasis** (*listOfPolynomials* [, *boolean*])

**factorGroebnerBasis** (*basis* [, *flag*]) checks whether the *basis* contains reducible polynomials and uses these to split the *basis*. Information about partial results is given if a second argument of *true* is given.

**factorials** (*expression* [, *symbol*])

**factorials** (*f* [, *x*]) rewrites the permutations and binomials in *f* in terms of factorials. If a symbol *x* is given as a second argument, the operation rewrites only those terms involving *x*.

**factorial** (*expression*)

**factorial** (*n*), where *n* is an integer, returns the integer value of  $n! = \prod_{i=1}^n i$ .

**factorial** (*n*), where *n* is an expression, returns a formal expression denoting *n*!. Note:  $n! = n(n-1)!$  when  $n > 0$ ; also,  $0! = 1$ .

**factorList** (*factoredForm*)

**factorList** (*f*), for a factored form *f*, returns list of records. Each record corresponds to a factor of *f* and has three fields: *flg*, *fctr*, and *xpnt*. The *fctr* lists the factor and *xpnt*, the exponent. The *flg* is one of the strings: "nil", "sqfr", "irred", or "prime".

**factorPolynomial** (*polynomial*)

**factorPolynomial** (*p*) returns the factorization of a sparse univariate polynomial *p* as a factored form.

**factors** (*factoredForm*)

**factors** (*u*) returns a list of the factors of a factored form *u* in a form as a list suitable for iteration. Each element in the list is a record containing both a *factor* and *exponent* field.

**factorsOfCyclicGroupSize** ()

**factorsOfCyclicGroupSize** () returns the factorization of *size* () - 1

**factorSquareFreePolynomial** (*polynomial*)

**factorSquareFreePolynomial** (*p*) factors the univariate polynomial *p* into irreducibles, where *p* is known to be square free and primitive with respect to its main variable.

**fibonacci** (*nonNegativeInteger*)

**fibonacci** (*n*) returns the *n*<sup>th</sup> Fibonacci number. The Fibonacci numbers *F*[*n*] are defined by *F*[0] = *F*[1] = 1 and *F*[*n*] = *F*[*n* - 1] + *F*[*n* - 2]. The algorithm has running time  $O(\log(n)^3)$ .

**filename** (*directory*, *name*, *extension*)

**filename** (*d*, *n*, *e*) creates a file name with string *d* as its directory, string *n* as its name and string *e* as its extension.

**fill!** (*aggregate*, *value*)

**fill!** (*a*, *x*) replaces each entry in aggregate *a* by *x*. The modified *a* is returned. If *a* is a domain of category *TwoDimensionalArrayCategory* such as a matrix, **fill!** (*a*, *x*) sets every element of *a* to *x*.

**filterUntil** (*predicate*, *stream*)

**filterUntil** (*p*, *s*) returns  $[x_0, x_1, \dots, x_n]$ , where stream  $s = [x_0, x_1, x_2, \dots]$  and *n* is the smallest index such that  $p(x_n) = \text{true}$ .

**filterWhile** (*predicate*, *stream*)

**filterWhile** (*pred*, *s*) returns  $[x_0, x_1, \dots, x_{(n-1)}]$  where  $s = [x_0, x_1, x_2, \dots]$  and *n* is the smallest index such that  $p(x_n) = \text{false}$ .

**find** (*predicate*, *aggregate*)

**find** (*pred*, *u*) returns the first *x* in *u* such that **pred** (*x*) is *true*, and "failed" if no such *x* exists.

**findCycle** (*nonNegativeInteger*, *stream*)

**findCycle** (*n*, *st*) determines if stream *st* is periodic within *n* terms. The operation returns a record with three fields: *cycle?*, *prefix*, and *period*. If *cycle?* has value *true*, *period* denotes the period of the cycle, and *prefix* gives the number of terms in the stream before the cycle begins.

**finite?** (*cardinalNumber*)

**finite?** (*f*) tests if expression *f* is finite.

**finite?** (*a*) tests if cardinal number *a* is a finite cardinal, that is, an integer.

**integrate** (*taylorSeries*, *symbol*, *coefficient*)

**integrate** (*s*, *v*, *c*) integrates the series *s* with respect to variable *v* and having *c* as the constant of integration.

**first** (*aggregate* [, *nonNegativeInteger*])

**first** (*u*) returns the first element *x* of aggregate *u*.

**first** (*u*, *n*) returns a copy of the first *n* elements of *u*.

**fixedPoint** (*function* [, *positiveInteger*])

**fixedPoint** (*f*), a function of type  $A \rightarrow A$ , is the fixed point of function *f*. That is,

**fixedPoint** (*f*) = **fixedPoint** (**fixedPoint** (*f*)).

**fixedPoint** (*f*, *n*), where *f* is a function of type  $\text{List}(A) \rightarrow \text{List}(A)$  and *n* is a positive integer, is the fixed point of function *f* which is assumed to transform a list of length *n*.

**fixedPoints** (*permutation*)

**fixedPoints** (*p*) returns the points fixed by the permutation *p*.

**flagFactor** (*base*, *exponent*, *flag*)

**flagFactor** (*base*, *exponent*, *flag*) creates a factored object with a single factor whose *base* is asserted to be properly described by the information *flag*: one of the strings "nil", "sqfr", "irred", and "prime".

**flatten** (*inputForm*)

**flatten** (*s*) returns an input form corresponding to *s* with all the nested operations flattened to triples using new local variables. This operation is used to optimize compiled code.

**flexible?** ()

**flexible?** ()\$*R* tests if **2associator** (*a*, *b*, *a*) = 0 for all *a*, *b* in a domain *R* of category *FiniteRankNonAssociativeAlgebra*. Note: only this can be tested since, in general, it is not known whether  $2a = 0$  implies  $a = 0$ .

**flexibleArray** (*listOfElements*)

**flexibleArray** (*ls*) creates a flexible array from a list of elements *ls*.

**float?** (*sExpression*)

**float?** (*s*) is *true* if *s* is an atom and belongs o *Flt*.

**float** (*integer*, *integer* [, *positiveinteger*])

**float** (*a*, *e*) returns  $\text{abase}()^e$  as a float.

**float** (*a*, *e*, *b*) returns  $ab^e$  as a float.

**floor** (*rationalNumber*)

**floor** (*fr*), where *fr* is a fraction, returns the largest integral element below *fr*.

**floor** (*fl*), where *fl* is a float, returns the largest integer  $\leq fl$ .

**formula** (*formulaFormat*)

**formula** (*t*) extracts the formula section of an IBM SCRIPT Formula formatted object *t*.

**fractionPart** (*fraction*)

**fractionPart** (*x*) returns the fractional part of *x*.

Argument *x* can be a fraction, a radix (binary, decimal, or hexadecimal) expansion, or a float. Note:  $x = \text{whole}(x) + \text{fractionPart}(x)$ .

**fractRadix** (*listOfIntegers*, *listOfIntegers*)

**fractRadix** (*pre*, *cyc*) creates a fractional radix expansion from a list of prefix ragits and a list of cyclic ragits. For example, **fractRadix** ([1], [6]) will return 0.16666666...

**fractRagits** (*radixExpansion*)

**fractRagits** (*rx*) returns the ragits of the fractional part of a radix expansion as a stream of integers.

**freeOf?** (*expression*, *kernel*)

**freeOf?** (*x*, *k*) tests if expression *x* does not contain any

operator whose name is the symbol or kernel  $k$ .

**Frobenius** (*element*)

**Frobenius** ( $a$ )\$ $F$  returns  $a^q$  where  $q$  is the **size** ()\$ $F$  of extension field  $F$ .

**front** (*queue*)

**front** ( $q$ ) returns the element at the front of the queue, or calls **error** if  $q$  is empty.

**frst** (*stream*)

**frst** ( $s$ ) returns the first element of stream  $s$ . Warning: this function should only be called after a *empty?* test has been made since there is no error check.

**function** (*expression*, *name* [ , *options*])

Most domains provide an operation which converts objects to type InputForm. Argument  $e$  below denotes an object from such a domain. These operations create user-functions from already computed results.

**function** ( $e, f$ ) creates a function  $f() == e$ .

**function** ( $e, f, [x_1, \dots, x_n]$ ) creates a function  $f(x_1, \dots, x_n) == e$ .

**function** ( $e, f, x$ ) creates a function  $f(x) == e$ .

**function** ( $e, f, x, y$ ) creates a function  $f(x, y) == e$ .

**function** (*expr*,  $[x_1, \dots, x_n]$ ,  $f$ ), where *expr* is an input form and where  $f$  and the  $x_i$ 's are symbols, returns the input form corresponding to  $f(x_1, \dots, x_n) == i$ . See also **unparse**.

**Gamma** (*smallFloat*)

**Gamma** ( $x$ ) is the Euler gamma function, **Gamma** ( $x$ ), defined by  $\Gamma(x) = \int_0^\infty t^{(x-1)} * \exp(-t) dt$ .

**gcdPolynomial** (*polynomial*, *polynomial*)

**gcdPolynomial** ( $p, q$ ) returns the **gcd** of the univariate polynomials  $p$  and  $q$ .

**gcd** (*element* [ , *element*, *element*])

**gcd** ( $x, y$ ) returns the greatest common divisor of  $x$  and  $y$ . Arguments  $x$  and  $y$  are elements of a domain of category *GcdDomain*.

**gcd** ( $[x_1, \dots, x_n]$ ) returns the common *gcd* of the elements of the list of  $x_i$ .

**gcd** ( $p_1, p_2, prime$ ) computes the *gcd* of the univariate polynomials  $p_1$  and  $p_2$  modulo the prime integer *prime*.

**generalizedContinuumHypothesisAssumed?** ([*bool*])

**generalizedContinuumHypothesisAssumed?** () tests if the hypothesis is currently assumed.

**generalizedContinuumHypothesisAssumed** (*bool*) dictates that the hypothesis is or is not to be assumed, according to whether *bool* is true or false.

**generalPosition** (*ideal*, *listOfVariables*)

**generalPosition** ( $I, listvar$ ) performs a random linear

transformation on the variables in *listvar* and returns the transformed ideal  $I$  along with the change of basis matrix.

**generate** (*function* [ , *element*])

**generate** ( $f$ ), where  $f$  is a function of no arguments, creates an infinite stream all of whose elements are equal to the value of  $f()$ . Note: **generate** ( $f$ ) =  $[f(), f(), f(), \dots]$ .

**generate** ( $f, x$ ), where  $f$  is a function of one argument, creates an infinite stream whose first element is  $x$  and whose  $n^{\text{th}}$  element ( $n > 1$ ) is  $f$  applied to the previous element. Note: **generate** ( $f, x$ ) =  $[x, f(x), f(f(x)), \dots]$ . See also **HallBasis**.

**generator** ()

**generator** ()\$ $R$  returns a root of the defining polynomial of a domain of category FiniteAlgebraicExtensionField  $R$ . This element generates the field as an algebra over the ground field.

See also **MonogenicAlgebra** and **FreeNilpotentLie**.

**generators** (*ideal*)

**generators** ( $I$ ) returns a list of generators for the ideal  $I$ .

**generators** (*gp*) returns the generators of a permutation group *gp*.

**genus** ()

**genus** ()\$ $R$  returns the genus of the algebraic function field  $R$ . If  $R$  has several absolutely irreducible components, then the genus of one of them is returned.

**getMultiplicationMatrix** ()

**getMultiplicationTable** ()

**getMultiplicationMatrix** ()\$ $R$  returns a matrix multiplication table for domain FiniteFieldNormalBasis( $p, n$ ), a finite extension field of degree  $n$  over the domain PrimeField( $p$ ) with  $p$  elements. Each element of the matrix is a member of the underlying prime field.

**getMultiplicationTable** ()\$ $R$  is similar except that the multiplication table for the normal basis of the field is represented by a vector of lists of records, each record having two fields: *value*, an element of the prime field over which the domain is built, and *index*, a small integer. This table is used to perform multiplications between field elements.

**getVariableOrder** ()

**getVariableOrder** () returns  $[[b_1, \dots, b_m], [a_1, \dots, a_n]]$  such that the ordering on the variables was given by **setVariableOrder** ( $[b_1, \dots, b_m], [a_1, \dots, a_n]$ ).

**getZechTable** ()

**getZechTable** ()\$ $F$  returns the Zech logarithm table of the field  $F$  where  $F$  is some domain FiniteFieldCyclicGroup( $p, extdeg$ ). This table is used to perform additions in the field quickly.



**gramschmidt** (*listOfMatrices*)

Argument *lv* has the form of a list of matrices of elements of type Expression.

**gramschmidt** (*lv*) converts the list of column vectors *lv* into a set of orthogonal column vectors of Euclidean length 1 using the Gram-Schmidt algorithm.

**graphs** (*integer*)

**graphs** (*n*) is the cycle index of the group induced on the edges of a graph by applying the symmetric function to the *n* nodes. See CycleIndicators for details.

**green** ()

**green** () returns the position of the green hue from total hues.

**groebner** (*listOfPolynomials*)

**groebner** (*lp*) computes a Gröbner basis for a polynomial ideal generated by the list of polynomials *lp*.

**groebner** (*I*) returns a set of generators of ideal *I* that are a Gröbner basis for *I*.

**groebner** (*lp*, *infoflag*) computes a Gröbner basis for a polynomial ideal generated by the list of polynomials *lp*. Argument *infoflag* is used to get information on the computation. If *infoflag* is "info", then summary information is displayed for each s-polynomial generated. If *infoflag* is "redcrit", the reduced critical pairs are displayed. To get the display of both kinds of information, use **groebner** (*lp*, "info", "redcrit").

**groebner?** (*ideal*)

**groebner?** (*I*) tests if the generators of the ideal *I* are a Gröbner basis.

**groebnerIdeal** (*listOfPolynomials*)

**groebnerIdeal** (*lp*) constructs the ideal generated by the list of polynomials *lp* assumed to be a Gröbner basis. Note: this operation avoids a Gröbner basis computation.

**groebnerFactorize** (*listOfPolynomials* [*options*])

**groebnerFactorize** (*lp*, [*bool*]) returns a list of list of polynomials, each inner list denoting a Gröbner basis. The union of the solutions of the bases is the solution of the system of equations given by *lp*. Information about partial results is printed if a second argument is given with value *true*.

**groebnerFactorize** (*lp*, *nonZeroRestrictions* [, *bool*]), where *nonZeroRestrictions* is a list of polynomials, is similar. Here, however, the solutions to the system of equations are computed under the restriction that the polynomials in the second argument do not vanish. Information about partial results is printed if a third argument with value *true* is given.

**ground** (*expression*)

**ground?** (*expression*)

**ground** (*p*) retracts expression polynomial *p* to the coefficient ring, or calls **error** if such a retraction is not possible.

**ground?** (*p*) tests if an expression or polynomial *p* is a member of the coefficient ring. See also **ground?**.

**harmonic** (*positiveInteger*)

**harmonic** (*n*) returns the *n*<sup>th</sup> harmonic number, defined by  $H[n] = \sum_{k=1}^n 1/k$ .

**has** (*domain*, *property*)

**has** (*R*, *prop*) tests if domain *R* has property *prop*.

Argument *prop* is either a category, operation, an attribute, or a combination of these. For example, **Integer** **has** **Ring** and **Integer** **has** **commutative**("\*").

**has?** (*operation*, *property*)

**has?** (*op*, *s*) tests if property *s* is attached to *op*.

**hash** (*number*)

**hash** (*n*) returns the hash code for *n*, an integer or a float.

**hasHi** (*segment*)

**hasHi** (*seg*) tests whether the segment *seg* has an upper bound. For example, **hasHi** (1..) = *false*.

**hasSolution?** (*matrix*, *vector*)

**hasSolution?** (*A*, *B*) tests if the linear system  $AX = B$  has a solution, where *A* is a matrix and *B* is a (column) vector.

**hconcat** (*outputForms* [, *outputForm*])

**hconcat** (*o*<sub>1</sub>, *o*<sub>2</sub>), where *o*<sub>1</sub> and *o*<sub>2</sub> are objects of type **OutputForm** (normally unexposed), returns an output form for the horizontal concatenation of forms *o*<sub>1</sub> and *o*<sub>2</sub>.

**hconcat** (*lof*), where *lof* is a list of objects of type **OutputForm** (normally unexposed), returns an output form for the horizontal concatenation of the elements of *lof*.

**heap** (*listOfElements*)

**heap** (*ls*) creates a Heap of elements consisting of the elements of *ls*.

**heapSort** (*predicate*, *aggregate*)

**heapSort** (*pred*, *agg*) sorts the aggregate *agg* with the ordering function *pred* using the heapsort algorithm.

**height** (*expression*)

**height** (*f*), where *f* is an expression, returns the highest nesting level appearing in *f*. Constants have height 0. Symbols have height 1. For any operator *op* and expressions *f*<sub>1</sub>, ..., *f*<sub>*n*</sub>, *op*(*f*<sub>1</sub>, ..., *f*<sub>*n*</sub>) has height equal to  $1 + \max(\text{height}(f_1), \dots, \text{height}(f_n))$ .

**height** (*d*) returns the number of elements in dequeue *d*. Note: **height** (*d*) = #*d*.

**hermiteH** (*nonNegativeInteger, element*)

**hermiteH** ( $n, x$ ) is the  $n^{\text{th}}$  Hermite polynomial,  $H[n](x)$ , defined by  $\exp(2tx - t^2) = \sum_{n=0}^{\infty} H[n](x)t^n/n!$ .

**hexDigit** ()

**hexDigit** () returns the class of all characters for which **hexDigit?** is *true*.

**hexDigit?** (*character*)

**hexDigit?** ( $c$ ) tests if  $c$  is a hexadecimal numeral, that is, one of 0..9, a..f or A..F.

**hex** (*rationalNumber*)

**hex** ( $r$ ) converts a rational number to a hexadecimal expansion.

**hi** (*segment*)

**hi** ( $s$ ) returns the second endpoint of segment  $s$ . For example, **hi** ( $l..h$ ) =  $h$ .

**horizConcat** (*matrix, matrix*)

**horizConcat** ( $x, y$ ) horizontally concatenates two matrices with an equal number of rows. The entries of  $y$  appear to the right of the entries of  $x$ . The operation calls **error** if the matrices do not have the same number of rows.

**htrigs** (*expression*)

**htrigs** ( $f$ ) converts all the exponentials in expression  $f$  into hyperbolic sines and cosines.

**hue** (*palette*)

**hue** ( $p$ ) returns the hue field of the indicated palette  $p$ .

**hue** (*color*)

**hue** ( $c$ ) returns the hue index of the indicated color  $c$ .

**hypergeometric0F1** (*complexDF, complexSF*)

**hypergeometric0F1** ( $c, z$ ) is the hypergeometric function  $0F1(c; z)$ . Arguments  $c$  and  $z$  are both either small floats or complex small floats.

**ideal** (*polyList*)

**ideal** (*polyList*) constructs the ideal generated by the list of polynomials *polyList*.

**imag** (*expression*)

**imagi** (*quaternionOrOctonion*)

**imagI** (*octonion*)

**imag** ( $x$ ) extracts the imaginary part of a complex value or expression  $x$ .

**imagI** ( $q$ ) extracts the  $i$  part of quaternion  $q$ . Similarly, operations **imagJ**, and **imagK** are used to extract the  $j$  and  $k$  parts.

**imagi** ( $o$ ) extracts the  $i$  part of octonion  $o$ . Similarly, **imagj**, **imagk**, **image**, **imagI**, **imagJ**, and **imagK** are used to extract other parts.

**implies** (*boolean, boolean*)

**implies** ( $a, b$ ) tests if boolean value  $a$  implies boolean value  $b$ . The result is *true* except when  $a$  is *true* and  $b$  is *false*.

**in?** (*ideal, ideal*)

**in?** ( $I, J$ ) tests if the ideal  $I$  is contained in the ideal  $J$ .

**inHallBasis** (*integer, integer, integer, integer*)

**inHallBasis?** ( $n, leftCandidate, rightCandidate, left$ )

tests to see if a new element should be added to the  $P$ .

Hall basis being constructed. The list

$[leftCandidate, wt, rightCandidate]$  is included in the

basis if in the unique factorization of  $rightCandidate$ , we

have left factor  $leftOfRight$ , and

$leftOfRight \leq leftCandidate$

**increasePrecision** (*integer*)

**increasePrecision** ( $n$ ) increases the current **precision** by  $n$  decimal digits.

**index** (*positiveInteger*)

**index** ( $i$ ) takes a positive integer  $i$  less than or equal to

**size** () and returns the  $i^{\text{th}}$  element of the set. This

operation establishes a bijection between the elements of the finite set and  $1..size()$ .

**index?** (*index, aggregate*)

**index?** ( $i, u$ ) tests if  $i$  is an index of aggregate  $u$ . For example, **index?**(2, [1, 2, 3]) is *true* but **index?**(4, [1, 2, 3]) is *false*.

**infieldIntegrate** (*rationalFunction, symbol*)

**infieldIntegrate** ( $f, x$ ), where  $f$  is a fraction of

polynomials, returns a fraction  $g$  such that  $\frac{dg}{dx} = f$  if  $g$  exists, and "failed" otherwise.

**infinite?** (*orderedCompletion*)

**infinite?** ( $x$ ) tests if  $x$  is infinite, where  $x$  is a member of the ordered completion of a domain. See OrderedCompletion using Browse.

**infinity** ()

**infinity** () returns *infinity* denoting  $+\infty$  as a one point completion of the integers. See OnePointCompletion using Browse. See also **minusInfinity** and **plusInfinity**.

**infix** (*outputForm, outputForms [ , OutputForm]*)

**infix** ( $o, lo$ ), where  $o$  is an object of type OutputForm

(normally unexposed) and  $lo$  is a list of objects of type

OutputForm, creates a form depicting the *nary* application of infix operation  $o$  to a tuple of arguments  $lo$ .

**infix** ( $o, a, b$ ), where  $o$ ,  $a$ , and  $b$  are objects of type OutputForm (normally unexposed), creates an output form which displays as:  $a \text{ op } b$ .

**initial** (*differentialPolynomial*)

**initial** ( $p$ ) returns the leading coefficient of differential polynomial  $p$  expressed as a univariate polynomial in its leader.

**initializeGroupForWordProblem** ( $group$  [,  $integer$ ,  $integer$ ])

**initializeGroupForWordProblem** ( $gp$  [,  $n$ ,  $m$ ]) initializes the group  $gp$  for the word problem. Consult `PermutationGroup` using `Browse` for details.

**input** ( $operator$  [,  $function$ ])

**input** ( $op$ ) returns the "%input" property of  $op$  if it has one attached, and "failed" otherwise.

**input** ( $op$ ,  $f$ ) attaches  $f$  as the "%input" property of  $op$ . If  $op$  has a "%input" property  $f$ , then  $op(a_1, \dots, a_n)$  is converted to `InputForm` using  $f(a_1, \dots, a_n)$ . Argument  $f$  must be a function with signature `List(InputForm) → InputForm`.

**inRadical?** ( $polynomial$ ,  $ideal$ )

**inRadical?** ( $f$ ,  $I$ ) tests if some power of the polynomial  $f$  belongs to the ideal  $I$ .

**insert** ( $x$ ,  $aggregate$  [,  $integer$ ])

**insert** ( $x$ ,  $u$ ,  $i$ ) returns a copy of  $u$  having  $x$  as its  $i^{\text{th}}$  element.

**insert** ( $v$ ,  $u$ ,  $k$ ) returns a copy of  $u$  having  $v$  inserted beginning at the  $i^{\text{th}}$  element.

**insert!** ( $x$ ,  $u$ ) destructively inserts item  $x$  into bag  $u$ .

**insert!** ( $x$ ,  $u$ ) destructively inserts item  $x$  as a leaf into binary search tree or binary tournament  $u$ .

**insert!** ( $x$ ,  $u$ ,  $i$ ) destructively inserts  $x$  into aggregate  $u$  at position  $i$ .

**insert!** ( $v$ ,  $u$ ,  $i$ ) destructively inserts aggregate  $v$  into  $u$  at position  $i$ .

**insert!** ( $x$ ,  $d$ ,  $n$ ) destructively inserts  $n$  copies of  $x$  into dictionary  $d$ .

**insertBottom!** ( $element$ ,  $queue$ )

**insertBottom!** ( $x$ ,  $d$ ) destructively inserts  $x$  into the dequeue  $d$  at the bottom (back) of the dequeue.

**insertTop!** ( $element$ ,  $dequeue$ )

**insertTop!** ( $x$ ,  $d$ ) destructively inserts  $x$  into the dequeue  $d$  at the top (front). The element previously at the top of the dequeue becomes the second in the dequeue, and so on.

**integer** ( $expression$ )

**integer?** ( $expression$ )

**integerIfCan** ( $expression$ )

**integer** ( $x$ ) returns  $x$  as an integer, or calls **error** if this is not possible.

**integer?** ( $x$ ) tests if expression  $x$  is an integer.

**integerIfCan** ( $x$ ) returns expression  $x$  as of type `Integer` or

else "failed" if it cannot.

**integerPart** ( $float$ )

**integerPart** ( $fl$ ) returns the integer part of the mantissa of float  $fl$ .

**integral** ( $expression$ ,  $symbol$ )

**integral** ( $expression$ ,  $segmentBinding$ )

**integral** ( $f$ ,  $x$ ) returns the formal integral  $\int f dx$ .

**integral** ( $f$ ,  $x = a..b$ ) returns the formal definite integral  $\int_a^b f(x) dx$ .

**integralBasis** ()

**integralBasisAtInfinity** ()

Domain  $F$  is the domain of functions on a fixed curve. See `FunctionFieldCategory` using `Browse`.

**integralBasisAtInfinity** ()\$ $F$  returns the local integral basis at infinity.

**integralBasis** ()\$ $F$  returns the integral basis for the curve.

**integralCoordinates** ( $function$ )

**integralCoordinates** ( $f$ ), where  $f$  is a function on a curve defined by domain  $F$ , returns the coordinates of  $f$  with respect to the **integralBasis** ()\$ $F$  as polynomials  $A_i$  together with a common denominator  $d$ . Specifically, the operation returns a record having selector  $num$  with value  $[A_1, \dots, A_n]$  and selector  $den$  with value  $d$  such that  $f = (A_1 w_1 + \dots + A_n w_n)/d$  where  $(w_1, \dots, w_n)$  is the integral basis. See `FunctionFieldCategory` using `Browse`.

**integralDerivationMatrix** ( $function$ )

**integralDerivationMatrix** ( $d$ ) extends the derivation  $d$  and returns the coordinates of the derivative of  $f$  with respect to the **integralBasis** ()\$ $F$  as a matrix of polynomials and a common denominator  $Q$ . Specifically, the operation returns a record having selector  $num$  with value  $M$  and selector  $den$  with value  $Q$  such that the  $i^{\text{th}}$  row of  $M$  divided by  $Q$  form the coordinates of  $f$  with respect to integral basis  $(w_1, \dots, w_n)$ . See `FunctionFieldCategory` using `Browse`.

**integralMatrix** ()

**integralMatrixAtInfinity** ()

Domain  $F$  is a domain of functions on a fixed curve. These operations return a matrix which transform the natural basis to an integral basis. See `FunctionFieldCategory` using `Browse`.

**integralMatrix** () returns  $M$  such that

$(w_1, \dots, w_n) = M(1, y, \dots, y^{n-1})$ , where  $(w_1, \dots, w_n)$  is the integral basis returned by **integralBasis** ()\$ $F$ .

**integralMatrixAtInfinity** ()\$ $F$  returns matrix  $M$  which transforms the natural basis such that  $(v_1, \dots, v_n) = M(1, y, \dots, y^{n-1})$  where  $(v_1, \dots, v_n)$  is the local integral basis at infinity returned by

**integralBasisAtInfinity** ()\$ $F$ .

**integralRepresents** (*vector*, *commonDenominator*)

**integralRepresents** ( $[A_1, \dots, A_n], d$ ) is the inverse of the operation **integralCoordinates** defined for domain  $F$ , a domain of functions on a fixed curve. Given the coordinates as polynomials  $[A_1, \dots, A_n]$  over a common denominator  $d$ , this operation returns the function represented as  $(A_1 w_1 + \dots + A_n w_n)/d$  where  $(w_1, \dots, w_n)$  is the integral basis returned by **integralBasis** ( $\%$  $F$ ). See **FunctionFieldCategory** using **Browse**.

**integrate** (*expression*)

**integrate** (*expression*, *variable* [*,* *options*])

**integrate** ( $f$ ) returns the integral of a univariate polynomial or power series  $f$  with respect to its distinguished variable.

**integrate** ( $f, x$ ) returns the integral of  $f(x)dx$ , where  $x$  is viewed as a real variable.

**integrate** ( $f, x = a..b$  [, "noPole"]) returns the integral of  $f(x)dx$  from  $a$  to  $b$ . If it is not possible to check whether  $f$  has a pole for  $x$  between  $a$  and  $b$ , then a third argument "noPole" will make this function assume that  $f$  has no such pole. This operation calls **error** if  $f$  has a pole for  $x$  between  $a$  and  $b$  or if a third argument different from "noPole" is given.

**interpret** (*inputForm*)

**interpret** ( $f$ ) passes  $f$  of type **InputForm** to the interpreter.

**interpret** ( $f$ ) $\%$  $P$ , where  $P$  is the package **InputFormFunctions1(R)** for some type  $R$ , passes  $f$  of type **InputForm** to the interpreter, and transforms the result into an object of type  $R$ .

**intersect** (*elements* [, *element*])

**intersect** ( $li$ ), where  $li$  is a list of ideals, computes the intersection of the list of ideals  $li$ .

**intersect** ( $u, v$ ), where  $u$  and  $v$  are sets, returns the set  $u$  consisting of elements common to both sets  $u$  and  $v$ . See also **Multiset**.

**intersect** ( $I, J$ ), where  $I$  and  $J$  are ideals, computes the intersection of the ideals  $I$  and  $J$ .

**inv** (*element*)

**inv** ( $x$ ) returns the multiplicative inverse of  $x$ , where  $x$  is an element of a domain of category **Group** or **DivisionRing**, or calls **error** if  $x$  is 0.

**inverse** (*matrix*)

**inverse** ( $A$ ) returns the inverse of the matrix  $A$ , or "failed" if the matrix is not invertible, or calls **error** if the matrix is not square.

**inverseColeman** (*listOfIntegers*, *listOfIntegers*, *matrix*)

**inverseColeman** ( $\alpha, \beta, C$ ) returns the lexicographically smallest permutation in a double coset of the symmetric group corresponding to a non-negative

Coleman-matrix. Consult

**SymmetricGroupCombinatoricFunctions** using **Browse** for details.

**inverseIntegralMatrix** ( $\%$ )

**inverseIntegralMatrixAtInfinity** ( $\%$ )

Domain  $F$  is a domain of functions on a fixed curve. These operations return a matrix which transform an integral basis to a natural basis. See **FunctionFieldCategory** using **Browse**.

**inverseIntegralMatrix** ( $\%$  $F$ ) returns  $M$  such that  $M(w_1, \dots, w_n) = (1, y, \dots, y^{n-1})$  where  $(w_1, \dots, w_n)$  is the integral basis returned by **integralBasis** ( $\%$  $F$ ). See also **integralMatrix**.

**inverseIntegralMatrixAtInfinity** ( $\%$ ) returns  $M$  such that  $M(v_1, \dots, v_n) = (1, y, \dots, y^{(n-1)})$  where  $(v_1, \dots, v_n)$  is the local integral basis at infinity returned by

**integralBasisAtInfinity** ( $\%$  $F$ ). See also

**integralMatrixAtInfinity**.

**inverseLaplace** (*expression*, *symbol*, *symbol*)

**inverseLaplace** ( $f, s, t$ ) returns the Inverse Laplace transform of  $f(s)$  using  $t$  as the new variable, or "failed" if unable to find a closed form.

**invmod** (*positiveInteger*, *positiveInteger*)

**invmod** ( $a, b$ ), for relatively prime positive integers  $a$  and  $b$  such that  $a < b$ , returns  $1/a \bmod b$ .

**iomode** (*file*)

**iomode** ( $f$ ) returns the status of the file  $f$  as one of the following strings: "input", "output" or "closed".

**irreducible?** (*polynomial*)

**irreducible?** ( $p$ ) tests whether the polynomial  $p$  is irreducible.

**irreducibleFactor** (*element*, *integer*)

**irreducibleFactor** ( $base, exponent$ ) creates a factored object with a single factor whose  $base$  is asserted to be irreducible (flag = "irred").

**irreducibleRepresentation** (*listOfIntegers* [, *permutations*])

**irreducibleRepresentation** ( $\lambda, \pi$ ) returns a matrix giving the irreducible representation corresponding to partition  $\lambda$ , represented as a list of integers, in Young's natural form of the permutation  $\pi$  in the symmetric group whose elements permute  $1, 2, \dots, n$ . If a second argument is not given, the permutation is taken to be the following two generators of the symmetric group, namely  $(12)$  (2-cycle) and  $(12 \dots n)$  (( $n$ )-cycle).

**is?** (*expression*, *pattern*)

**is?** ( $expr, pat$ ) tests if the expression  $expr$  matches the pattern  $pat$ .

**is?** (*expression*, *op*) tests if *expression* is a kernel and is its operator is *op*.

**isAbsolutelyIrreducible?** (*listOfMatrices*, *integer*)

**isAbsolutelyIrreducible?** (*aG*, *numberOfTries*) uses Norton's irreducibility test to check for absolute irreducibility. Consult RepresentationPackage2 using Browse for details.

**isExpt** (*expression* [, *operator*])

**isExpt** (*p* [, *op*]) returns a record with two fields: *var* denoting a kernel *x*, and *exponent* denoting an integer *n*, if expression *p* has the form  $p = x^n$  and  $n \neq 0$ . If a second argument *op* is given, *x* must have the form *op*(*a*) for some *a*.

**isMult** (*expression*)

**isMult** (*p*) returns a record with two fields: *coef* denoting an integer *n*, and *var* denoting a kernel *x*, if *p* has the form  $n * x$  and  $n \neq 0$ , and "failed" if this is not possible.

**isobaric?** (*differentialPolynomial*)

**isobaric?** (*p*) tests if every differential monomial appearing in the differential polynomial *p* has the same weight.

**isPlus** (*expression*)

**isPlus** (*p*) returns  $[m_1, \dots, m_n]$  if *p* has the form  $m_1 + \dots + m_n$  for  $n > 1$  and  $m_i \neq 0$ , and "failed" if this is not possible.

**isTimes** (*expression*)

**isTimes** (*p*) returns  $[a_1, \dots, a_n]$  if *p* has the form  $a_1 * \dots * a_n$  for  $n > 1$  and  $m_i \neq 1$ , and "failed" if this is not possible.

**Is** (*subject*, *pattern*)

**Is**(*expr*, *pat*) matches the pattern *pat* on the expression *expr* and returns a list of matches  $[v_1 = e_1, \dots, v_n = e_n]$  or "failed" if matching fails. An empty list is returned if either *expr* is exactly equal to *pat* or if *pat* does not match *expr*.

**jacobi** (*integer*, *integer*)

**jacobi** (*a*, *b*) returns the Jacobi symbol  $J(a/b)$ . When *b* is odd,  $J(a/b) = \prod_{p \in \text{factors}(b)} L(a/p)$ . Note: by convention, 0 is returned if  $\text{gcd}(a, b) \neq 1$ .

**jacobiIdentity?** ()

**jacobiIdentity?** () tests if  $(ab)c + (bc)a + (ca)b = 0$  for all *a*, *b*, *c* in a domain of FiniteRankNonAssociativeAlgebra. For example, this relation holds for crossed products of three-dimensional vectors.

**janko2** ([*listOfIntegers*])

**janko2** () constructs the janko group acting on the integers  $1, \dots, 100$ .

**janko2** ([*li*]) constructs the janko group acting on the 100 integers given in the list *li*. The default value of *li* is  $[1, \dots, 100]$ . This operation removes duplicates in the list and calls **error** if *li* does not have exactly 100 distinct entries.

**jordanAdmissible?** ()

**jordanAlgebra?** ()

**jordanAdmissible?** ()\$*F*, where *F* is a member of FiniteRankNonAssociativeAlgebra(*R*) over a commutative ring *R*, tests if 2 is invertible in *R* and if the algebra defined by  $\{a, b\}$  defined by  $(1/2)(ab + ba)$  is a Jordan algebra, that is, satisfies the Jordan identity.

**jordanAlgebra?** ()\$*F* tests if the algebra is commutative, that **characteristic** ()\$*F*  $\neq 2$ , and  $(ab)a^2 - a(ba^2) = 0$  for all *a* and *b* in the algebra (Jordan identity). Example: for every associative algebra (*A*, +, @), you can construct a Jordan algebra (*A*, +, \*), where  $a * b := (a @ b + b @ a) / 2$ .

**kernel** (*operator*, *expression*)

**kernel** (*op*, *x*) constructs *op*(*x*) without evaluating it.

**kernel** (*op*, [*f*<sub>1</sub>, ..., *f*<sub>*n*</sub>]) constructs *op*(*f*<sub>1</sub>, ..., *f*<sub>*n*</sub>) without evaluating it.

**kernels** (*expression*)

**kernels** (*f*) returns the list of all the top-level kernels appearing in expression *f*, but not the ones appearing in the arguments of the top-level kernels.

**key?** (*key*, *dictionary*)

**keys** (*dictionary*)

**key?** (*k*, *d*) tests if *k* is a key in dictionary *d*. Dictionary *d* is an element of a domain of category KeyedDictionary(*K*, *E*), where *K* and *E* denote the domains of keys and entries.

**keys** (*d*) returns the list the keys in table *d*.

**kronckerDelta** ([*integer*, *integer*])

**kronckerDelta** () is the rank 2 tensor defined by **kronckerDelta** (*i*, *j*) = 1 if *i* = *j*, and 0 otherwise.

**label** (*outputForm*, *outputForm*)

**label** (*o*<sub>1</sub>, *o*<sub>2</sub>), where *o*<sub>1</sub> and *o*<sub>2</sub> are objects of type OutputForm (normally unexposed), returns an output form displaying equation *o*<sub>2</sub> with label *o*<sub>1</sub>.

**laguerreL** (*nonNegativeInteger*, *x*)

**laguerreL** (*nonNegativeInteger*, *nonNegativeInteger*, *x*)

**laguerreL** (*n*, *x*) is the *n*<sup>th</sup> Laguerre polynomial,  $L[n](x)$ , defined by  $\exp(\frac{-tx}{1-t}) / (1-t) = \sum_{n=0}^{\infty} L[n](x) t^n / n!$ .

**laguerreL** (*m*, *n*, *x*) is the associated Laguerre polynomial,  $L_m[n](x)$ , defined as the *m*<sup>th</sup> derivative of  $L[n](x)$ .

**lambda** (*inputForm*, *listOfSymbols*)

**lambda** (*i*, [*x*<sub>1</sub>, ..., *x*<sub>*n*</sub>]) returns the input form corresponding to  $(x_1, \dots, x_n) \mapsto i$  if  $n > 1$ . See also

**compiledFunction**, **flatten**, and **unparse**.

**laplace** (*expression*, *symbol*, *symbol*)

**laplace** ( $f, t, s$ ) returns the Laplace transform of  $f(t)$ , defined by  $\int_{t=0}^{\infty} \exp(-st)f(t)dt$ . If the transform cannot be computed, the formal object **laplace** ( $f, t, s$ ) is returned.

**last** (*indexedAggregate* [, *nonNegativeInteger*])

**last** ( $u$ ) returns the last element of  $u$ .

**last** ( $u, n$ ) returns a copy of the last  $n$  ( $n \geq 0$ ) elements of  $u$ .

**laurent** (*expression*)

**laurentIfCan** (*expression*)

**laurent** ( $u$ ) converts  $u$  to a Laurent series, or calls **error** if this is not possible.

**laurentIfCan** ( $u$ ) converts the Puiseux series  $u$  to a Laurent series, or returns "failed" if this is not possible.

**laurent** ( $f, x = a$ ) expands the expression  $f$  as a Laurent series in powers of  $(x - a)$ .

**laurent** ( $f, n$ ) expands the expression  $f$  as a Laurent series in powers of  $x$ ; at least  $n$  terms are computed.

**laurent** ( $n \mapsto a_n, x = a, n_0..[n_1]$ ) returns a Laurent series defined by  $\sum_{n=n_0}^{n_1} a_n(x - a)^n$ , where  $n_1$  is  $\infty$  by default.

**laurent** ( $a_n, n, x = a, n_0..[n_1]$ ) returns a Laurent series defined by  $\sum_{n=n_0}^{n_1} a_n(x - a)^n$ , where  $n_1$  is  $\infty$  by default.

**laurentRep** (*expression*)

**laurentRep** ( $f(x)$ ) returns  $g(x)$  where the Puiseux series  $f(x) = g(x^r)$  is represented by  $[r, g(x)]$ .

**lazy?** (*stream*)

**lazy?** ( $s$ ) tests if the first node of the stream  $s$  is a lazy evaluation mechanism which could produce an additional entry to  $s$ .

**lazyEvaluate** (*stream*)

**lazyEvaluate** ( $s$ ) causes one lazy evaluation of stream  $s$ . Caution:  $s$  must be a "lazy node" satisfying **lazy?** ( $s$ ) = *true*, as there is no error check. A call to this function may or may not produce an explicit first entry.

**lcm** (*elements* [, *element*])

**lcm** ( $x, y$ ) returns the least common multiple of  $x$  and  $y$ .

**lcm** ( $lx$ ) returns the least common multiple of the elements of the list  $lx$ .

**ldexquo** (*lodOperator*, *lodOperator*)

**ldexquo** ( $a, b$ ) returns  $q$  such that  $a = b * q$ , or "failed" if no such  $q$  exists.

**leftDivide** (*lodOperator*, *lodOperator*)

**leftQuotient** (*lodOperator*, *lodOperator*)

**leftRemainder** (*lodOperator*, *lodOperator*)

**leftDivide** ( $a, b$ ) returns a record with two fields:

"quotient"  $q$  and "remainder"  $r$  such that  $a = bq + r$  and the degree of  $r$  is less than the degree of  $b$ . This operation is called "left division." Operation **leftQuotient** ( $a, b$ ) returns  $q$ , and **leftRemainder** ( $a, b$ ) returns  $r$ .

**leader** (*differentialPolynomial*)

**leader** ( $p$ ) returns the derivative of the highest rank appearing in the differential polynomial  $p$ , or calls **error** if  $p$  is in the ground ring.

**leadingCoefficient** (*polynomial*)

**leadingCoefficient** ( $p$ ) returns the coefficient of the highest degree term of polynomial  $p$ . See also **IndexedDirectProductCategory** and **MonogenicLinearOperator**.

**leadingIdeal** (*ideal*)

**leadingIdeal** ( $I$ ) is the ideal generated by the leading terms of the elements of the ideal  $I$ .

**leadingMonomial** (*polynomial*)

**leadingMonomial** ( $p$ ) returns the monomial of polynomial  $p$  with the highest degree.

**leaf?** (*aggregate*)

**leafValues** (*aggregate*)

**leaves** (*aggregate*)

These operations apply to a recursive aggregate  $a$ . See, for example, **BinaryTree**.

**leaf?** ( $a$ ) tests if  $a$  is a terminal node.

**leaves** ( $a$ ) returns the list of values at the leaf nodes in left-to-right order.

**left** (*binaryRecursiveAggregate*)

**left** ( $a$ ) returns the left child of binary aggregate  $a$ .

**leftAlternative?** ()

**leftAlternative?** ()\$ $F$ , where  $F$  is a domain of **FiniteRankNonAssociativeAlgebra**, tests if  $2 * \text{assciator}(a, a, b) = 0$  for all  $a, b$  in  $F$ . Note: in general, you do not know whether  $2 * a = 0$  implies  $a = 0$ .

**leftCharacteristicPolynomial** (*polynomial*)

**leftCharacteristicPolynomial** ( $p$ )\$ $F$  returns the characteristic polynomial of the left regular representation of  $p$  of domain  $F$  with respect to any basis. Argument  $p$  is a member of a domain of category **FiniteRankNonAssociativeAlgebra**( $R$ ) where  $R$  is a commutative ring.

**leftDiscriminant** ([*listOfVectors*])

**leftDiscriminant** ( $[v_1, \dots, v_n]$ )\$ $F$  where  $F$  is a domain of category **FramedNonAssociativeAlgebra** over a commutative ring  $R$ , returns the determinant of the  $n$ -by- $n$  matrix whose element at the  $i$ <sup>th</sup> row and  $j$ <sup>th</sup> column is given by the left trace of the product  $v_i * v_j$ . Same as

**determinant**(**leftTraceMatrix** ( $[v_1, \dots, v_n]$ )). If no argument is given,  $v_1, \dots, v_n$  are taken to elements of the fixed  $R$ -basis.

**leftGcd** (*lodOperator*, *lodOperator*)

**leftGcd** ( $a, b$ ) computes the value  $g$  of highest degree such that  $a = aa * g$  and  $b = bb * g$  for some values  $aa$  and  $bb$ . The value  $g$  is computed using left-division.

**leftLcm** (*lodOperator*, *lodOperator*)

**leftLcm** ( $a, b$ ) computes the value  $m$  of lowest degree such that  $m = a * aa = b * bb$  for some values  $aa$  and  $bb$ . The value  $m$  is computed using left-division.

**leftMinimalPolynomial** (*element*)

**leftMinimalPolynomial** ( $a$ ) returns the polynomial determined by the smallest non-trivial linear combination of left powers of  $a$ , an element of a domain of category `FiniteRankNonAssociativeAlgebra`. Note: the polynomial has no a constant term because, in general, the algebra has no unit.

**leftNorm** (*element*)

**leftNorm** ( $a$ ) returns the determinant of the left regular representation of  $a$ , an element of a domain of category `FiniteRankNonAssociativeAlgebra`.

**leftPower** (*monad*, *nonNegativeInteger*)

**leftPower** ( $a, n$ ) returns the  $n^{\text{th}}$  left power of monad  $a$ , that is, **leftPower** ( $a, n$ ) := **leftPower** ( $a, n - 1$ ). If the monad has a unit then **leftPower** ( $a, 0$ ) := 1. Otherwise, define **leftPower** ( $a, 1$ ) =  $a$ . See `Monad` and `MonadWithUnit` for details. See also **leftRecip**.

**leftRankPolynomial** ()

**leftRankPolynomial** ()\$ $F$  calculates the left minimal polynomial of a generic element of an algebra of domain  $F$ , a domain of category `FramedNonAssociativeAlgebra` over a commutative ring  $R$ . This generic element is an element of the algebra defined by the same structural constants over the polynomial ring in symbolic coefficients with respect to the fixed basis.

**leftRank** (*element*)

**leftRank** ( $x$ ) returns the number of linearly independent elements in  $xb_1, \dots, xb_n$ , where  $b = [b_1, \dots, b_n]$  is a basis. Argument  $x$  is an element of a domain of category `FramedNonAssociativeAlgebra` over a commutative ring  $R$ .

**leftRecip** (*element*)

**leftRecip** ( $a$ ) returns an element that is a left inverse of  $a$ , or "failed", if there is no unit element, such an element does not exist, or the left reciprocal cannot be determined (see `unitsKnown`).

**leftRecip** (*element*)

**leftRecip** ( $a$ ) returns an element, which is a left inverse of  $a$ , or "failed" if such an element doesn't exist or cannot be determined (see `unitsKnown`).

**leftRegularRepresentation** (*element* [, *vectorOfElements*])

This operation is defined on a domain  $F$  of category `NonAssociativeAlgebra`.

**leftRegularRepresentation** ( $a$  [,  $[v_1, \dots, v_n]$ ]) returns the matrix of the linear map defined by left multiplication by  $a$  with respect to the basis  $[v_1, \dots, v_n]$ . If a second argument is missing, the basis is taken to be the fixed basis for  $F$ .

**leftTraceMatrix** ([*vectorOfElements*])

This operation is defined on a domain  $F$  of category `NonAssociativeAlgebra`.

**leftTraceMatrix** ( $[v]$ ), where  $v$  is an optional vector  $[v_1, \dots, v_n]$ , returns the  $n$ -by- $n$  matrix  $M$  such that  $M_{i,j}$  is the left trace of the product  $v_i * v_j$  of elements from the basis  $[v_1, \dots, v_n]$ . If the argument is missing, the basis is taken to be the fixed basis for  $F$ .

**leftTrace** (*element*)

**leftTrace** ( $a$ ) returns the trace of the left regular representation of  $a$ , an element of a domain of category `FiniteRankNonAssociativeAlgebra`.

**leftTrim** (*string*, *various*)

**leftTrim** ( $s, c$ ) returns string  $s$  with all leading characters  $c$  deleted. For example, **leftTrim** (" abc ", " ") returns "abc ".

**leftTrim** ( $s, cc$ ) returns  $s$  with all leading characters in  $cc$  deleted. For example, **leftTrim** ("abc)", `charClass` "()") returns "abc".

**leftUnit** ()

**leftUnits** ()

These operations are defined on a domain  $F$  of category `NonAssociativeAlgebra`.

**leftUnit** ()\$ $F$  returns a left unit of the algebra (not necessarily unique), or "failed" if there is none.

**leftUnits** ()\$ $F$  returns the affine space of all left units of an algebra  $F$ , or "failed" if there is none, where  $F$  is a domain of category `FiniteRankNonAssociativeAlgebra`. The normal result is returned as a record with selector *particular* for an element of  $F$ , and *basis* for a list of elements of  $F$ .

**legendreSymbol** (*integer*, *integer*)

**legendreSymbol** ( $a, p$ ) returns the Legendre symbol  $L(a/p)$ ,  $L(a/p) = (-1)^{(p-1)/2} \text{ mod } p$  for prime  $p$ . This is 0 if  $a = 0$ , 1 if  $a$  is a quadratic residue **mod**  $p$ , and -1 otherwise. Note: because the primality test is expensive, use **jacobi** ( $a, p$ ) if you know that  $p$  is prime.

**LegendreP** (*nonNegativeInteger*, *element*)

**LegendreP** ( $n, x$ ) is the  $n^{\text{th}}$  Legendre polynomial,  $P[n](x)$ , defined by  $\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P[n](x)t^n$ .

**length** (*various*)

**length** ( $a$ ) returns the length of integer  $a$  in digits.

**less?** (*aggregate*, *nonNegativeInteger*)

**less?** ( $u, n$ ) tests if  $u$  has less than  $n$  elements.

**leviCivitaSymbol** ()

**leviCivitaSymbol** () is the rank  $\text{dim}$  tensor defined by **leviCivitaSymbol** ( $i_1, \dots, i_{\text{dim}}$ ), which is  $+1$ ,  $-1$  or  $0$  according to whether the permutation  $i_1, \dots, i_{\text{dim}}$  is an even permutation, an odd permutation, or not a permutation of  $i_0, \dots, i_0 + \text{dim} - 1$ , respectively, where  $i_0$  is the minimum index.

**lexGroebner** (*listOfPolynomials*, *listOfSymbols*)

**lexGroebner** ( $lp, lv$ ) computes a Gröbner basis for the list of polynomials  $lp$  in lexicographic order. The variables  $lv$  are ordered by their position in the list  $lp$ .

**lhs** (*equationOrRewriteRule*)

**lhs** ( $x$ ) returns the left hand side of an equation or rewrite-rule.

**library** (*filename*)

**library** ( $name$ ) creates a new library file with filename  $name$ .

**lieAdmissible?** ()

**lieAdmissible?** ( $\$F$ ) tests if the algebra defined by the commutators is a Lie algebra. The domain  $F$  is a member of the category `FiniteRankNonAssociativeAlgebra(R)`. The property of anticommutativity follows from the definition.

**lieAlgebra?** ()

**lieAlgebra?** ( $\$F$ ) tests if the algebra of  $F$  is anticommutative and that the Jacobi identity  $(a * b) * c + (b * c) * a + (c * a) * b = 0$  is satisfied for all  $a, b, c$  in  $F$ .

**light** (*color*)

**light** ( $c$ ) sets the shade of a hue  $c$  to its highest value.

**limit** (*expression*, *equation* [, *direction*])

**limit** ( $f(x), x = a$ ) computes the real two-sided limit of  $f$  as its argument  $x$  approaches  $a$ .

**limit** ( $f(x), x = a, "left"$ ) computes the real limit of  $f$  as its argument  $x$  approaches  $a$  from the left.

**limit** ( $f(x), x = a, "right"$ ) computes the corresponding limit as  $x$  approaches  $a$  from the right.

**limitedIntegrate** (*rationalFunction*, *symbol*,

*listOfRationalFunctions*)

**limitedIntegrate** ( $f, x, [g_1, \dots, g_n]$ ) returns fractions  $[h, [c_i, g_i]]$  such that the  $g_i$ 's are among  $[g_1, \dots, g_n]$ ,  $dc_i/dx = 0$ , and  $d(h + \sum_i c_i \log g_i)/dx = f$  if possible, "failed" otherwise.

**linearDependenceOverZ** (*vector*)

**linearlyDependentOverZ?** (*vector*)

**linearlyDependentOverZ** ( $[v_1, \dots, v_n]$ ) tests if the elements  $v_i$  of a ring (typically algebraic numbers or Expressions) are linearly dependent over the integers. If so, the operation returns  $[c_1, \dots, c_n]$  such that  $c_1 v_1 + \dots + c_n v_n = 0$  (for which not all the  $c_i$ 's are 0). If linearly independent over the integers, "failed" is returned.

**linearlyDependentOverZ?** ( $[v_1, \dots, v_n]$ ) returns *true* if the  $v_i$ 's are linearly dependent over the integers, and *false* otherwise.

**lineColorDefault** ([*palette*])

**lineColorDefault** () returns the default color of lines connecting points in a two-dimensional viewport.

**lineColorDefault** ( $p$ ) sets the default color of lines connecting points in a two-dimensional viewport to the palette  $p$ .

**linSolve** (*listOfPolynomials*, *listOfVariables*)

**linSolve** ( $lp, lvar$ ) finds the solutions of the linear system of polynomials  $lp = 0$  with respect to the list of symbols  $lvar$ .

**li** (*expression*)

**li** ( $x$ ) returns the logarithmic integral of  $x$  defined by,  $\int \frac{dx}{\log(x)}$ .

**list** (*element*)

**list** ( $x$ ) creates a list consisting of the one element  $x$ .

**list?** (*sExpression*)

**list?** ( $s$ ) tests if SExpression value  $s$  is a Lisp list, possibly the null list.

**listBranches** (*listOfListsOfPoints*)

**listBranches** ( $c$ ) returns a list of lists of points representing the branches of the curve  $c$ .

**listRepresentation** (*permutation*)

**listRepresentation** ( $p$ ) produces a representation *rep* of the permutation  $p$  as a list of preimages and images  $i$ , that is, permutation  $p$  maps (*rep.preimage*). $k$  to (*rep.image*). $k$  for all indices  $k$ .

**listYoungTableaus** (*listOfIntegers*)

**listYoungTableaus** ( $\lambda$ ), where  $\lambda$  is a proper partition, generates the list of all standard tableaux of shape  $\lambda$  by means of lattice permutations. The



numbers of the lattice permutation are interpreted as column labels.

**listOfComponents** (*threeSpace*)

**listOfComponents** (*sp*) returns a list of list of list of points for threeSpace object *sp* assumed to be composed of a list of components, each a list of curves, which in turn is each a list of points, or calls **error** if this is not possible.

**listOfCurves** (*sp*) returns a list of list of subspace component properties for threeSpace object *sp* assumed to be a list of curves, each of which is a list of subspace components, or calls **error** if this is not possible.

**lo** (*segment*)

**lo** (*s*) returns the first endpoint of *s*. For example,  
**lo**(1..h) = 1.

**log** (*expression*)

**logIfCan** (*expression*)

**log** (*x*) returns the natural logarithm of *x*.

**logIfCan** (*z*) returns **log** (*z*) if possible, and "failed" otherwise.

**log2** ([*float*])

**log2** () returns  $\ln(2) = 0.6931471805\dots$

**log2** (*x*) computes the base 2 logarithm for *x*.

**log10** ([*float*])

**log10** () returns  $\ln(10) = 2.3025809299\dots$

**log10** (*x*) computes the base 10 logarithm for *x*.

**logGamma** (*float*)

**logGamma** (*x*) is the natural log of  $\Gamma(x)$ . Note: this can often be computed even if  $\Gamma(x)$  cannot.

**lowerCase** ([*string*])

**lowerCase?** (*character*)

**lowerCase** () returns the class of all characters for which **lowerCase?** is *true*.

**lowerCase** (*c*) returns a corresponding lower case alphabetic character *c* if *c* is an upper case alphabetic character, and *c* otherwise.

**lowerCase** (*s*) returns the string with all characters in lower case.

**lowerCase?** (*c*) tests if character *c* is an lower case letter, that is, one of *a...z*.

**listOfProperties** (*threeSpace*)

**listOfProperties** (*sp*) returns a list of subspace component properties for *sp* of type ThreeSpace, or calls **error** if this is not possible.

**listOfPoints** (*threeSpace*)

**listOfPoints** (*sp*), where *sp* is a ThreeSpace object, returns the list of points component contained in *sp*.

**mainKernel** (*expression*)

**mainKernel** (*f*) returns a kernel of *f* with maximum nesting level, or "failed" if *f* has no kernels (that is, *f* is a constant).

**mainVariable** (*polynomial*)

**mainVariable** (*u*) returns the variable of highest ordering that actually occurs in the polynomial *p*, or "failed" if no variables are present. Argument *u* can be either a polynomial or a rational function.

**makeFloatFunction** (*expression*, *symbol* [, *symbol*])

Argument *expr* may be of any type that is coercible to type InputForm (objects of the most common types can be so coerced).

**makeFloatFunction** (*expr*, *x*) returns an anonymous function of type  $\text{Float} \rightarrow \text{Float}$  defined by  $x \mapsto \text{expr}$ .

**makeFloatFunction** (*expr*, *x*, *y*) returns an anonymous function of type  $(\text{Float}, \text{Float}) \rightarrow \text{Float}$  defined by  $(x, y) \mapsto \text{expr}$ .

**makeVariable** (*element*)

**makeVariable** (*s*), where *s* is a symbol, differential indeterminate, or a differential polynomial, returns a function *f* defined on the non-negative integers such that *f*(*n*) returns the *n*<sup>th</sup> derivative of *s*.

**makeVariable** (*s*, *n*) returns the *n*<sup>th</sup> derivative of a differential indeterminate *s* as an algebraic indeterminate.

**makeObject** (*functions*, *range* [, *range*])

Arguments *f*, *g*, and *h* appearing below with arguments (for example, *f*(*x*, *y*)) denote symbolic expressions involving those arguments.

Arguments *f*, *g*, and *h* appearing below as symbols without arguments denote user-defined functions which map one or more DoubleFloat values to DoubleFloat values.

Values *a*, *b*, *c*, and *d* denote numerical values.

**makeObject** (*curve*(*f*, *g*, *h*), *a..b*) returns the space *sp* of the domain ThreeSpace with the addition of the graph of the parametric curve  $x = f(t)$ ,  $y = g(t)$ ,  $z = h(t)$  as *t* ranges from **min** (*a*, *b*) to **max** (*a*, *b*).

**makeObject** (*curve*(*f*(*t*), *g*(*t*), *h*(*t*)), *t = a..b*) returns the space *sp* of the domain ThreeSpace with the addition of the graph of the parametric curve  $x = f(t)$ ,  $y = g(t)$ ,  $z = h(t)$  as *t* ranges from **min** (*a*, *b*) to **max** (*a*, *b*).

**makeObject** (*f*, *a..b*, *c..d*) returns the space *sp* of the domain ThreeSpace with the addition of the graph of  $z = f(x, y)$  as *x* ranges from **min** (*a*, *b*) to **max** (*a*, *b*) and *y* ranges from **min** (*c*, *d*) to **max** (*c*, *d*).

**makeObject** (*f*(*x*, *y*),  $x = a..b$ ,  $y = c..d$ ) returns the space *sp* of the domain ThreeSpace with the addition of the graph of  $z = f(x, y)$  as *x* ranges from **min** (*a*, *b*) to **max** (*a*, *b*) and *y* ranges from **min** (*c*, *d*) to **max** (*c*, *d*).

**makeObject** (*surface*(*f*, *g*, *h*), *a..b*, *c..d*) returns the space *sp* of the domain *ThreeSpace* with the addition of the graph of the parametric surface  $x = f(u, v)$ ,  $y = g(u, v)$ ,  $z = h(u, v)$  as *u* ranges from **min**(*a*, *b*) to **max**(*a*, *b*) and *v* ranges from **min**(*c*, *d*) to **max**(*c*, *d*).

*makeObject*(*surface*(*f*(*u*, *v*), *g*(*u*, *v*), *h*(*u*, *v*)), *u = a..b*, *v = c..d*) returns the space *sp* of the domain *ThreeSpace* with the addition of the graph of the parametric surface  $x = f(u, v)$ ,  $y = g(u, v)$ ,  $z = h(u, v)$  as *u* ranges from **min**(*a*, *b*) to **max**(*a*, *b*) and *v* ranges from **min**(*c*, *d*) to **max**(*c*, *d*).

**makeYoungTableau** (*listOfIntegers*, *listOfIntegers*)

**makeYoungTableau** (*lambda*, *gitter*) computes for a given lattice permutation *gitter* and for an improper partition *lambda* the corresponding standard tableau of shape *lambda*. See **listYoungTableaus**.

**mantissa** (*float*)

**mantissa** (*x*) returns the mantissa part of *x*.

**map** (*function*, *structure*[ , *structure*])

**map!** (*function*, *structure*)

**map** (*fn*, *u*) maps the one-argument function *fn* onto the components of a structure, returning a new structure. Most structures allow *f* to have different source and target domains. Specifically, the function *f* is mapped onto the following components of the structure as follows. If *u* is:

- a series: the coefficients of the series.
- a polynomial: the coefficients of the non-zero monomials.
- a direct product of elements: the elements.
- an aggregate, tuple, table, or a matrix: all its elements.
- an operation of the form *op*(*a*<sub>1</sub>, ..., *a*<sub>*n*</sub>): each *a*<sub>*i*</sub>, returning *op*(*f*(*a*<sub>1</sub>), ..., *f*(*a*<sub>*n*</sub>)).
- a fraction: the numerator and denominator.
- complex: the real and imaginary parts.
- a quaternion or octonion: the real and all imaginary parts.
- a finite or infinite series or stream: all the coefficients.
- a factored object: onto all the factors.
- a segment *a..b* or a segment binding of the form *x = a..b*: each of the elements from *a* to *b*.
- an equation: both sides of the equation.

**map** (*fn*, *u*, *v*) maps the two argument function *fn* onto the components of a structure, returning a new structure. Arguments *u* and *v* can be matrices, finite aggregates such as lists, tables, and vectors, and infinite aggregates such as streams and series.

**map!** (*f*, *u*), where *u* is homogeneous aggregate, destructively replaces each element *x* of *u* by *f*(*x*).

See also **match**.

**mapCoef** (*function*, *freeAbelianMonoid*)

**mapGen** (*function*, *freeAbelianMonoid*)

**mapCoeff** (*f*, *m*) maps unary function *f* onto the coefficients of a free abelian monoid of the form  $e_1 a_1 + \dots + e_n a_n$  returning  $f(e_1) a_1 + \dots + f(e_n) a_n$ .

**mapGen** (*fn*, *m*) similarly returns  $e_1 f(a_1) + \dots + e_n f(a_n)$ . See *FreeAbelianMonoidCategory* using *Browse*.

**mapDown!** (*tree*, *value*, *function*)

These operations make a preorder traversal (node then left branch then right branch) of a tree *t* of type *BalancedBinaryTree*(*S*), destructively mapping values of type *S* from the root to the leaves of the tree, then returning the modified tree as value; *p* is a value of type *S*.

**mapDown!** (*t*, *p*, *f*), where *f* is a function of type (*S*, *S*) → *S*, replaces the successive interior nodes of *t* as follows. The root value *x* is replaced by  $q = f(x, p)$ . Then **mapDown!** is recursively applied to (*l*, *q*, *f*) and (*r*, *q*, *f*) where *l* and *r* are respectively the left and right subtrees of *t*.

**mapDown!** (*t*, *p*, *f*), where *f* is a function of type (*S*, *S*, *S*) → *List S*, is similar. The root value of *t* is first replaced by *p*. Then *f* is applied to three values: the value at the current, left, and right node (in that order) to produce a list of two values *l* and *r*, which are then passed recursively as the second argument of **mapDown!** to the left and right subtrees.

**mapExponents** (*function*, *polynomial*)

**mapExponents** (*fn*, *u*) maps function *fn* onto the exponents of the non-zero monomials of polynomial *u*.

**mapUp!** ([*tree*, ]*tree*, *function*)

These operations make an endorder traversal (left branch then right branch then node) of a tree *t* of type *BalancedBinaryTree*(*S*), destructively mapping values of type *S* from the leaves to the root of the tree, then returning the modified tree as value; *p* is a value of type *S*.

**mapUp!** (*t*, *f*), where *f* has type (*S*, *S*) → *S*, replaces the value at each interior node by  $f(l, r)$ , where *l* and *r* are the values at the immediate left and right nodes.

**mapUp!** (*t*, *t*<sub>1</sub>, *f*) makes an endorder traversal of both *t* and *t*<sub>1</sub> (of identical shape) in parallel. The value at each successive interior node of *t* is replaced by  $f(l, r, l_1, r_1)$ , where *l* and *r* are the values at the immediate left and right nodes of *t*, and *l*<sub>1</sub> and *r*<sub>1</sub> are corresponding values of *t*<sub>1</sub>.

**mask** (*integer*)

**mask** (*n*) returns  $2^n - 1$  (an *n*-bit mask).

**match?** (*string*, *string*, *character*)

**match?** (*s*, *t*, *char*) tests if *s* matches *t* except perhaps for multiple and consecutive occurrences of character *char*. Typically *char* is the blank character.

**match** (*list*, *list*[ , *option*])

**match** (*la*, *lb*[, *u*]), where *la* and *lb* are lists of equal length,

creates a function that can be used by **map**. The target of a source value  $x$  in  $la$  is the value  $y$  with the corresponding index in  $lb$ . Optional argument  $u$  defines the target for a source value  $a$  which is not in  $la$ . If  $u$  is a value of the source domain, then  $a$  is replaced by  $u$ , which must be a member of  $la$ . If  $u$  is a value of the target domain, the value returned by the map for  $a$  is  $u$ . If  $u$  is a function  $f$ , then the value returned is  $f(a)$ . If no third argument is given, an error occurs when such a  $a$  is found.

**mathieu11** ( $[listOfIntegers]$ )

**mathieu12** ( $[listOfIntegers]$ )

**mathieu22** ( $[listOfIntegers]$ )

**mathieu23** ( $[listOfIntegers]$ )

**mathieu24** ( $[listOfIntegers]$ )

**mathieu11** ( $[li]$ ) constructs the mathieu group acting on the eleven integers given in the list  $li$ . Duplicates in the list will be removed and **error** will be called if  $li$  has fewer or more than eleven different entries. The default value of  $li$  is  $[1, \dots, 11]$ . Operations **mathieu12**, **mathieu22**, and **mathieu23** and **mathieu24** are similar. These operations provide examples of permutation groups in AXIOM.

**matrix** ( $listOfLists$ )

**matrix** ( $l$ ) converts the list of lists  $l$  to a matrix, where the list of lists is viewed as a list of the rows of the matrix.

**matrix** ( $llo$ ), where  $llo$  is a list of list of objects of type **OutputForm** (normally unexposed), returns an output form displaying  $llo$  as a matrix.

**max** ( $[various]$ )

**max** () returns the largest small integer.

**max** ( $u$ ) returns the largest element of aggregate  $u$ .

**max** ( $x, y$ ) returns the maximum of  $x$  and  $y$  relative to a total ordering “ $<$ ”.

**maxColIndex** ( $matrix$ )

**maxColIndex** ( $m$ ) returns the index of the last column of the matrix or two-dimensional array  $m$ .

**maxIndex** ( $aggregate$ )

**maxIndex** ( $u$ ) returns the maximum index  $i$  of indexed aggregate  $u$ . For most indexed aggregates (vectors, strings, lists), **maxIndex** ( $u$ ) is equivalent to  $\#u$ .

**maxRowIndex** ( $matrix$ )

**maxRowIndex** ( $m$ ) returns the index of the “last” row of the matrix or two-dimensional array  $m$ .

**meatAxe** ( $listOfListsOfMatrices$  [ $, boolean, integer, integer$ ])

**meatAxe** ( $aG$ ,  $randomElts$ ,  $numOfTries$ ,  $maxTests$ ) tries to split the representation given by  $aG$  and returns a 2-list of representations. All matrices of argument  $aG$  are assumed to be square and of equal size. The default values of arguments  $randomElts$ ,  $numOfTries$  and  $maxTests$

are *false*, 25, and 7, respectively.

**member?** ( $element, aggregate$ )

**member?** ( $x, u$ ) tests if  $x$  is a member of  $u$ .

**member?** ( $pp, gp$ ), where  $pp$  is a permutation and  $gp$  is a group, tests whether  $pp$  is in the group  $gp$ .

**merge** ( $various$ )

**merge!** ( $various$ )

**merge** ( $[s1, s2, \dots, sn]$ ) will create a new **ThreeSpace** object that has the components of all the ones in the list; groupings of components into composites are maintained.

**merge** ( $s1, s2$ ) will create a new **ThreeSpace** object that has the components of  $s1$  and  $s2$ ; groupings of components into composites are maintained.

**merge** ( $[p, ]a, b$ ) returns an aggregate  $c$  which merges  $a$  and  $b$ . The result is produced by examining each element  $x$  of  $a$  and  $y$  of  $b$  successively. If  $p(x, y)$  is *true*, then  $x$  is inserted into the result. Otherwise  $y$  is inserted. If  $x$  is chosen, the next element of  $a$  is examined, and so on. When all the elements of one aggregate are examined, the remaining elements of the other are appended. For example, **merge** ( $<, [1, 3], [2, 7, 5]$ ) returns  $[1, 2, 3, 7, 5]$ . By default, function  $p$  is  $\leq$ .

**merge!** ( $[p], u, v$ ) destructively merges the elements  $u$  and  $v$  into  $u$  using comparison function  $p$ . Function  $p$  is  $\leq$  by default.

**mesh** ( $u$ ,  $v, w, x$ )

Argument  $sp$  below is a **ThreeSpace** object  $sp$ . Argument  $lc$  is a list of curves. Each curve is either a list of points (objects of type **Point**) or else a list of lists of small floats. **mesh** ( $lc$ ) returns a **ThreeSpace** object defined by  $lc$ .

**mesh** ( $sp$ ) returns the list of curves contained in space  $sp$ .

**mesh** ( $[sp, ], lc, close1, close2$ ) adds the list of curves  $lc$  to the **ThreeSpace** object  $sp$ . Boolean arguments  $close1$  and  $close2$  tell how the curves and surface are to be closed. If  $close1$  is *true*, each individual curve will be closed, that is, the last point of the list will be connected to the first point. If  $close2$  is *true*, the first and last curves are regarded as boundaries and are connected. By default, the argument  $sp$  is empty.

**midpoints** ( $listOfIntervals$ )

These operations are defined on “intervals” represented by records with keys *right* and *left*, and rational number values.

**midpoints** ( $isolist$ ) returns the list of midpoints for the list of intervals  $isolist$ .

**midpoint** ( $int$ ) returns the midpoint of the interval  $int$ .

**min** ( $[u, v]$ )

**min** () returns the element of type **SingleInteger**.

**min** ( $u$ ) returns the smallest element of aggregate  $u$ .

**min** ( $x, y$ ) returns the minimum of  $x$  and  $y$  relative to total ordering  $<$ .

**minColIndex** (*matrix*)

**minColIndex** (*m*) returns the index of the “first” column of the matrix or two-dimensional array *m*.

**minimalPolynomial** (*element*, *positiveInteger*)

**minimalPolynomial** (*x*, *n*) computes the minimal polynomial of *x* over the field of extension degree *n* over the ground field *F*. The default value of *n* is 1.

**minimalPolynomial** (*element*)

**minimalPolynomial** (*a*) returns the minimal polynomial of element *a* of a finite rank algebra. See **FiniteRankAlgebra** using **Browse**.

**minimumDegree** (*polynomial*, *variable*)

**minimumDegree** (*p*, *v*) gives the minimum degree of polynomial *p* with respect to *v*, that is, viewed as a univariate polynomial in *v*.

**minimumDegree** (*p*, *lv*) gives the list of minimum degrees of the polynomial *p* with respect to each of the variables in the list *lv*.

See also **FiniteAbelianMonoidRing** and **MonogenicLinearOperator**.

**minIndex** (*aggregate*)

**minIndex** (*aggregate*) returns the minimum index *i* of aggregate *u*. Note: the **minIndex** of most system-defined indexed aggregates is 1. See also **PointCategory**.

**minordet** (*matrix*)

**minordet** (*m*) computes the determinant of the matrix *m* using minors, or calls **error** if the matrix is not square.

**minPoly** (*expression*)

**minPoly** (*k*) returns polynomial *p* such that  $p(k) = 0$ .

**minRowIndex** (*matrix*)

**minRowIndex** (*m*) returns the index of the “first” row of the matrix or two-dimensional array *m*.

**minusInfinity** ()

**minusInfinity** () returns **%minusInfinity**, the AXIOM name for  $-\infty$ .

**modifyPointData** (*space*, *nonNegativeInteger*, *point*)

**modifyPointData** (*sp*, *i*, *p*) changes the point at the indexed location *i* in the **ThreeSpace** object *sp* to *p*. This operation is useful for making changes to existing data.

**moduloP** (*integer*)

**moduloP** (*x*), such that  $p = \mathbf{modulus}()$ , returns *a*, where  $x = a + bp$  where *x* is a *p*-adic integer. See **PAdicIntegerCategory** using **Browse**.

**modulus** ()

**modulus** ()\$*R* returns the value of the modulus *p* of a

*p*-adic integer domain *R*. See **PAdicIntegerCategory** using **Browse**.

**moebiusMu** (*integer*)

**moebiusMu** (*n*) returns the Moebius function  $\mu(n)$ , defined as  $-1$ ,  $0$  or  $1$  as follows:  $\mu(n) = 0$  if *n* is divisible by a square  $> 1$ , and  $(-1)^k$  if *n* is square-free and has *k* distinct prime divisors.

**monicDivide** (*polynomial*, *polynomial* [ , *variable*])

**monicDivide** (*p*, *q*, [*v*]) divides the polynomial *p* by the monic polynomial *q*, returning the record containing a *quotient* and *remainder*. For multivariate polynomials, the polynomials are viewed as a univariate polynomials in *v*. If *p* and *q* are univariate polynomials, then the third argument may be omitted. The operation calls **error** if *q* is not monic with respect to *v*.

**monomial** (*coefficient*, *exponent* [ , *option*])

**monomial** (*coef*, *exp*) creates a term of a univariate polynomial or series object from a coefficient *coef* and exponent *exp*. The variable name must be given by context (as through a declaration for the result).

**monomial** (*c*, [*x*<sub>1</sub>, ..., *x*<sub>*k*</sub>], [*n*<sub>1</sub>, ..., *n*<sub>*k*</sub>]) creates a term  $cx_1^{n_1} \dots x_k^{n_k}$  of a multivariate power series or polynomial from coefficient *c*, variables *x*<sub>*j*</sub> and exponents *n*<sub>*j*</sub>.

**monomial** (*c*, *x*, *n*) creates a term  $cx^n$  of a polynomial or series from a coefficient *c*, variable *x*, and exponent *n*.

**monomial** (*c*, [*n*<sub>1</sub>, ..., *n*<sub>*k*</sub>]) creates a **CliffordAlgebra** element  $ce(n_1), \dots, ce(n_k)$  from a coefficient *c* and basis elements  $c(i_j)$

**monomial?** (*polynomialOrSeries*)

**monomial?** (*p*) tests if polynomial or series *p* is a single monomial.

**monomials** (*polynomial*)

**monomials** (*p*) returns the list of non-zero monomials of polynomial *p*, [*a*<sub>1</sub>*X*<sup>(1)</sup>, ..., *a*<sub>*n*</sub>*X*<sup>(*n*)</sup>].

**more?** (*aggregate*, *nonNegativeInteger*)

**more?** (*u*, *n*) tests if *u* has greater than *n* elements.

**movedPoints** (*permutation*)

**movedPoints** (*p*) returns the set of points moved by the permutation *p*.

**movedPoints** (*gp*) returns the points moved by the group *gp*.

**mulmod** (*integer*, *integer*, *integer*)

**mulmod** (*a*, *b*, *p*), where *a*, *b* are non-negative integers both  $<$  integer *p*, returns  $ab \bmod p$ .

**multiEuclidean** (*listOfElements*, *element*)

**multiEuclidean** (*[f*<sub>1</sub>, ..., *f*<sub>*n*</sub>], *z*) returns a list of coefficients [*a*<sub>1</sub>, ..., *a*<sub>*n*</sub>] such that  $z/\prod_{i=1}^n f_i = \sum_{j=1}^n a_j/f_j$ .

If no such list of coefficients exists, "failed" is returned.

**multinomial** (*integer*, *listOfIntegers*)

**multinomial** ( $n, [m_1, m_2, \dots, m_k]$ ) returns the multinomial coefficient  $n!/(m_1!m_2!\dots m_k!)$ .

**multiple** (*expression*)

**multiple** ( $x$ ) directs the pattern matcher that  $x$  should preferably match a multi-term quantity in a sum or product. For matching on lists, **multiple**( $x$ ) tells the pattern matcher that  $x$  should match a list instead of an element of a list. This operation calls **error** if  $x$  is not a symbol.

**multiplyCoefficients** (*function*, *series*)

**multiplyCoefficients** ( $f, s$ ) returns  $\sum_{n=0}^{\infty} f(n)a_nx^n$  where  $s$  is the series  $\sum_{n=0}^{\infty} a_nx^n$ .

**multiplyExponents** (*various*, *nonNegativeInteger*)

**multiplyExponents** ( $p, n$ ), where  $p$  is a univariate polynomial or series, returns a new polynomial or series resulting from multiplying all exponents by the non negative integer  $n$ .

**multiset** (*listOfElements*)

**multiset** ( $ls$ ) creates a multiset with elements from  $ls$ .

**multivariate** (*polynomial*, *symbol*)

**multivariate** ( $p, v$ ) converts an anonymous univariate polynomial  $p$  to a polynomial in the variable  $v$ .

**name** (*various*)

**name** ( $f$ ) returns the name part of the file name for file  $f$ .

**name** ( $op$ ) returns the name of basic operator  $op$ .

**name** ( $s$ ) returns symbol  $s$  without its scripts.

**nand** (*boolean*, *boolean*)

**nand** ( $a, b$ ) returns the logical negation of  $a$  and  $b$ , either booleans or bit aggregates. Note: **nand** ( $a, b$ ) = *true* if and only if one of  $a$  and  $b$  is *false*.

**nary?** (*basicOperator*)

**nary?** ( $op$ ) tests if  $op$  accepts an arbitrary number of arguments.

**ncols** (*matrix*)

**ncols** ( $m$ ) returns the number of columns in the matrix or two-dimensional array  $m$ .

**new** ([*various*])

**new** ()\$ $R$  create a new object of type  $R$ . When  $R$  is an aggregate, **new** creates an empty object. Other variations are as follows:

**new** ( $s$ ), where  $s$  is a symbol, returns a new symbol whose name starts with % $s$ .

**new** ( $n, x$ ) returns **fill!** (**new**( $n$ ),  $x$ ), an aggregate of  $n$  elements, each with value  $x$ .

**new** ( $m, n, r$ )\$ $R$  creates an  $m$ -by- $n$  array or matrix of type  $R$  all of whose entries are  $r$ .

**new** ( $d, pre, e$ ), where  $d$ , *smathpre*, and *smathe* are strings, constructs the name of a new writable file with  $d$  as its directory,  $pre$  as a prefix of its name and  $e$  as its extension. When  $d$  or  $e$  is the empty string, a default is used. The operation calls **error** if a new file cannot be written in the given directory.

**newLine** ()

**newLine** () sends a new line command to output. See DisplayPackage.

**nextColeman** (*listOfIntegers*, *listOfIntegers*, *matrix*)

**nextColeman** ( $alpha, beta, C$ ) generates the next Coleman-matrix of column sums  $alpha$  and row sums  $beta$  according to the lexicographical order from bottom-to-top. The first Coleman matrix is created using  $C = \mathbf{new}(1, 1, 0)$ . Also, **new** ( $1, 1, 0$ ) indicates that  $C$  is the last Coleman matrix. See SymmetricGroupCombinatoricFunctions for details.

**nextLatticePermutation** (*integers*, *integers*, *boolean*)

**nextLatticePermutation** ( $lambda, lattP, constructNotFirst$ ) generates the lattice permutation according to the proper partition  $lambda$  succeeding the lattice permutation  $lattP$  in lexicographical order as long as *constructNotFirst* is *true*. If *constructNotFirst* is *false*, the first lattice permutation is returned. The result *nil* indicates that  $lattP$  has no successor. See SymmetricGroupCombinatoricFunctions for details.

**nextPartition** (*vectorOfIntegers*, *vectorOfIntegers*, *integer*)

**nextPartition** ( $gamma, part, number$ ) generates the partition of  $number$  which follows  $part$  according to the right-to-left lexicographical order. The partition has the property that its components do not exceed the corresponding components of  $gamma$ . the first partition is achieved by  $part = []$ . Also,  $[]$  indicates that  $part$  is the last partition. See SymmetricGroupCombinatoricFunctions for details.

**nextPrime** (*positiveInteger*)

**nextPrime** ( $n$ ) returns the smallest prime strictly larger than  $n$ .

**nil** ()

**nil** ()\$ $R$  returns the empty list of type  $R$ .

**nilFactor** (*element*, *nonNegativeInteger*)

**nilFactor** ( $base, exponent$ ) creates a factored object with a single factor with no information about the kind of  $base$ . See Factored for details.

**node?** (*aggregate*, *aggregate*)

**node?** (*u*, *v*) tests if node *u* is contained in node *v* (either as a child, a child of a child, etc.).

**nodes** (*recursiveAggregate*)

**nodes** (*a*) returns a list of all the nodes of aggregate *a*.

**noncommutativeJordanAlgebra?** ()

**noncommutativeJordanAlgebra?** ()\$*F* tests if the algebra *F* is flexible and Jordan admissible. See `FiniteRankNonAssociativeAlgebra`.

**nor** (*boolean*, *boolean*)

**nor** (*a*, *b*) returns the logical *nor* of booleans or bit aggregates *a* and *b*. Note: **nor** (*a*, *b*) = true if and only if both *a* and *b* are *false*.

**norm** (*element* [, *option*])

**norm** (*x*) returns:

for complex *x*: **conjugate** (*x*) .

for floats: the absolute value.

for quaternions or octonions: the sum of the squares of its coefficients.

for a domain of category `FiniteRankAlgebra`: the determinant of the regular representation of *x* with respect to any basis.

**norm** (*x* [, *p*]), where *p* is a positiveInteger and *x* is an element of a domain of category `FiniteAlgebraExtensionField` over ground field *F*, returns the norm of *x* with respect to the field of extension degree *d* over the ground field of size. The default value of *p* is 1. The operation calls **error** if *p* does not divide the extension degree of *x*. Note:

$$\mathbf{norm}(x, p) = \prod_{i=0}^{n/p} x^{q^{pi}}$$

**normal?** (*element*)

**normal?** (*a*), where *a* is a member of a domain of category `FiniteAlgebraicExtensionField` over a field *F*, tests whether the element *a* is normal over the ground field *F*, that is, if  $a^{q^i}$ ,  $0 \leq i \leq \mathbf{extensionDegree}() - 1$  is an *F*-basis, where  $q = \mathbf{size}()$ .

**normalElement** ()

**normalElement** ()\$*R*, where *R* is a domain of category `FiniteAlgebraicExtensionField` over a field *F*, returns a element, normal over the ground field *F*, that is,  $a^{q^i}$ ,  $0 \leq i < \mathbf{extensionDegree}()$  is an *F*-basis, where  $q = \mathbf{size}()$ . At the first call, the element is computed by **createNormalElement** then cached in a global variable. On subsequent calls, the element is retrieved by referencing the global variable.

**normalForm** (*polynomial*, *listOfpolynomials*)

**normalForm** (*poly*, *gb*) reduces the polynomial *poly* modulo the precomputed Gröbner basis *gb* giving a

canonical representative of the residue class.

**normalise** (*element*)

**normalise** (*v*) returns the column vector *v* divided by its Euclidean norm; when possible, the vector *v* is expressed in terms of radicals.

**normalize** (*element* [, *option*])

**normalize** (*flt*) normalizes float *flt* at current precision.

**normalize** (*f* [, *x*]) rewrites *f* using the least possible number of real algebraically independent kernels involving symbol *x*. If no symbol *x* is given, the operation rewrites *f* using the least possible number of real algebraically independent kernels.

**normalizeAtInfinity** (*vectorOfFunctions*)

**normalizeAtInfinity** (*v*) makes *v* normal at infinity, where *v* is a vector of functions defined on a curve.

**not** (*boolean*)

**not** (*n*) returns the negation of boolean or bit aggregate *n*.

**not** (*n*) returns the bit-by-bit logical *not* of the small integer *n*.

**nrows** (*matrix*)

**nrows** (*m*) returns the number of rows in the matrix or two-dimensional array *m*.

**nthExponent** (*factored*, *positiveInteger*)

**nthExponent** (*u*, *n*) returns the exponent of the *n*<sup>th</sup> factor of *u*, or 0 if *u* has no such factor.

**nthFactor** (*factor*, *positiveInteger*)

**nthFactor** (*u*, *n*) returns the base of the *n*<sup>th</sup> factor of *u*, or 1 if *n* is not a valid index for a factor. If *u* consists only of a unit, the unit is returned.

**nthFlag** (*factored*, *positiveInteger*)

**nthFlag** (*u*, *n*) returns the information flag of the *n*<sup>th</sup> factor of *u*, "nil" if *n* is not a valid index for a factor.

**nthFractionalTerm** (*partialFraction*, *integer*)

**nthFractionalTerm** (*p*, *n*) extracts the *n*<sup>th</sup> fractional term from the partial fraction *p*, or 0 if the index *n* is out of range.

**nthRoot** (*expression*, *integer*)

**nthRootIfCan** (*expression*, *integer*)

Argument *x* can be of type `Expression`, `Complex`, `Float` and `DoubleFloat`, or a series.

**nthRoot** (*x*, *n*) returns the *n*<sup>th</sup> root of *x*. If *x* is not an expression, the operation calls **error** if this is not possible.

**nthRootIfCan** (*z*, *n*) returns the *n*<sup>th</sup> root of *z* if possible, and "failed" otherwise.

**null?** (*sExpression*)

**null?** (*s*) is *true* if *s* is the SExpression object ().

**nullary** ()

**nullary** (*x*), where *x* has type *R*, returns a function *f* of type  $\rightarrow R$  such that *f*() returns the value *c*. See also **constant** for a similar operation.

**nullary?** (*basicOperator*)

**nullary?** (*op*) tests if basic operator *op* is nullary.

**nullity** (*matrix*)

**nullity** (*m*) returns the dimension of the null space of the matrix *m*.

**nullSpace** (*matrix*)

**nullSpace** (*m*) returns a basis for the null space of the matrix *m*.

**numberOfComponents** ([*threeSpace*])

**numberOfComponents** ()\$*F* returns the number of absolutely irreducible components for a domain *F* of functions defined over a curve.

**numberOfComponents** (*sp*) returns the number of distinct object components in the ThreeSpace object *s* such as points, curves, and polygons.

**numberOfComputedEntries** (*stream*)

**numberOfComputedEntries** (*st*) returns the number of explicitly computed entries of stream *st*.

**numberOfCycles** (*permutation*)

**numberOfCycles** (*p*) returns the number of non-trivial cycles of the permutation *p*.

**numberOfDivisors** (*integer*)

**numberOfDivisors** (*n*) returns the number of integers between 1 and *n* inclusive which divide *n*. The number of divisors of *n* is often denoted by  $\tau(n)$ .

**numberOfFactors** (*factored*)

**numberOfFactors** (*u*) returns the number of factors in factored form *u*.

**numberOfFractionalTerms** (*partialFraction*)

**numberOfFractionalTerms** (*p*) computes the number of fractional terms in *p*, or 0 if there is no fractional part.

**numberOfHues** ()

**numberOfHues** () returns the number of total hues. See also **totalHues**.

**numberOfImproperPartitions** (*integer*, *integer*)

**numberOfImproperPartitions** (*n*, *m*) computes the number of partitions of the nonnegative integer *n* in *m* nonnegative parts with regarding the order (improper

partitions). Example: **numberOfImproperPartitions** (3, 3) is 10, since [0, 0, 3], [0, 1, 2], [0, 2, 1], [0, 3, 0], [1, 0, 2], [1, 1, 1], [1, 2, 0], [2, 0, 1], [2, 1, 0], [3, 0, 0] are the possibilities. Note: this operation has a recursive implementation.

**numberOfMonomials** (*polynomial*)

**numberOfMonomials** (*p*) gives the number of non-zero monomials in polynomial *p*.

**numer** (*fraction*)

**numerator** (*fraction*)

Argument *x* is from domain Fraction(*R*) for some domain *R*, or of type Expression

**numer** (*x*) returns the numerator of *x* as an object of domain *R*; if *x* is of type Expression, it returns an object of domain SMP(*D*, Kernel(Expression *R*)).

**numerator** (*x*) returns the numerator of *x* as an element of Fraction(*R*); if *x* if of type Expression, it returns an object of domain Expression.

**numerators** (*continuedFraction*)

**numerators** (*cf*) returns the stream of numerators of the approximants of the continued fraction *cf*. If the continued fraction is finite, then the stream will be finite.

**numeric** (*expression* [, *n*])

**numeric** (*x*, *n*) returns a float approximation of expression *x* to *n* decimal digits accuracy.

**objectOf** (*typeAnyObject*)

**objectOf** (*a*) returns a printable form of an object of type Any.

**objects** (*threeSpace*)

**objects** (*sp*) returns the ThreeSpace object *sp*. The result is returned as record with fields: *points*, the number of points; *curves*, the number of curves; *polygons*, the number of polygons; and *constructs*, the number of constructs.

**oblateSpheroidal** (*function*)

**oblateSpheroidal** (*a*), where *a* is a small float, returns a function to map the point  $(\xi, \eta, \phi)$  to cartesian coordinates  $x = a \sinh(\xi) \sin(\eta) \cos(\phi)$ ,  $y = a \sinh(\xi) \sin(\eta) \sin(\phi)$ ,  $z = a \cosh(\xi) \cos(\eta)$ .

**octon** (*element*, *element* [, *elements*])

**octon** (*q<sub>e</sub>*, *q<sub>E</sub>*) constructs an octonion whose first 4 components are given by a quaternion *q<sub>e</sub>* and whose last 4 components are given by a quaternion *q<sub>E</sub>*.

**octon** (*r<sub>e</sub>*, *r<sub>i</sub>*, *r<sub>j</sub>*, *r<sub>k</sub>*, *r<sub>E</sub>*, *r<sub>I</sub>*, *r<sub>J</sub>*, *r<sub>K</sub>*) constructs an octonion from scalars.

**odd?** (*x*)

**odd?** (*n*) tests if integer *n* is odd.

**odd?** (*p*) tests if *p* is an odd permutation, that is, **sign** (*p*)

is  $-1$ .

**oneDimensionalArray** (*[integer, ]elements*)

**oneDimensionalArray** (*ls*) creates a one-dimensional array consisting of the elements of list *ls*.

**oneDimensionalArray** (*n, s*) creates a one-dimensional array of *n* elements, each with value *s*.

**one?** (*element*)

**one?** (*a*) tests whether *a* is the unit 1.

**open** (*file* [, *string*])

**open** (*s* [, *mode*]) returns the file *s* open in the indicated mode: "input" or "output". Argument *mode* is "output" by default.

**operator** (*symbol* [, *nonNegativeInteger*])

**operator** (*f, n*) makes *f* into an *n*-ary operator. If the second argument *n* is omitted, *f* has arbitrary *arity*, that is, *f* takes an arbitrary number of arguments.

**operators** (*expression*)

**operators** (*f*) returns a list of all basic operators in *f*, regardless of level.

**optional** (*symbol*)

**optional** (*x*) tells the pattern matcher that *x* can match an identity (0 in a sum, 1 in a product or exponentiation), or calls **error** if *x* is not a symbol.

**or** (*boolean, boolean*)

*a* **or** *b* returns the logical *or* of booleans or bit aggregates *a* and *b*.

*n* **or** *m* returns the bit-by-bit logical *or* of the small integers *n* and *m*.

**orbit** (*group, elements*)

**orbit** (*gp, el*) returns the orbit of the element *el* under the permutation group *gp*, that is, the set of all points gained by applying each group element to *el*.

**orbit** (*gp, ls*), where *ls* is a list or unordered set of elements, returns the orbit of *ls* under the permutation group *gp*.

**orbits** (*group*)

**orbits** (*gp*) returns the orbits of the permutation group *gp*.

**ord** (*character*)

**ord** (*c*) returns an integer code corresponding to the character *c*.

**order** (*element*)

**order** (*p*) returns:

if *p* is a float: the magnitude of *p* (Note:  $\text{base}^{\text{order}(x)} \leq |x| < \text{base}^{(1+\text{order}(x))}$ .)

if *p* is a differential polynomial: the maximum number of differentiations of a differential indeterminate among all those appearing in *p*.

if *p* is a differential variable: the number of differentiations of the differential indeterminate appearing in *p*.

if *p* is an element of finite field: the order of an element in the multiplicative group of the field (the function calls **error** if *p* is 0).

if *p* is a univariate power series: the degree of the lowest order non-zero term in *f*. (A call to this operation results in an infinite loop if *f* has no non-zero terms.)

if *p* is a *q*-adic integer: the exponent of the highest power of *q* dividing *p* (see `PAdicIntegerCategory`).

if *p* is a permutation: the order of a permutation *p* as a group element.

if *p* is permutation group: the order of the group.

**order** (*p, q*) returns the order of the differential polynomial *p* in differential indeterminate *q*.

**order** (*p, q*) returns the order of multivariate series *p* viewed as a series in *q* (this operation results in an infinite loop if *f* has no non-zero terms).

**order** (*p, q*) returns the largest *n* such that  $q^n$  divides polynomial *p*, that is, the order of  $p(x)$  at  $q(x) = 0$ .

**orthonormalBasis** (*matrix*)

**orthonormalBasis** (*M*) returns the orthogonal matrix *B* such that  $BMB^{-1}$  is diagonal, or calls **error** if *M* is not a symmetric matrix.

**output** (*x*)

**output** (*x*) displays *x* on the "algebra output" stream defined by `)set output algebra`.

**outputAsFortran** (*outputForms*)

**outputAsFortran** (*f*) outputs `OutputForm` object *f* in FORTRAN format to the destination defined by the system command `)set output fortran`. If *f* is a list of `OutputForm` objects, each expression in *f* is output in order.

**outputAsFortran** (*s, f*), where *s* is a string, outputs  $s = f$ , but is otherwise identical.

**outputAsTex** (*outputForms*)

**outputAsTex** (*f*) outputs `OutputForm` object *f* in  $\text{\TeX}$  format to the destination defined by the system command `)set output tex`. If *f* is a list of `OutputForm` objects, each expression in *f* is output in order.

**outputFixed** (*[nonNegativeInteger]*)

**outputFixed** (*[n]*) sets the output mode of floats to fixed point notation, that is, as an integer, a decimal point, and a number of digits. If *n* is given, then *n* digits are displayed after the decimal point.



**outputFloating** (*[nonNegativeInteger]*)

**outputFloating** (*[n]*) sets the output mode to floating (scientific) notation, that is,  $m10^e$  is displayed as **mEe**. If *n* is given, *n* digits will be displayed after the decimal point.

**outputForm** (*various*)

**outputForm** (*x*) creates an object of type *OutputForm* from *x*, an object of type *Integer*, *DoubleFloat*, *String*, or *Symbol*.

**outputGeneral** (*[nonNegativeInteger]*)

**outputGeneral** (*[n]*) sets the output mode (default mode) to general notation, that is, numbers will be displayed in either fixed or floating (scientific) notation depending on the magnitude. If *n* is given, *n* digits are displayed after the decimal point.

**outputSpacing** (*nonNegativeInteger*)

**outputSpacing** (*n*) inserts a space after *n* digits on output. **outputSpacing** (0) means no spaces are inserted. By default, *n* = 10.

**over** (*outputForm*, *outputForm*)

**over** (*o*<sub>1</sub>, *o*<sub>2</sub>), where *o*<sub>1</sub> and *o*<sub>2</sub> are objects of type *OutputForm* (normally unexposed), creates an output form for the vertical fraction of *o*<sub>1</sub> over *o*<sub>2</sub>.

**overbar** (*outputForm*)

**overbar** (*o*), where *o* is an object of type *OutputForm* (normally unexposed), creates the output form *o* with an overbar.

**pack!** (*file*)

**pack!** (*f*) reorganizes the file *f* on disk to recover unused space.

**packageCall** ()

**packageCall** (*f*)\$*P*, where *P* is the package *InputFormFunctions1*(*R*) for some type *R*, returns the input form corresponding to *f*\$*R*. See also **interpret**.

**pade** (*integer*, *integer*, *series* [ , *series*])

**pade** (*nd*, *dd*, *s* [ , *ds*]) computes the quotient of polynomials (if it exists) with numerator degree at most *nd* and denominator degree at most *dd*. If a single univariate Taylor series *s* is given, the quotient approximate must match the series *s* to order *nd* + *dd*. If two series *s* and *ds* are given, *ns* is the numerator series of the function and *ds* is the denominator series.

**padicFraction** (*partialFraction*)

**padicFraction** (*q*) expands the fraction *p*-adically in the primes *p* in the denominator of *q*. For example, **padicFraction** (3/(2<sup>2</sup>)) = 1/2 + 1/(2<sup>2</sup>). Use **compactFraction** to return to compact form.

**pair?** (*sExpression*)

**pair?** (*s*) tests if *SExpression* object is a non-null Lisp object.

**parabolic** (*point*)

**parabolic** (*pt*) transforms *pt* from parabolic coordinates to Cartesian coordinates: the function produced will map the point (*u*, *v*) to  $x = 1/2(u^2 - v^2)$ ,  $y = uv$ .

**parabolicCylindrical** (*point*)

**parabolicCylindrical** (*pt*) transforms *pt* from parabolic cylindrical coordinates to Cartesian coordinates: the function produced will map the point (*u*, *v*, *z*) to  $x = 1/2(u^2 - v^2)$ ,  $y = uv$ ,  $z = z$ .

**paraboloidal** (*point*)

**paraboloidal** (*pt*) transforms *pt* from paraboloidal coordinates to Cartesian coordinates: the function produced will map the point (*u*, *v*, *phi*) to  $x = uv\cos(\phi)$ ,  $y = uv\sin(\phi)$ ,  $z = 1/2(u^2 - v^2)$ .

**paren** (*expressions*)

**paren** (*f*) returns (*f*) unless *f* is a list [*f*<sub>1</sub>, ..., *f*<sub>*n*</sub>] in which case it returns (*f*<sub>1</sub>, ..., *f*<sub>*n*</sub>). This prevents *f* or the constituent *f*<sub>*i*</sub> from being evaluated when operators are applied to it. For example, **log**(1) returns 0, but **log**(**paren** 1) returns the formal kernel **log**((1)). Also, **atan**(**paren** [*x*, 2]) returns the formal kernel **atan**((*x*, 2)).

**partialDenominators** (*continuedFraction*)

**partialDenominators** (*x*) extracts the denominators in *x*. If  $x = \text{continuedFraction}(b_0, [a_1, \dots], [b_1, \dots])$ , then **partialDenominators** (*x*) = [*b*<sub>1</sub>, *b*<sub>2</sub> ...].

**partialFraction** (*element*, *factored*)

**partialFraction** (*numer*, *denom*) is the main function for constructing partial fractions. The second argument *denom* is the denominator and should be factored.

**partialNumerators** (*continuedFraction*)

**partialNumerators** (*x*) extracts the numerators in *x*, if  $x = \text{continuedFraction}(b_0, [a_1, \dots], [b_1, \dots], \dots)$ , then **partialNumerators** (*x*) = [*a*<sub>1</sub>, ...].

**partialQuotients** (*continuedFraction*)

**partialQuotients** (*x*) extracts the partial quotients in *x*, if  $x = \text{continuedFraction}(b_0, [a_1, \dots], [b_1, \dots], \dots)$ , then **partialQuotients** (*x*) = [*b*<sub>0</sub>, *b*<sub>1</sub>, ...].

**particularSolution** (*matrix*, *vector*)

**aSolution** (*M*, *v*) finds a particular solution *x* of the linear system  $Mx = v$ . The result *x* is returned as a vector, or "failed" if no solution exists.

**partition** (*integer*)

**partition** (*n*) returns the number of partitions of the

integer  $n$ . This is the number of distinct ways that  $n$  can be written as a sum of positive integers.

**partitions** (*integer* [, *integer*, *integer*])

**partitions** ( $i, j$ ) is the stream of all partitions whose number of parts and largest part are no greater than  $i$  and  $j$ .

**partitions** ( $n$ ) is the stream of all partitions of integer  $n$ .

**partitions** ( $p, l, n$ ) is the stream of partitions of  $n$  whose number of parts is no greater than  $p$  and whose largest part is no greater than  $l$ .

**parts** (*aggregate*)

**parts** ( $u$ ) returns a list of the consecutive elements of  $u$ . Note: if  $u$  is a list, **parts** ( $u$ ) =  $u$ .

**pastel** (*color*)

**pastel** ( $c$ ) sets the shade of a hue  $c$  above “bright” but below “light”.

**pattern** (*rewriteRule*)

**pattern** ( $r$ ) returns the pattern corresponding to the left hand side of the rewrite rule  $r$ .

**patternMatch** (*expression*, *expression*, *patternMatchResult*)

**patternMatch** ( $expr, pat, res$ ) matches the pattern  $pat$  to the expression  $expr$ . The argument  $res$  contains the variables of  $pat$  which are already matched and their matches. Initially,  $res$  is the result of **new**(), an empty list of matches.

**perfectNthPower?** (*integer*, *nonNegativeInteger*)

**perfectNthPower?** ( $n, r$ ) tests if  $n$  is an  $r^{\text{th}}$  power.

**perfectNthRoot** (*integer* [, *nonNegativeInteger*])

**perfectNthRoot** ( $n$ ) returns a record with fields “base”  $x$  and “exponent”  $r$  such that  $n = x^r$  and  $r$  is the largest integer such that  $n$  is a perfect  $r^{\text{th}}$  power.

**perfectNthRoot** ( $n, r$ ) returns the  $r^{\text{th}}$  root of  $n$  if  $n$  is an  $r^{\text{th}}$  power, and “failed” otherwise.

**perfectSqrt** (*integer*)

**perfectSqrt** ( $n$ ) returns the square root of  $n$  if  $n$  is a perfect square, and “failed” otherwise.

**perfectSquare?** (*integer*)

**perfectSquare?** ( $n$ ) tests if  $n$  is a perfect square.

**permanent** (*matrix*)

**permanent** ( $x$ ) returns the permanent of a square matrix  $x$ , equivalent to the **determinant** except that coefficients have no change of sign.

**permutation** (*integer*, *integer*)

**permutation** ( $n, m$ ) returns the number of permutations of  $n$  objects taken  $m$  at a time. Note:

**permutation** ( $n, m$ ) =  $n!/(n - m)!$ .

**permutationGroup** (*listPermutations*)

**permutationGroup** ( $ls$ ) coerces a list of permutations  $ls$  to the group generated by this list.

**permutationRepresentation** (*permutations* [,  $n$ ])

**permutationRepresentation** ( $pi, n$ ) returns the matrix  $\delta_{i, pi(i)}$  (Kronecker delta) if the permutation  $pi$  is in list notation and permutes  $1, 2, \dots, n$ . Argument  $pi$  may either be permutation or a list of integers describing a permutation by list notation.

**permutationRepresentation** ( $[pi_1, \dots, pi_k], n$ ) returns the list of matrices  $[(\delta_{i, pi_1(i)}), \dots, (\delta_{i, pi_k(i)})]$  (Kronecker delta) for permutations  $pi_1, \dots, pi_k$  of  $1, 2, \dots, n$ .

**permutations** (*integer*)

**permutations** ( $n$ ) returns the stream of permutations formed from  $1, 2, \dots, n$ .

**physicalLength** (*flexibleArray*)

**physicalLength!** (*flexibleArray*, *positiveInteger*)

These operations apply to a flexible array  $a$  and concern the “physical length” of  $a$ , the maximum number of elements that  $a$  can hold. When a destructive operation (such as **concat!**) is applied that increases the number of elements of  $a$  beyond this number, new storage is allocated (generally to be about 50% larger than current storage allocation) and the elements from the old storage are copied over to the new storage area.

**physicalLength** ( $a$ ) returns the physical length of  $a$ .

**physicalLength!** ( $a, n$ ) causes new storage to be allocated for the elements of  $a$  with a physical length of  $n$ . The **maxIndex** elements from the old storage area are copied. An **error** is called if  $n$  is less than **maxIndex**( $a$ ).

**pi** ()

**pi** () returns  $\pi$ , also denoted by the special symbol %pi.

**pile** (*listOfOutputForms*)

**pile** ( $lo$ ), where  $lo$  is a list of objects of type **OutputForm** (normally unexposed), creates the output form consisting of the elements of  $lo$  displayed as a pile, that is, each element begins on a new line and is indented right to the same margin.

**plenaryPower** (*element*, *positiveInteger*)

Argument  $a$  is a member of a domain of category **NonAssociativeAlgebra**

**plenaryPower** ( $a, n$ ) is recursively defined to be

**plenaryPower** ( $a, n - 1$ ) \* **plenaryPower** ( $a, n - 1$ ) for

$n > 1$  and  $a$  for  $n = 1$ .

**plusInfinity** ()

**plusInfinity** () returns the constant `%plusInfinity` denoting  $+\infty$ .

**point** ( $u$ , *option*)

**point** ( $p$ ) returns a ThreeSpace object which is composed of one component, the point  $p$ . **point** ( $l$ ) creates a point defined by a list  $l$ .

**point** ( $sp$ ) checks to see if the ThreeSpace object  $sp$  is composed of only a single point and, if so, returns the point, or calls **error** if  $sp$  has more than one point.

**point** ( $sp, l$ ) adds a point component defined by a list  $l$  to the ThreeSpace object  $sp$ .

**point** ( $sp, i$ ) adds a point component into a component list of the ThreeSpace object  $sp$  at the index given by  $i$ .

**point** ( $sp, p$ ) adds a point component defined by the point  $p$  described as a list, to the ThreeSpace object  $sp$ .

**point?** (*space*)

**point?** ( $sp$ ) queries whether the ThreeSpace object  $sp$ , is composed of a single component which is a point.

**pointColor** (*palette*)

**pointColor** ( $v$ ) specifies a color  $v$  for two-dimensional graph points. This option is expressed in the form `pointColor == v` in the **draw** command. Argument  $p$  is either a palette or a float.

**pointColorDefault** ([*palette*])

**pointColorDefault** () returns the default color of points in a two-dimensional viewport.

**pointColorDefault** ( $p$ ) sets the default color of points in a two-dimensional viewport to the palette  $p$ .

**pointSizeDefault** ([*positiveInteger*])

**pointSizeDefault** () returns the default size of the points in a two-dimensional viewport.

**pointSizeDefault** ( $i$ ) sets the default size of the points in a two-dimensional viewport to  $i$ .

**polarCoordinates** ( $x$ )

**polarCoordinates** ( $x$ ) returns a record with components  $(r, \phi)$  such that  $x = re^{i\phi}$ .

**polar** (*point*)

**polar** ( $pt$ ) transforms point  $pt$  from polar coordinates to Cartesian coordinates. The function produced will map the point  $(r, \theta)$  to  $x = r\cos(\theta)$ ,  $y = r\sin(\theta)$ .

**pole?** (*series*)

**pole?** ( $f$ ) tests if the power series  $f$  has a pole.

**polygamma** ( $k, x$ )

**polygamma** ( $k, x$ ) is the  $k^{\text{th}}$  derivative of **digamma** ( $x$ ), often written  $\psi(k, x)$  in the literature.

**polygon** ([ $sp$ , ]*listOfPoints*)

**polygon?** (*space*)

**polygon** ([ $sp, lp$ ]) adds a polygon defined by  $lp$  to the ThreeSpace object  $sp$ . Each  $lp$  is either a list of points (objects of type Point) or else a list of small floats. If  $sp$  is omitted, it is understood to be empty.

**polygon** ( $sp$ ) returns ThreeSpace object  $sp$  as a list of polygons, or an error if  $sp$  is not composed of a single polygon.

**polygon?** ( $sp$ ) tests if the ThreeSpace object  $sp$  contains a single polygon component.

**polynomial** (*series*, *integer* [ , *integer*])

**polynomial** ( $s, k$ ) returns a polynomial consisting of the sum of all terms of series  $s$  of degree  $\leq k$  and greater than or equal to 0.

**polynomial** ( $s, k_1, k_2$ ) returns a polynomial consisting of the sum of all terms of Taylor series  $s$  of degree  $d$  with  $0 \leq k_1 \leq d \leq k_2$ .

**pop!** (*stack*)

**pop!** ( $s$ ) returns the top element  $x$  from stack  $s$ , destructively removing it from  $s$ , or calls **error** if  $s$  is empty. Note: Use **top** ( $s$ ) to obtain  $x$  without removing it from  $s$ .

**position** (*aggregate*, *aggregate* [ , *index*])

**position** ( $x, a$ , [ $n$ ]) returns the index  $i$  of the first occurrence of  $x$  in  $a$  where  $i \geq n$ , and **minIndex** ( $a$ ) - 1 if no such  $x$  is found. The default value of  $n$  is 1.

**position** ( $cc, t, i$ ) returns the position  $j \geq i$  in  $t$  of the first character belonging to character class  $cc$ .

**positive?** (*orderedSetElement*)

**positive?** ( $x$ ) tests if  $x$  is strictly greater than 0.

**positiveRemainder** (*integer*, *integer*)

**positiveRemainder** ( $a, b$ ), where  $b > 1$ , yields  $r$  where  $0 \leq r < b$  and  $r = a \bmod b$ .

**possiblyInfinite?** (*stream*)

**possiblyInfinite?** ( $s$ ) tests if the stream  $s$  could possibly have an infinite number of elements. Note: for many datatypes, **possiblyInfinite?** ( $s$ ) = **not explicitlyFinite?** ( $s$ ).

**postfix** (*outputForm*, *outputForm*)

**postfix** ( $op, a$ ), where  $op$  and  $a$  are objects of type OutputForm (normally unexposed), creates an output form which prints as:  $a \ op$ .

**powerAssociative?** ()

**powerAssociative?** ( $\$F$ , where  $F$  is a domain of category FiniteRankNonAssociativeAlgebra, tests if all subalgebras generated by a single element are associative.

**powerSum** (*integer*)

**powerSum** ( $n$ ) is the  $n$  th power sum symmetric function. See CycleIndicators for details.

**powmod** (*integer, integer, integer*)

**powmod** ( $a, b, p$ ), where  $a$  and  $b$  are non-negative integers, each  $< p$ , returns  $a^b \bmod p$ .

**precision** (*[positiveInteger]*)

**precision** () returns the precision of Float values in decimal digits.

**precision** ( $n$ ) set the precision in the base to  $n$  decimal digits.

**prefix** (*outputForm, listOfOutputForms*)

**prefix** ( $o, lo$ ), where  $o$  is an object of type OutputForm (normally unexposed) and  $lo$  is a list of objects of type OutputForm, creates an output form depicting the nary prefix application of  $o$  to a tuple of arguments given by list  $lo$ .

**prefix?** (*string, string*)

**prefix?** ( $s, t$ ) tests if the string  $s$  is the initial substring of  $t$ .

**prefixRagits** (*listOfIntegers*)

**prefixRagits** ( $rx$ ) returns the non-cyclic part of the ragits of the fractional part of a radix expansion. For example, if  $x = 3/28 = 0.10714285714285\dots$ , then

**prefixRagits** ( $x$ ) =  $[1, 0]$ .

**presub** (*outputForm, outputForm*)

**presub** ( $o_1, o_2$ ), where  $o_1$  and  $o_2$  are objects of type OutputForm (normally unexposed), creates an output form for  $o_1$  presubscripted by  $o_2$ .

**presuper** (*outputForm, outputForm*)

**presuper** ( $o_1, o_2$ ), where  $o_1$  and  $o_2$  are objects of type OutputForm (normally unexposed), creates an output form for  $o_1$  presuperscripted by  $o_2$ .

**primaryDecomp** (*ideal*)

**primaryDecomp** ( $I$ ) returns a list of primary ideals such that their intersection is the ideal  $I$ .

**prime** (*outputForm* [, *positiveInteger*])

**prime** ( $o, n$ ), where  $o$  is an object of type OutputForm (normally unexposed), creates an output form for  $o$  following by  $n$  primes (that is, a prime like “ ’ ”). By default,  $n = 1$ .

**prime?** (*element*)

**prime?** ( $x$ ) tests if  $x$  cannot be written as the product of two non-units, that is,  $x$  is an irreducible element. Argument  $x$  may be an integer, a polynomial, an ideal, or, in general, any element of a domain of category UniqueFactorizationDomain.

**primeFactor** (*element, integer*)

**primeFactor** ( $base, exponent$ ) creates a factored object with a single factor whose *base* is asserted to be prime (flag = “prime”).

**primeFrobenius** (*finiteFieldElement* [, *nonNegativeInteger*])

Argument  $a$  is a member of a domain of category FieldOfPrimeCharacteristic( $p$ ).

**primeFrobenius** ( $a, s$ ) returns  $a^{p^s}$ . The default value of  $s$  is 1.

**primes** (*integer, integer*)

**primes** ( $a, b$ ) returns a list of all primes  $p$  with  $a \leq p \leq b$ .

**primitive?** (*finiteFieldElement*)

**primitive?** ( $b$ ) tests whether the element  $b$  of a finite field is a generator of the (cyclic) multiplicative group of the field, that is, is a primitive element.

**primitiveElement** (*expressions* [, *expression*])

**primitiveElement** ( $a_1, a_2$ ) returns a record with four components: a primitive element  $a$  with selector *primelt*, and three polynomials  $q_1$ ,  $q_2$ , and  $q$  with selectors *pol1*, *pol2*, and *prim*. The prime element  $a$  is such that the algebraic extension generated by  $a_1$  and  $a_2$  is the same as that generated by  $a$ ,  $a_i = q_i(a)$  and  $q(a) = 0$ . The minimal polynomial for  $a_2$  may involve  $a_1$ , but the minimal polynomial for  $a_1$  may not involve  $a_2$ . This operations uses **resultant**.

**primitiveMonomials** (*polynomial*)

**primitiveMonomials** ( $p$ ) gives the list of monomials of the polynomial  $p$  with their coefficients removed. Note:

**primitiveMonomials** ( $\sum a_i X^{(i)} = [X^{(1)}, \dots, X^{(n)}]$ ).

**primitivePart** (*polynomial* [, *symbol*])

**primitivePart** ( $p, v$ ) returns the unit normalized form of polynomial  $p$  divided by the **content** of  $p$  with respect to variable  $v$ . If no  $v$  is given, the content is removed with respect to all variables.

**principalIdeal** (*listOfPolynomials*)

**principalIdeal** ( $[f_1, \dots, f_n]$ ) returns a record whose “generator” component is a generator of the ideal generated by  $[f_1, \dots, f_n]$  whose “coef” component is a list of coefficients  $[c_1, \dots, c_n]$  such that *generator* =  $\sum_i c_i f_i$ .

**print** (*outputForm*)

**print** (*o*) writes the output form *o* on standard output using the two-dimensional formatter.

**product** (*element*, *element*)

**product** ( $f(n)$ ,  $n = a..b$ ) returns  $\prod_{n=a}^b f(n)$  as a formal product.

**product** ( $f(n)$ ,  $n$ ) returns the formal product  $P(n)$  verifying  $P(n+1)/P(n) = f(n)$ .

**product** ( $s$ ,  $t$ ), where  $s$  and  $t$  are cartesian tensors, returns the outer product of  $s$  and  $t$ . For example, if

$r = \text{product}(s, t)$  for rank 2 tensors  $s$  and  $t$ , then  $r$  is a rank 4 tensor given by  $r_{i,j,k,l} = s_{i,j}t_{k,l}$ .

**product** ( $a$ ,  $b$ ), where  $a$  and  $b$  are elements of a graded algebra returns the degree-preserving linear product. See GradedAlgebra for details.

**prolateSpheroidal** (*smallFloat*)

**prolateSpheroidal** ( $a$ ) returns a function to transform prolate spheroidal coordinates to Cartesian coordinates. This function will map the point  $(\xi, \eta, \phi)$  to  $x = a \sinh(\xi) \sin(\eta) \cos(\phi)$ ,  $y = a \sinh(\xi) \sin(\eta) \sin(\phi)$ ,  $z = a \cosh(\xi) \cos(\eta)$ .

**prologue** (*text*)

**prologue** ( $t$ ) extracts the prologue section of a IBM SCRIPT Formula Formatter or T<sub>E</sub>X formatted object  $t$ .

**properties** (*basicOperator* [, *prop*])

**properties** ( $op$ ) returns the list of all the properties currently attached to  $op$ .

**property** ( $op$ ,  $s$ ) returns the value of property  $s$  if it is attached to  $op$ , and "failed" otherwise.

**pseudoDivide** (*polynomial*, *polynomial*)

**pseudoDivide** ( $p$ ,  $q$ ) returns  $(c, q, r)$ , when  $p' := p \text{ leadingCoefficient}(q)^{\deg(p) - \deg(q) + 1} = cp$  is pseudo right-divided by  $q$ , that is,  $p' = sq + r$ .

**pseudoQuotient** (*polynomial*, *polynomial*)

**pseudoQuotient** ( $p$ ,  $q$ ) returns  $r$ , the quotient when  $p' := p \text{ leadingCoefficient}(q)^{\deg(p) - \deg(q) + 1}$  is pseudo right-divided by  $q$ , that is,  $p' = sq + r$ .

**pseudoRemainder** (*polynomial*, *polynomial*)

**pseudoRemainder** ( $p$ ,  $q$ ) =  $r$ , for polynomials  $p$  and  $q$ , returns the remainder when  $p' := p \text{ leadingCoefficient}(q)^{\deg(p) - \deg(q) + 1}$  is pseudo right-divided by  $q$ , that is,  $p' = sq + r$ .

**puiseux** (*expression* [, *options*])

**puiseux** ( $f$ ) returns a Puiseux expansion of the expression  $f$ . Note:  $f$  should have only one variable; the series will be expanded in powers of that variable. Also, if  $x$  is a symbol,

**puiseux** ( $x$ ) returns  $x$  as a Puiseux series.

**puiseux** ( $f$ ,  $x = a$ ) expands the expression  $f$  as a Puiseux series in powers of  $(x - a)$ .

**puiseux** ( $f$ ,  $n$ ) returns a Puiseux expansion of the expression  $f$ . Note:  $f$  should have only one variable; the series will be expanded in powers of that variable and terms will be computed up to order at least  $n$ .

**puiseux** ( $f$ ,  $x = a$ ,  $n$ ) expands the expression  $f$  as a Puiseux series in powers of  $(x - a)$ ; terms will be computed up to order at least  $n$ .

**puiseux** ( $n+->a(n)$ ,  $x = a$ ,  $r_0..r$ ) returns

$\sum_{n=r_0, r_0+r, r_0+2r, \dots} a(n)(x-a)^n$ .

**puiseux** ( $a(n)$ ,  $n$ ,  $x = a$ ,  $r_0..r$ ) returns

$\sum_{n=r_0, r_0+r, r_0+2r, \dots} a(n)(x-a)^n$ .

Note: Each of the last two commands have alternate forms whose third argument is the finite segment  $r_0..r_1$  producing a similar series with a finite number of terms.

**push!** (*element*, *stack*)

**push!** ( $x$ ,  $s$ ) pushes  $x$  onto stack  $s$ , that is, destructively changing  $s$  so as to have a new first (top) element  $x$ .

**pushdown** (*polynomial*, *symbol*)

**pushdterm** (*monomial*, *symbol*)

**pushdown** ( $prf$ ,  $var$ ) pushes all top level occurrences of the variable  $var$  into the coefficient domain for the polynomial  $prf$ .

**pushdterm** ( $monom$ ,  $var$ ) pushes all top level occurrences of the variable  $var$  into the coefficient domain for the monomial  $monom$ .

**pushucoef** (*polynomial*, *variable*)

**pushucoef** ( $upoly$ ,  $var$ ) converts the anonymous univariate polynomial  $upoly$  to a polynomial in  $var$  over rational functions.

**pushuconst** (*rationalFunction*, *variable*)

**pushuconst** ( $r$ ,  $var$ ) takes a rational function and raises all occurrences of the variable  $var$  to the polynomial level.

**pushup** (*polynomial*, *variable*)

**pushup** ( $prf$ ,  $var$ ) raises all occurrences of the variable  $var$  in the coefficients of the polynomial  $prf$  back to the polynomial level.

**qelt** ( $u$  [, *options*])

**qelt** ( $u$ ,  $p$  [, *options*]) is equivalent to a corresponding **elt** form except that it performs no check that indices are in range. Use Browse to discover if a given domain has this alternative operation.

**qsetelt!** ( $u$ ,  $x$ ,  $y$  [,  $z$ ])

**qsetelt!** ( $u$ ,  $x$ ,  $y$  [,  $z$ ]) is equivalent to a corresponding **setelt** form except that it performs no check that indices are in range.

**quadraticForm** (*matrix*)

**quadraticForm** (*m*) creates a quadratic form from a symmetric, square matrix *m*.

**quatern** (*element, element, element, element*)

**quatern** (*r, i, j, k*) constructs a quaternion from scalars.

**queue** (*[listOfElements]*)

**queue** ()\$*R* returns an empty queue of type *R*.

**queue** (*[x, y, ..., z]*) creates a queue with first (top) element *x*, second element *y*, ..., and last (bottom) element *z*.

**quickSort** (*predicate, aggregate*)

**quickSort** (*f, agg*) sorts the aggregate *agg* with the ordering predicate *f* using the quicksort algorithm.

**quo** (*integer, integer*)

*a quo b* returns the quotient of *a* and *b* discarding the remainder.

**quoByVar** (*series*)

**quoByVar** ( $a_0 + a_1x + a_2x^2 + \dots$ ) returns  $a_1 + a_2x + a_3x^2 + \dots$ . Thus, this function subtracts the constant term and divides by the series variable. This function is used when Laurent series are represented by a Taylor series and an order.

**quote** (*outputForm*)

**quote** (*o*), where *o* is an object of type *OutputForm* (normally unexposed), creates an output form *o* with a prefix quote.

**quotedOperators** (*rewriteRule*)

**quotedOperators** (*r*), where *r* is a rewrite rule, returns the list of operators on the right-hand side of *r* that are considered quoted, that is, they are not evaluated during any rewrite, but applied formally to their arguments.

**quotient** (*ideal, polynomial*)

**quotient** (*I, f*) computes the quotient of the ideal *I* by the principal ideal generated by the polynomial *f*, (*I : (f)*).

**quotient** (*I, J*) computes the quotient of the ideals *I* and *J*, (*I : J*).

**radical** (*ideal*)

**radical** (*I*) returns the radical of the ideal *I*.

**radicalEigenvalues** (*matrix*)

**radicalEigenvalues** (*m*) computes the eigenvalues of the matrix *m*; when possible, the eigenvalues are expressed in terms of radicals.

**radicalEigenvectors** (*matrix*)

**radicalEigenvectors** (*m*) computes the eigenvalues and the corresponding eigenvectors of the matrix *m*; when

possible, values are expressed in terms of radicals.

**radicalEigenvector** (*eigenvalue, matrix*)

**radicalEigenvector** (*c, m*) computes the eigenvector(*s*) of the matrix *m* corresponding to the eigenvalue *c*; when possible, values are expressed in terms of radicals.

**radicalOfLeftTraceForm** ()

**radicalOfLeftTraceForm** ()\$*F* returns the basis for the null space of **leftTraceMatrix** ()\$*F*, where *F* is a domain of category *FramedNonAssociativeAlgebra*. If the algebra is associative, alternative or a Jordan algebra, then this space equals the radical (maximal nil ideal) of the algebra.

**radicalRoots** (*fractions*)

**radicalRoots** (*rf, v*) finds the roots expressed in terms of radicals of the rational function *rf* with respect to the symbol *v*.

**radicalRoots** (*lrf, lv*) finds the roots expressed in terms of radicals of the list of rational functions *lrf* with respect to the list of symbols *lv*.

**radicalSolve** (*eq, x*)

See **solve** (*u, v*).

**radix** (*rationalNumber, integer*)

**radix** (*rn, b*) converts rational number *rn* to a radix expansion in base *b*.

**ramified?** (*polynomial*)

**ramifiedAtInfinity?** ()

Domain *F* is a domain of functions on a fixed curve.

**ramified?** (*p*)\$*F* tests whether  $p(x) = 0$  is ramified.

**ramifiedAtInfinity?** () tests if infinity is ramified.

**random** (*[u, v]*)

**random** ()\$*R* creates a random element from domain *D*.

**random** (*gp, i*) returns a random product of maximal *i* generators of the permutation group *gp*. The value of *i* is 20 by default.

**range** (*listOfSegments*)

**range** (*ls*), where *ls* is a list of segments of the form  $[a_1..b_1, \dots, a_n..b_n]$ , provides a user-specified range for clipping for the **draw** command. This command may also be expressed locally to the **draw** command as the option *range == ls*. The values  $a_i$  and  $b_i$  are either given as floats or rational numbers.

**ranges** (*listOfSegments*)

**ranges** (*l*) provides a list of user-specified ranges for the **draw** command. This command may also be expressed as an option to the **draw** command in the form **ranges == l**.

**rank** (*matrix*)

**rank** (*m*) returns the rank of the matrix *m*. Also:

**rank** ( $A, B$ ) computes the rank of the complete matrix  $(A|B)$  of the linear system  $AX = B$ .

**rank** ( $t$ ), where  $t$  is a Cartesian tensor, returns the tensorial rank of  $t$  (that is, the number of indices). See also `FiniteRankAlgebra` and `FiniteRankNonAssociativeAlgebra`.

**rarrow** (*outputForm*, *outputForm*)

**rarrow** ( $o_1, o_2$ ), where  $o_1$  and  $o_2$  are objects of type `OutputForm` (normally unexposed), creates a form for the mapping  $o_1 \rightarrow o_2$ .

**ratDenom** (*expression* [, *option*])

**ratDenom** ( $f$ ,  $u$ ) rationalizes the denominators appearing in  $f$ . If no second argument is given, then all algebraic quantities are moved into the numerators. If the second argument is given as an algebraic kernel  $a$ , then  $a$  is removed from the denominators. Similarly, if  $u$  is a list of algebraic kernels  $[a_1, \dots, a_n]$ , the operation removes the  $a_i$ 's from the denominators in  $f$ .

**rational?** (*element*)

**rationalIfCan** (*element*)

**rational** (*element*)

**rational?** ( $x$ ) tests if  $x$  is a rational number, that is, that it can be converted to type `Fraction(Integer)`. Specifically, if  $x$  is complex, a quaternion, or an octonion, it tests that all imaginary parts are 0.

**rationalIfCan** ( $x$ ) returns  $x$  as a rational number if possible, and "failed" if it is not.

**rational** ( $x$ ) returns  $x$  as a rational number if possible, and calls **error** if it is not.

**rationalApproximation** (*float*, *nonNegativeInteger* [, *positiveInteger*])

**rationalApproximation** ( $f$ ,  $n$ ,  $b$ ) computes a rational approximation  $r$  to  $f$  with relative error  $< b^{-n}$ , that is  $|(r - f)/f| < b^{-n}$ , for some positive integer base  $b$ . By default,  $b = 10$ . The first argument  $f$  is either a float or small float.

**rationalFunction** (*series*, *integer*, *integer*)

**rationalFunction** ( $f$ ,  $m$ ,  $n$ ) returns a rational function consisting of the sum of all terms of  $f$  of degree  $d$  with  $m \leq d \leq n$ . By default,  $n$  is the maximum degree of  $f$ .

**rationalPoint?** (*value*, *value*)

**rationalPoint?** ( $a, b$ )\$ $F$  tests if  $(x = a, y = b)$  is on the curve defining function field  $F$ . See `FunctionFieldCategory`.

**rationalPoints** ()

**rationalPoints** ()\$ returns the list of all the affine rational points on the curve defining function field  $F$ . See `FunctionFieldCategory`.

**rationalPower** (*puiseuxSeries*)

**rationalPower** ( $f(x)$ ) returns  $r$  where the Puiseux series

$f(x) = g(x^r)$ .

**ratPoly** (*expression*)

**ratPoly** ( $f$ ) returns a polynomial  $p$  such that  $p$  has no algebraic coefficients, and  $p(f) = 0$ .

**rdexquo** (*lodOperator*)

**rdexquo** ( $a, b$ ), where  $a$  and  $b$  are linear ordinary differential operators, returns  $q$  such that  $a = bq$ , or "failed" if no such  $q$  exists.

**rightDivide** (*lodOperator*, *lodOperator*)

**rightQuotient** (*lodOperator*, *lodOperator*)

**rightRemainder** (*lodOperator*, *lodOperator*)

**rightDivide** ( $a, b$ ) returns the pair  $q, r$  such that  $a = qb + r$  and the degree of  $r$  is less than the degree of  $b$ . The pair is returned as a record with fields *quotient* and *remainder*. This process is called "right division". Also: **rightQuotient** ( $a, b$ ) returns only  $q$ . **rightRemainder** ( $a, b$ ) returns only  $r$ .

**read!** (*file*)

**readIfCan!** (*file*)

**read!** ( $f$ ) extracts a value from file  $f$ . The state of  $f$  is modified so a subsequent call to **read!** will return the next element.

**readIfCan!** ( $f$ ) returns a value from the file  $f$  or "failed" if this is not possible (that is, either  $f$  is not open for reading, or  $f$  is at the end of the file).

**readable?** (*file*)

**readable?** ( $f$ ) tests if the named file exists and can be opened for reading.

**readLine!** (*file*)

**readLineIfCan!** (*file*)

**readLineIfCan!** ( $f$ ) returns a string of the contents of a line from file  $f$ , or "failed" if this is not possible (if  $f$  is not readable or is positioned at the end of file).

**readLine!** ( $f$ ) returns a string of the contents of a line from the file  $f$ , and calls **error** if this is not possible.

**real** ( $x$ )

**real?** (*expression*)

**real** ( $x$ ) returns real part of  $x$ . Argument  $x$  can be an expression or a complex value, quaternion, or octonion.

**real?** ( $f$ ) tests if expression  $f = \text{real}(f)$ .

**realEigenvectors** (*matrix*, *float*)

**realEigenvectors** ( $m, \text{eps}$ ) returns a list of records, each containing a real eigenvalue, its algebraic multiplicity, and a list of associated eigenvectors. All these results are computed to precision *eps* as floats or rational numbers depending on the type of *eps*. Argument  $m$  is a matrix of rational functions.

**realElementary** (*expression* [, *symbol*])

**realElementary** (*f*, *sy*) rewrites the kernels of *f* involving *sy* in terms of the 4 fundamental real transcendental elementary functions: *log*, *exp*, *tan*, *atan*. If *sy* is omitted, all kernels of *f* are rewritten.

**realRoots** (*rationalfunctions*, *v* [, *w*])

**realRoots** (*rf*, *eps*) finds the real zeros of a univariate rational function *rf* with precision given by *eps*.

**realRoots** (*lp*, *lv*, *eps*) computes the list of the real solutions of the list *lp* of rational functions with rational coefficients with respect to the variables in *lv*, with precision *eps*. Each solution is expressed as a list of numbers in order corresponding to the variables in *lv*.

**realZeros** (*polynomial*, *rationalNumber* [, *option*])

**realZeros** (*pol*) returns a list of isolating intervals for all the real zeros of the univariate polynomial *pol*.

**realZeros** (*pol*, *eps*) returns a list of intervals of length less than the rational number *eps* for all the real roots of the polynomial *pol*. The default value of *eps* is ???.

**realZeros** (*pol*, *int* [, *eps*]) returns a list of intervals of length less than the rational number *eps* for all the real roots of the polynomial *pol* which lie in the interval expressed by the record *int*. The default value of *eps* is ???.

**recip** (*element*)

**recip** (*x*) returns the multiplicative inverse for *x*, or "failed" if no inverse can be found. See also `FiniteRankNonAssociativeAlgebra` and `MonadWithUnit`.

**recur** (*function*)

**recur** (*f*), where *f* is a function of type  $(\text{NonNegativeInteger}, R) \rightarrow R$  for some domain *R*, returns the function *g* such that  $g(n, x) = f(n, f(n-1, \dots f(1, x) \dots))$ . For related functions, see `MappingPackage`.

**red** ()

**red** () returns the position of the red hue from total hues.

**reduce** (*op*, *aggregate* [, *identity*, *element*])

**reduce** (*f*, *u* [, *ident*, *a*]) reduces the binary operation *f* across *u*. For example, if *u* is  $[x_1, x_2, \dots, x_n]$  then **reduce** (*f*, *u*) returns  $f(\dots f(x_1, x_2), \dots, x_n)$ .

An optional identity element of *f* provided as a third argument affects the result if *u* has less than two elements. If *u* is empty, the third argument is returned if given, and a call to **error** occurs otherwise. If *u* has one element and the third argument is given, the value returned is  $f(\text{ident}, x_1)$ . Otherwise  $x_1$  is returned. Thus both **reduce** (+, *u*) and **reduce** (+, *u*, 0) return  $\sum_{i=1}^n x_i$ . Similarly, **reduce** (\*, *u*) and **reduce** (\*, *u*, 1) return  $\prod_{i=1}^n x_i$ .

An optional fourth argument *z* acts as an "absorbing

element". **reduce** (*f*, *u*, *x*, *z*) reduces the binary operation *f* across *u*, stopping when an "absorbing element" *z* is encountered. For example **reduce** (*or*, *u*, *false*, *true*) will stop iterating across *u* returning *true* as soon as an  $x_i = \text{true}$  is found. Note: if *u* has one element *x*, **reduce** (*f*, *u*) returns *x*, or calls **error** if *u* is empty.

**reduceBasisAtInfinity** (*basis*)

**reduceBasisAtInfinity** ( $b_1, \dots, b_n$ ), where the  $b_i$  are functions on a fixed curve, returns  $(x^i b_j)$  for all  $i, j$  such that  $x^i b_j$  is locally integral at infinity. See `FunctionFieldCategory` using `Browse`.

**reducedContinuedFraction** (*element*, *stream*)

**reducedContinuedFraction** ( $b_0, b$ ) returns a continued fraction constructed as follows. If  $b = [b_1, b_2, \dots]$  then the result is the continued fraction  $b_0 + 1/(b_1 + 1/(b_2 + \dots))$ . That is, the result is the same as **continuedFraction** ( $b_0, [1, 1, 1, \dots], [b_1, b_2, b_3, \dots]$ ).

**reducedForm** (*continuedFraction*)

**reducedForm** (*x*) puts the continued fraction *x* in reduced form, that is, the function returns an equivalent continued fraction of the form **continuedFraction** ( $b_0, [1, 1, 1, \dots], [b_1, b_2, b_3, \dots]$ ).

**reducedSystem** (*matrix* [, *vector*])

**reducedSystem** (*A*, *v*) returns a matrix *B* such that  $Ax = v$  and  $Bx = v$  have the same solutions. By default,  $v = 0$ .

**reductum** (*polynomial*)

**reductum** (*p*) returns polynomial *p* minus its leading monomial, or zero if handed the zero element. See also `IndexedDirectProductCategory` and `MonogenicLinearOperator`.

**refine** (*polynomial*, *interval*, *precision*)

**refine** (*pol*, *int*, *tolerance*) refines the interval *int* containing exactly one root of the univariate polynomial *pol* to size less than the indicated *tolerance*. Argument *int* is an interval denoted by a record with selectors *left* and *right*, each with rational number values. The tolerance is either a rational number or another interval. In the latter case, "failed" is returned if no such isolating interval exists.

**regularRepresentation** (*element*, *basis*)

**regularRepresentation** (*a*, *basis*) returns the matrix of the linear map defined by left multiplication by *a* with respect to basis *basis*. Element *a* is a complex element or an element of a domain *R* of category `FramedAlgebra`. The second argument may be omitted when a fixed basis is defined for *R*.



**reindex** (*cartesianTensor*, *listOfIntegers*)

**reindex** ( $t, [i_1, \dots, i_{\text{dim}}]$ ) permutes the indices of cartesian tensor  $t$ . For example, if  $r = \text{reindex}(t, [4, 1, 2, 3])$  for a rank 4 tensor  $t$ , then  $r$  is the rank 4 tensor given by  $r(i, j, k, l) = t(l, i, j, k)$ .

**relationsIdeal** (*listOfPolynomials*)

**relationsIdeal** (*polyList*) returns the ideal of relations among the polynomials in *polyList*.

**relererror** (*float*, *float*)

**relererror** ( $x, y$ ), where  $x$  and  $y$  are floats, computes the absolute value of  $x - y$  divided by  $y$ , when  $y \neq 0$ .

**rem** (*element*, *element*)

$a \text{ rem } b$  returns the remainder of  $a$  and  $b$ .

**remove** (*predicate*, *aggregate*)

Argument  $u$  is any extensible aggregate such as a list.

**remove** ( $\text{pred}, u$ ) returns a copy of  $u$  removing all elements  $x$  such that  $\text{pred}(x)$  is *true*. Argument  $u$  may be any homogeneous aggregate including infinite streams. Note: for lists and streams,  $\text{remove}(p, u) == [x \text{ for } x \text{ in } u \mid \text{not } p(x)]$ .

**remove!** ( $\text{pred}, u$ ) destructively removes all elements  $x$  of  $u$  such that  $\text{pred}(x)$  is *true*. The value of  $u$  after all such elements are removed is returned.

**remove!** ( $x, u$ ) destructively removes all values  $x$  from  $u$ .

**remove!** ( $k, t$ ), where  $t$  is a keyed dictionary, searches the table  $t$  for the key  $k$ , removing and returning the entry if there. If  $t$  has no such key, it returns "failed".

**removeCoshSq** (*expression*)

**removeCoshSq** ( $f$ ) converts every  $\cosh(u)^2$  appearing in  $f$  into  $1 - \sinh(x)^2$ , and also reduces higher powers of  $\cosh(u)$  with that formula.

**removeDuplicates** (*aggregate*)

**removeDuplicates!** (*aggregate*)

**removeDuplicates** ( $u$ ) returns a copy of  $u$  with all duplicates removed.

**removeDuplicates!** ( $u$ ) destructively removes duplicates from  $u$ .

**removeSinhSq** (*expression*)

**removeSinhSq** ( $f$ ) converts every  $\sinh(u)^2$  appearing in  $f$  into  $1 - \cosh(x)^2$ , and also reduces higher powers of  $\sinh(u)$  with that formula.

**removeSinSq** (*expression*)

**removeSinSq** ( $f$ ) converts every  $\sin(u)^2$  appearing in  $f$  into  $1 - \cos(x)^2$ , and also reduces higher powers of  $\sin(u)$  with that formula.

**removeZeroes** (*[integer, ]laurentSeries*)

**removeZeroes** ( $[n, ]f(x)$ ) removes up to  $n$  leading zeroes

from the Laurent series  $f(x)$ . If no integer  $n$  is given, all leading zeroes are removed.

**reopen!** (*file*, *string*)

**reopen!** ( $f, \text{mode}$ ) returns a file  $f$  reopened for operation in the indicated mode: "input" or "output". For example, **reopen!** ( $f, \text{"input"}$ ) will reopen the file  $f$  for input.

**repeating** (*listOfElements* [, *stream*])

**repeating?** (*stream*)

**repeating** ( $l$ ) is a repeating stream whose period is the list  $l$ .

**repeating?** ( $l, s$ ) tests if a stream  $s$  is periodic with period  $l$ .

**replace** (*string*, *segment*, *string*)

**replace** ( $s, i..j, t$ ) replaces the substring  $s(i..j)$  of  $s$  by string  $t$ .

**represents** (*listOfElements* [, *listOfBasisElements*])

**represents** ( $[a^1, \dots, a^n], [v^1, \dots, v^n]$ ) returns  $a^1 v^1 + \dots + a^n v^n$ . Arguments  $v_i$  are elements of a domain of category FiniteRankAlgebra or FiniteRankNonAssociativeAlgebra built over a ring  $R$ . The  $a_i$  are elements of  $R$ . In a framed algebra or finite algebra extension field domain with a fixed basis,  $[v_1, \dots, v_n]$  defaults to the elements of the fixed basis. See FramedAlgebra, FramedNonAssociateAlgebra, and FiniteAlgebraicExtensionField. See also FunctionFieldCategory.

**resetNew** ()

**resetNew** () resets the internal counter that **new** () uses.

**resetVariableOrder** ()

**resetVariableOrder** () cancels any previous use of **setVariableOrder** and returns to the default system ordering.

**rest** (*aggregate* [, *nonNegativeInteger*])

**rest** ( $u$ ) returns an aggregate consisting of all but the first element of  $u$  (equivalently, the next node of  $u$ ).

**rest** ( $u, n$ ) returns the  $n^{\text{th}}$  node of  $u$ . Note:

**rest** ( $u, 0$ ) =  $u$ .

**resultant** (*polynomial*, *polynomial* [, *variable*])

**resultant** ( $p, q, v$ ) returns the resultant of the polynomials  $p$  and  $q$  with respect to the variable  $v$ . If  $p$  and  $q$  are univariate polynomials, the variable  $v$  defaults to the unique variable.

**retract** (*element*)

**retractIfCan** (*element*)

**retractIfCan** ( $a$ )@ $S$  returns  $a$  as an object of type  $S$ , or "failed" if this is not possible.

**retract** ( $a$ )@ $S$  transforms  $a$  into an element of  $S$ , or calls

**error** if this is not possible.

**retractable?** (*typeAnyObject*)

**retractable?** (*a*)\$*S* tests if object *a* of type *Any* can be converted into an object of type *S*.

**reverse** (*linearAggregate*)

**reverse!** (*linearAggregate*)

**reverse** (*a*) returns a copy of linear aggregate *a* with elements in reverse order.

**reverse!** (*a*) destructively puts the elements of linear aggregate *a* in reverse order.

**rightGcd** (*lodOperator*, *lodOperator*)

**rightGcd** (*a*, *b*), where *a* and *b* are linear ordinary differential operators, computes the value *g* of highest degree such that  $a = g * aa$  and  $b = g * bb$  for some values *aa* and *bb*. The value *g* is computed using right-division.

**rhs** (*rewriteRuleOrEquation*)

**rhs** (*u*) returns the right-hand side of the rewrite rule or equation *u*.

**right** (*binaryRecursiveAggregate*)

**right** (*a*) returns the right child.

**rightAlternative?** ()

See **leftAlternative?**.

**rightCharacteristicPolynomial** (*element*)

See **leftCharacteristicPolynomial**.

**rightDiscriminant** (*basis*)

See **leftDiscriminant**.

**rightMinimalPolynomial** (*element*)

See **leftMinimalPolynomial**.

**rightNorm** (*element*)

See **leftNorm**.

**rightPower** (*monad*, *nonNegativeInteger*)

See **rightPower**.

**rightRankPolynomial** ()

See **leftRankPolynomial**.

**rightRank** (*basis*)

See **leftRank**.

**rightRecip** (*element*)

See **leftRecip**.

**rightRegularRepresentation** (*element* [, *basis*])

See **leftRegularRepresentation**.

**rightTraceMatrix** ( [*basis*])

See **leftTraceMatrix**.

**rightTrim** (*string*, *various*)

See **leftTrim**.

**rightUnits** ()

See **leftUnits**.

**rischNormalize** (*expression*, *x*)

**rischNormalize** (*f*, *x*) returns  $[g, [k_1, \dots, k_n], [h_1, \dots, h_n]]$  such that  $g = \text{normalize}(f, x)$  and each  $k_i$  was rewritten as  $h_i$  during the normalization.

**rightLcm** (*lodOperator*, *lodOperator*)

**rightLcm** (*a*, *b*), where *a* and *b* are linear ordinary differential operators, computes the value *m* of lowest degree such that  $m = aa * a = bb * b$  for some values *aa* and *bb*. The value *m* is computed using right-division.

**roman** (*integerOrSymbol*)

**roman** (*x*) creates a roman numeral for integer or symbol *x*.

**romberg** (*floatFunction*, *fourFloats*, *threeIntegers*)

**rombergOpen** (*floatFunction*, *fourFloats*, *twoIntegers*)

**rombergClose** (*floatFunction*, *fourFloats*, *twoIntegers*)

**romberg** (*fn*, *a*, *b*, *epsrel*, *epsabs*, *nmin*, *nmax*, *nint*) uses an adaptive romberg method to numerically integrate function *fn* over the closed interval from *a* to *b*, with relative accuracy *epsrel* and absolute accuracy *epsabs*; the refinement levels for the checking of convergence vary from *nmin* to *nmax*. The method is called “adaptive” since it requires an additional parameter *nint* giving the number of subintervals over which the integrator independently applies the convergence criteria using *nmin* and *nmax*. This is useful when a large number of points are needed only in a small fraction of the entire interval. Parameter *fn* is a function of type *Float* → *Float*; *a*, *b*, *epsrel*, and *epsabs* are floats; *nmin*, *nmax*, and *nint* are integers. The operation returns a record containing: **value**, an estimate of the integral; **error**, an estimate of the error in the computation; **totalpts**, the total integral number of function evaluations, and **success**, a boolean value that is *true* if the integral was computed within the user specified error criterion. See **NumericalQuadrature** for details.

**rombergClosed** (*fn*, *a*, *b*, *epsrel*, *epsabs*, *nmin*, *nmax*)

similarly uses the Romberg method to numerically integrate function *fn* over the closed interval *a* to *b*, but is not adaptive.

**rombergOpen** (*fn*, *a*, *b*, *epsrel*, *epsabs*, *nmin*, *nmax*)

is similar to **rombergClosed**, except that it integrates function *fn* over the open interval from *a* to *b*.

**root** (*outputForm* [, *positiveInteger*])

**root** (*o* [, *n*]), where *o* and *n* are objects of type *OutputForm* (normally unexposed), creates an output form for the  $n^{\text{th}}$  root of the form *o*. By default,  $n = 2$ .

**rootOfIrreduciblePoly** (*polynomial*)

**rootOfIrreduciblePoly** (*f*) computes one root of the monic, irreducible polynomial *f*, whose degree must divide the extension degree of *F* over *GF*. That is, *f* splits into linear factors over *F*.

**rootOf** (*polynomial* [, *variable*])

**rootOf** (*p* [, *y*]) returns *y* such that  $p(y) = 0$ . The object returned displays as *'y*. The second argument may be omitted when *p* is a polynomial in a unique variable *y*.

**rootSimp** (*expression*)

**rootSimp** (*f*) transforms every radical of the form  $(ab^{qn+r})^{1/n}$  appearing in expression *f* into  $b^q(ab^r)^{1/n}$ . This transformation is not in general valid for all complex numbers *b*.

**rootsOf** (*polynomialOrExpression* [, *symbol*])

**rootsOf** (*p* [, *y*]) returns the value of  $[y_1, \dots, y_n]$  such that  $p(y_i) = 0$ . The  $y_i$  are symbols of the form *%y* with a suffix number which are bound in the interpreter to respective root values. Argument *p* is either an expression or a polynomial. Argument *y* may be omitted in which case *p* must contain exactly one symbol.

**rootSplit** (*expression*)

**rootSplit** (*f*) transforms every radical of the form  $(a/b)^{1/n}$  appearing in *f* into  $a^{1/n}/b^{1/n}$ . This transformation is not in general valid for all complex numbers *a* and *b*.

**rotate!** (*queue*)

**rotate!** (*q*) rotates queue *q* so that the element at the front of the queue goes to the back of the queue.

**round** (*float*)

**round** (*x*) computes the integer closest to *x*.

**row** (*matrix*, *positiveInteger*)

**row** (*m*, *i*) returns the  $i^{\text{th}}$  row of the matrix or two-dimensional array *m*.

**rowEchelon** (*matrix*)

**rowEchelon** (*m*) returns the row echelon form of the matrix *m*.

**rst** (*stream*)

**rst** (*s*) returns a pointer to the next node of stream *s*. Caution: this function should only be called after a **empty?** test returns *true* since no error check is performed.

**rubiksGroup** ()

**rubiksGroup** () constructs the permutation group representing Rubik's Cube acting on integers  $10i + j$  for  $1 \leq i \leq 6, 1 \leq j \leq 8$ . The faces of Rubik's Cube are labelled: Front, Right, Up, Down, Left, Back and numbered from 1 to 6. The pieces on each face (except the unmoveable center piece) are clockwise numbered from 1 to 8 starting with the piece in the upper left corner. The moves of the cube are represented as permutations on these pieces, represented as a two digit integer *ij* where *i* is the number of the face and *j* is the number of the piece on this face. The remaining ambiguities are resolved by looking at the 6 generators representing 90-degree turns of the faces.

**rule** (*various*)

See Section 6.21 on page 228.

**rules** (*ruleset*)

**rules** (*r*) returns the list of rewrite rules contained in ruleset *r*.

**ruleset** (*listOfRules*)

**ruleset** ( $[r_1, \dots, r_n]$ ) creates a ruleset from a list of rewrite rules  $r_1, \dots, r_n$ .

**rungaKutta** (*vector*, *integer*, *fourFloats*, *integer*, *function*)

**rungaKuttaFixed** (*vector*, *integer*, *float*, *float*, *integer*, *function*)

**rungaKutta** (*y*, *n*, *a*, *b*, *eps*, *h*, *ncalls*, *derivs*) uses a 4-th order Runge-Kutta method to numerically integrate the ordinary differential equation  $dy/dx = f(y, x)$  from  $x_1$  to  $x_2$ , where *y* is an *n*-vector of *n* variables. Initial and final values are provided by solution vector *y*. The local truncation error is kept within *eps* by changing the local step size. Argument *h* is a trial step size and *ncalls* is the maximum number of single steps the integrator is allowed to take. Argument *derivs* is a function of type (*Vector Float*, *Vector Float*, *Float*)  $\rightarrow$  *Void*, which computes the right-hand side of the ordinary differential equation, then replaces the elements of the first argument by updated elements.

**rungaKuttaFixed** (*y*, *n*,  $x_1, x_2, ns$ , *derivs*) is similar to **rungaKutta** except that it uses *ns* fixed steps to integrate the solution vector *y* from  $x_1$  to  $x_2$ , returning the values in *y*.

**saturate** (*ideal*, *polynomial* [, *listOfVariables*])

**saturate** (*I*, *f* [, *lvar*]) is the saturation of the ideal *I* with respect to the multiplicative set generated by the polynomial *f* in the variables given by *lvar*, a list of variables. Argument *lvar* may be omitted in which case *lvar* is taken to be the list of all variables appearing in *f*.

**say** (*strings*)

**say** (*u*) sends a string or a list of strings *u* to output.

**sayLength** (*listOfStrings*)

**sayLength** (*ls*) returns the total number of characters in the list of strings *ls*.

**scalarMatrix** (*scalar* [, *dimension*])

**scalarMatrix** (*r* [, *n*]) returns an *n*-by-*n* matrix with scalar *r* on the diagonal and zero elsewhere. The dimension may be omitted if the result is to be an object of type **SquareMatrix** (*n*, *R*) for some *n*.

**scan** (*binaryFunction*, *aggregate*, *element*)

**scan** (*f*, *a*, *r*) successively applies **reduce** (*f*, *x*, *r*) to more and more leading sub-aggregates *x* of aggregate *a*. More precisely, if *a* is [*a*<sub>1</sub>, *a*<sub>2</sub>, ...], then **scan** (*f*, *a*, *r*) returns [*reduce*(*f*, [*a*<sub>1</sub>], *r*), *reduce*(*f*, [*a*<sub>1</sub>, *a*<sub>2</sub>], *r*), ...]. Argument *a* can be any linear aggregate including streams. For example, if *a* is a list or an infinite stream of the form [*x*<sub>1</sub>, *x*<sub>2</sub>, ...], then **scan**(+, *a*, 0) returns a list or stream of the form [*x*<sub>1</sub>, *x*<sub>1</sub> + *x*<sub>2</sub>, ...].

**scanOneDimSubspaces** (*listOfVectors*, *integer*)

**scanOneDimSubspaces** (*basis*, *n*) gives a canonical representative of the *n*<sup>th</sup> one-dimensional subspace of the vector space generated by the elements of *basis*. Consult RepresentationPackage2 using details.

**script** (*symbol*, *listOfListsOfOutputForms*)

**script** (*sy*, [*a*, *b*, *c*, *d*, *e*]) returns *sy* with subscripts *a*, superscripts *b*, pre-superscripts *c*, pre-subscripts *d*, and argument-scripts *e*. Omitted components are taken to be empty. For example, **script** (*s*, [*a*, *b*, *c*]) is equivalent to **script** (*s*, [*a*, *b*, *c*, [], []]).

**scripted?** (*symbol*)

**scripted?** (*sy*) tests if *sy* has been given any scripts.

**scripts** (*symbolOrOutputForm* [, *listOfOutputForms*])

**scripts** (*o*, *lo*), where *o* is an object of type **OutputForm** (normally unexposed) and *lo* is a list [*sub*, *super*, *presuper*, *presub*] of four objects of type **OutputForm** (normally unexposed), creates a form for *o* with scripts on all four corners.

**scripts** (*s*) returns all the scripts of *s* as a record with selectors *sub*, *sup*, *presup*, *presub*, and *args*, each with a list of output forms as a value.

**search** (*key*, *table*)

**search** (*k*, *t*) searches the table *t* for the key *k*, returning the entry stored in *t* for key *k*, or "failed" if *t* has no such key.

**sec** (*expression*)

**secIfCan** (*expression*)

**sec** (*x*) returns the secant of *x*.

**secIfCan** (*z*) returns **sec** (*z*) if possible, and "failed" otherwise.

**sec2cos** (*expression*)

**sec2cos** (*f*) converts every **sec** (*u*) appearing in *f* into 1/cos(*u*).

**sech** (*expression*)

**sechIfCan** (*expression*)

**sech** (*x*) returns the hyperbolic secant of *x*.

**sechIfCan** (*z*) returns **sech** (*z*) if possible, and "failed" otherwise.

**sech2cosh** (*expression*)

**sech2cosh** (*f*) converts every **sech** (*u*) appearing in *f* into 1/cosh(*u*).

**second** (*aggregate*)

**second** (*u*) returns the second element of recursive aggregate *u*. Note: **second** (*u*) = **first**(**rest**(*u*)).

**segment** (*integer* [, *integer*])

**segment** (*i* [, *j*]) returns the segment *i*..*j*. If not qualified by a **by** clause, this notation for integers *i* and *j* denotes the tuple of integers *i*, *i* + 1, ..., *j*. When *j* is omitted, **segment** (*i*) denotes the half open segment *i*..., that is, a segment with no upper bound.

**segment** (*x* = *bd*), where *bd* is a binding, returns *bd*. For example, **segment** (*x* = *a*..*b*) returns *a*..*b*.

**select** (*pred*, *aggregate*)

**select!** (*pred*, *aggregate*)

**select** (*p*, *u*) returns a copy of *u* containing only those elements *x* such that *p*(*x*) is true. For a list *l*, **select**(*p*, *l*) == [*x* for *x* in *l* | *p*(*x*)]. Argument *u* may be any finite aggregate or infinite stream.

**select!** (*p*, *u*) destructively changes *u* by keeping only values *x* such that *p*(*x*) is true. Argument *u* can be any extensible linear aggregate or dictionary.

**semicolonSeparate** (*listOfOutputForms*)

**semicolonSeparate** (*lo*), where *lo* is a list of objects of type **OutputForm** (normally unexposed), returns an output form which separates the elements of *lo* by semicolons.

**separant** (*differentialPolynomial*)

**separant** (*polynomial*) returns the partial derivative of the differential polynomial *p* with respect to its leader.

**separate** (*polynomial*, *polynomial*)

**separate** (*p*, *q*) returns (*a*, *b*) such that polynomial *p* = *ab* and *a* is relatively prime to *q*. The result produced is a record with selectors *primePart* and *commonPart* with value *a* and *b* respectively.

**separateDegrees** (*polynomial*)

**separateDegrees** (*p*) splits the polynomial *p* into factors. Each factor is a record with selector *deg*, a non-negative integer, and *prod*, a product of irreducible polynomials of degree *deg*.

**separateFactors** (*listOfRecords*, *polynomial*)

**separateFactors** (*lfact*, *p*) takes the list produced by **separateDegrees** along with the original polynomial *p*, and produces the complete list of factors.

**separateFactors** (*listOfRecords*, *integer*)

**separateFactors** (*ddl*, *p*) refines the distinct degree factorization produced by **ddFact** to give a complete list of factors.

**sequences** (*listOfIntegers*)

**sequences** (*listOfIntegers*, *listOfIntegers*)

**sequences** ( $[l_0, l_1, l_2, \dots, l_n]$ ) is the set of all sequences formed from  $l_0$  0's,  $l_1$  1's,  $l_2$  2's, ...,  $l_n$   $n$ 's.

**sequences** ( $l1, l2$ ) is the stream of all sequences that can be composed from the multiset defined from two lists of integers  $l1$  and  $l2$ . For example, the pair  $([1, 2, 4], [2, 3, 5])$  represents multiset with 1 2, 2 3's, and 4 5's.

**series** (*specifications* [, ...])

**series** (*expression*) returns a series expansion of the expression *f*. Note: *f* must have only one variable. The series will be expanded in powers of that variable.

**series** (*sy*), where *sy* is a symbol, returns *sy* as a series.

**series** (*st*), where *t* is a stream  $[a_0, a_1, a_2, \dots]$  of coefficients  $a_i$  from some ring, creates the Taylor series  $a_0 + a_1x + a_2x^2 + \dots$ . Also, if *st* is a stream of elements of type `Record(k:NonNegativeInteger, c:R)`, where *k* denotes an exponent and *c*, a non-zero coefficient from some ring *R*, it creates a stream of non-zero terms. The terms in *st* must be ordered by increasing order of exponents.

**series** (*f*,  $x = a[n]$ ) expands the expression *f* as a series in powers of  $(x - a)$  with *n* terms. If *n* is missing, the number of terms is governed by the value set by the system command `set streams calculate`.

**series** (*f*, *n*) returns a series expansion of the expression *f*. Note: *f* should have only one variable; the series will be expanded in powers of that variable and terms will be computed up to order at least *n*.

**series** ( $i+->a(i), x = a, m..[n, k]$ ) creates the series  $\sum_{i=m..n \text{ by } k} a(i)(x - a)^i$ . Here *m*, *n*, and *k* are rational numbers. Upper-limit *n* and stepsize *k* are optional and have default values  $n = \infty$  and  $k = 1$ .

**series** ( $a(i), i, x = a, m..[n, k]$ ) returns  $\sum_{i=m..n \text{ by } k} a(n)(x - a)^n$ .

**seriesSolve** (*eq*, *y*, *x*, *c*)

*eq* denotes an equation to be solved; alternatively, an expression *u* may be given for *eq* in which case the

equation *eq* is defined as  $u = 0$ .

*leq* denotes a list  $[eq_1 \dots eq_n]$  of equations; alternatively, a list of expressions  $[u_1 \dots u_n]$  may be given of *leq* in which case the equations *eq<sub>i</sub>* are defined by  $u_i = 0$ .

**seriesSolve** (*eq*, *y*,  $x = a, [y(a) = b]$ ) returns a Taylor series solution of *eq* around  $x = a$  with initial condition  $y(a) = b$ . Note: *eq* must be of the form  $f(x, y)y'(x) + g(x, y) = h(x, y)$ .

**seriesSolve** (*eq*, *y*,  $x = a, [b_0, \dots, b_{(n-1)}]$ ) returns a Taylor series solution of *eq* around  $x = a$  with initial conditions  $y(a) = b_0, y'(a) = b_1, \dots, y^{(n-1)}(a) = b_{(n-1)}$ . Equation *eq* must be of the form  $f(x, y, y', \dots, y^{(n-1)}) * y^{(n)}(x) + g(x, y, x', \dots, y^{(n-1)}) = h(x, y, y', \dots, y^{(n-1)})$ .

**seriesSolve** (*leq*,  $[y_1, \dots, y_n], x = a, [y_1(a) = b_1, \dots, y_n(a) = b_n]$ ) returns a Taylor series solution of the equations *eq<sub>i</sub>* around  $x = a$  with initial conditions  $y_i(a) = b_i$ . Note: each *eq<sub>i</sub>* must be of the form  $f_i(x, y_1, y_2, \dots, y_n)y'_1(x) + g_i(x, y_1, y_2, \dots, y_n) = h(x, y_1, y_2, \dots, y_n)$ .

**seriesSolve** (*leq*,  $[y_1, \dots, y_n], x = a, [b_1, \dots, b_n]$ ) is equivalent to the same command with fourth argument  $[y_1(a) = b_1, \dots, y_n(a) = b_n]$ .

**setchildren!** (*recursiveAggregate*)

**setchildren!** (*u*, *v*) replaces the current children of node *u* with the members of *v* in left-to-right order.

**setColumn!** (*matrix*)

**setColumn!** (*m*, *j*, *v*) sets the  $j^{\text{th}}$  column of matrix or two-dimensional array *m* to *v*.

**setDifference** (*list*, *list*)

**setDifference** ( $l_1, l_2$ ) returns a list of the elements of  $l_1$  that are not also in  $l_2$ . The order of elements in the resulting list is unspecified.

**setelt** (*structure*, *index*, *value* [, *option*])

**setelt** (*u*, *x*, *y*), also written  $u.x := y$ , sets the image of *x* to be *y* under *u*, regarded as a function mapping values from the domain of *x* to the domain of *y*. Specifically, if *u* is:

a list:  $u.first := x$  is equivalent to **setfirst!** (*u*, *x*). Also,  $u.rest := x$  is equivalent to **setrest!** (*u*, *x*), and  $u.last := x$  is equivalent to **setlast!** (*u*, *x*).

a linear aggregate,  $u(i..j) := x$  destructively replaces each element in the segment  $u(i..j)$  by *x*. The value *x* is returned. Note: This function has the same effect as **for k in i..j repeat u.k := x; x**. The length of *u* is unchanged.

a recursive aggregate,  $u.value := x$  is equivalent to **setvalue!** (*u*, *x*) and sets the value part of node *u* to *x*. Also, if *u* is a `BinaryTreeAggregate`,  $u.left := x$  is equivalent to **setleft!** (*u*, *x*) and sets the left child of *u*

to  $x$ . Similarly,  $u.right := x$  is equivalent to **setright!**( $u, x$ ). See also **setchildren!**.

a table of category TableAggregate(Key, Entry):  $u(k) := e$  is equivalent to (**insert**!([ $k, e, t$ ];  $e$ ), where  $k$  is a key and  $e$  is an entry.

a library:  $u.k := v$  saves the value  $v$  in the library  $u$ , so that it can later be extracted by  $u.k$ .

**setelt** ( $u, i, j, r$ ), also written,  $u(i, j) := r$ , sets the element in the  $i^{\text{th}}$  row and  $j^{\text{th}}$  column of matrix or two-dimensional array  $u$  to  $r$ .

**setelt** ( $u, rowList, colList, r$ ), also written  $u([i_1, i_2, \dots, i_m], [j_1, j_2, \dots, j_n]) := r$ , where  $u$  is a matrix or two-dimensional array and  $r$  is another  $m$  by  $n$  matrix or array, destructively alters the matrix  $u$ : the  $x_{i_k, j_l}$  is set to  $r(k, l)$ .

**setEpilogue!** (*formattedObject*, *listOfStrings*)

**setEpilogue!** ( $t, strings$ ) sets the epilogue section of a formatted object  $t$  to  $strings$ . Argument  $t$  is either an IBM SCRIPT Formula Formatted or TeX formatted object.

**setfirst!** (*aggregate*, *value*)

**setfirst!** ( $a, x$ ) destructively changes the first element of recursive aggregate  $a$  to  $x$ .

**setFormula!** (*formattedObject*, *listOfStrings*)

**setFormula!** ( $t, strings$ ) sets the formula section of a formatted object  $t$  to  $strings$ .

**setIntersection** (*list*, *list*)

**setIntersection** ( $l_1, l_2$ ) returns a list of the elements that lists  $l_1$  and  $l_2$  have in common. The order of elements in the resulting list is unspecified.

**setlast!** (*aggregate*, *value*)

**setlast!** ( $u, x$ ) destructively changes the last element of  $u$  to  $x$ . Note:  $u.last := x$  is equivalent.

**setleaves!** (*balancedBinaryTree*, *listOfElements*)

**setleaves!** ( $t, ls$ ) sets the leaves of balanced binary tree  $t$  in left-to-right order to the elements of  $ls$ .

**setleft!** (*binaryRecursiveAggregate*)

**setleft!** ( $a, b$ ) sets the left child of  $a$  to be  $b$ .

**setPrologue!** (*formattedObject*, *listOfStrings*)

**setPrologue!** ( $t, strings$ ) sets the prologue section of a formatted object  $t$  to  $strings$ . Argument  $t$  is either an IBM SCRIPT Formula Formatted or TeX formatted object.

**setProperty!** (*basicOperator*, *associationList*)

**setProperty!** ( $op, al$ ) sets the property list of basic operator  $op$  to association list  $l$ . Note: argument  $op$  is modified “in place”, that is, no copy is made.

**setProperty!** (*basicOperator*, *string*, *value*)

**setProperty!** ( $op, s, v$ ) attaches property  $s$  to  $op$ , and sets its value to  $v$ . Argument  $op$  is modified “in place”, that is, no copy is made.

**setrest!** (*aggregate*[, *integer*], *aggregate*)

Arguments  $u$  and  $v$  are finite or infinite aggregates of the same type.

**setrest!** ( $u, v$ ) destructively changes the rest of  $u$  to  $v$ .

**setrest!** ( $x, n, y$ ) destructively changes  $x$  so that **rest** ( $x, n$ ), that is,  $x$  after the  $n^{\text{th}}$  element, equals  $y$ . The function will expand cycles if necessary.

**setright!** (*binaryRecursiveAggregate*)

**setright!** ( $a, x$ ) sets the right child of  $t$  to be  $x$ .

**setRow!** (*matrix*, *integer*, *row*)

**setRow!** ( $m, i, v$ ) sets the  $i^{\text{th}}$  row of matrix or two-dimensional array  $m$  to  $v$ .

**setsubMatrix!** (*matrix*, *integer*, *integer*, *matrix*)

**setsubMatrix** ( $x, i_1, j_1, y$ ) destructively alters the matrix  $x$ . Here  $x(i, j)$  is set to  $y(i - i_1 + 1, j - j_1 + 1)$  for  $i = i_1, \dots, i_1 - 1 + \mathbf{nrows}(y)$  and  $j = j_1, \dots, j_1 - 1 + \mathbf{ncols}(y)$ .

**setTex!** (*text*, *listOfStrings*)

**setTex!** ( $t, strings$ ) sets the TeX section of a TeX form  $t$  to  $strings$ .

**setUnion** (*list*, *list*)

**setUnion** ( $l_1, l_2$ ) appends the two lists  $l_1$  and  $l_2$ , then removes all duplicates. The order of elements in the resulting list is unspecified.

**setvalue!** (*aggregate*, *value*)

**setvalue!** ( $u, x$ ) destructively changes the value of node  $u$  to  $x$ .

**setVariableOrder** (*listOfSymbols*[, *listOfSymbols*])

**setVariableOrder** ( $[a_1, \dots, a_m], [b_1, \dots, b_n]$ ) defines an ordering on the variables given by  $a_1 > a_2 > \dots > a_m >$  other variables  $b_1 > b_2 > \dots > b_n$ .

**setVariableOrder** ( $[a_1, \dots, a_n]$ ) defines an ordering given by  $a_1 > a_2 > \dots > a_n >$  all other variables.

**sFunction** (*listOfIntegers*)

**sFunction** ( $li$ ) is the S-function of the partition given by list of integer  $li$ , expressed in terms of power sum symmetric functions. See CycleIndicators for details.

**shade** (*palette*)

**shade** ( $p$ ) returns the shade index of the indicated palette  $p$ .

**shellSort** (*sortingFunction*, *aggregate*)

**shellSort** ( $f, a$ ) sorts the aggregate  $a$  using the shellSort algorithm with sorting function  $f$ . Aggregate  $a$  can be any finite linear aggregate which is mutable (for example, lists, vectors, and strings). The sorting function  $f$  has type  $(R, R) \rightarrow \text{Boolean}$  where  $R$  is the domain of the elements of  $a$ .

**shift** (*integerNumber*, *integer*)

**shift** ( $a, i$ ) shifts integer number or float  $a$  by  $i$  digits.

**showAll?** ()

**showAll?** () tests if all computed entries of streams will be displayed according to system command `)set streams showall`.

**showAllElements** (*stream*)

**showAllElements** ( $s$ ) creates an output form displaying all the already computed elements of stream  $s$ . This command will not result in any further computation of elements of  $s$ . Also, the command has no effect if the user has previously entered `)set streams showall true`.

**showTypeInOut** (*boolean*)

**showTypeInOut** ( $bool$ ) affects the way objects of Any are displayed. If  $bool$  is *true*, the type of the original object that was converted to Any will be printed. If  $bool$  is *false*, it will not be printed.

**shrinkable** (*boolean*)

**shrinkable** ( $b$ )\$ $R$  tells AXIOM that flexible arrays of domain  $R$  are or are not allowed to shrink (reduce their **physicalLength**) according to whether  $b$  is *true* or *false*.

**shufflein** (*listOfIntegers*, *streamOfListsOfIntegers*)

**shufflein** ( $li, sli$ ) maps **shuffle** ( $li, u$ ) onto all members  $u$  of  $sli$ , concatenating the results. See `PartitionsAndPermutations`.

**shuffle** (*listOfIntegers*, *listOfIntegers*)

**shuffle** ( $l1, l2$ ) forms the stream of all shuffles of  $l1$  and  $l2$ , that is, all sequences that can be formed from merging  $l1$  and  $l2$ . See `PartitionsAndPermutations`.

**sign** (*various* [, ...])

**sign** ( $x$ ), where  $x$  is an element of an ordered ring, returns 1 if  $x$  is positive, -1 if  $x$  is negative, 0 if  $x$  equals 0.

**sign** ( $p$ ), where  $p$  is a permutation, returns 1, if  $p$  is an even permutation, or -1, if it is odd.

**sign** ( $f, x, a, s$ ) returns the sign of rational function  $f$  as symbol  $x$  nears  $a$ , a real value represented by either a rational function or one of the values `%plusInfinity` or `%minusInfinity`. If  $s$  is:

the string `"left"`: from the left (below).

the string `"right"`: from the right (above).

not given: from both sides if  $a$  is finite.

**simplify** (*expression*)

**simplify** ( $f$ ) performs the following simplifications on  $f$ :

rewrites trigs and hyperbolic trigs in terms of *sin*, *cos*, *sinh*, *cosh*.

rewrites  $\sin^2$  and  $\sinh^2$  in terms of *cos* and *cosh*.

rewrites  $e^a e^b$  as  $e^{a+b}$ .

**simplifyExp** (*expression*)

**simplifyExp** ( $f$ ) converts every product  $e^a e^b$  appearing in  $f$  into  $e^{a+b}$ .

**simpson** (*floatFunction*, *fourFloats*, *threeIntegers*)

**simpsonClosed** (*floatFunction*, *fourFloats*, *twoIntegers*)

**simpsonOpen** (*floatFunction*, *fourFloats*, *twoIntegers*)

**simpson** ( $fn, a, b, epsrel, epsabs, nmin, nmax, nint$ ) uses the adaptive simpson method to numerically integrate function  $fn$  over the closed interval from  $a$  to  $b$ , with relative accuracy *epsrel* and absolute accuracy *epsabs*; the refinement levels for the checking of convergence vary from *nmin* to *nmax*. The method is called "adaptive" since it requires an additional parameter *nint* giving the number of subintervals over which the integrator independently applies the convergence criteria using *nmin* and *nmax*. This is useful when a large number of points are needed only in a small fraction of the entire interval. Parameter  $fn$  is a function of type  $\text{Float} \rightarrow \text{Float}$ ;  $a, b, epsrel$ , and *epsabs* are floats; *nmin*, *nmax*, and *nint* are integers. The operation returns a record containing: **value**, an estimate of the integral; **error**, an estimate of the error in the computation; **totalpts**, the total integral number of function evaluations, and **success**, a boolean value which is *true* if the integral was computed within the user specified error criterion. See `NumericalQuadrature` for details.

**simpsonClosed** ( $fn, a, b, epsrel, epsabs, nmin, nmax$ )

similarly uses the Simpson method to numerically integrate function  $fn$  over the closed interval  $a$  to  $b$ , but is not adaptive.

**simpsonOpen** ( $fn, a, b, epsrel, epsabs, nmin, nmax$ )

is similar to **simpsonClosed**, except that it integrates function  $fn$  over the open interval from  $a$  to  $b$ .

**sin** (*expression*)

Argument  $x$  can be a Complex, Float, DoubleFloat, or Expression value or a series.

**sin** ( $x$ ) returns the sine of  $x$  if possible, and calls **error** otherwise.

**sinIfCan** ( $x$ ) returns **sin** ( $x$ ) if possible, and `"failed"` otherwise.

**sin2csc** (*expression*)

**sin2csc** ( $f$ ) converts every **sin** ( $u$ ) appearing in  $f$  into  $1/\text{csc}(u)$ .

**singular?** (*polynomialOrFunctionField*)

**singularAtInfinity?** ()

**singular?** ( $p$ ) tests whether  $p(x) = 0$  is singular.

**singular?** ( $a$ ) $\$F$  tests if  $x = a$  is a singularity of the algebraic function field  $F$  (a domain of FunctionFieldCategory).

**singularAtInfinity?** () $\$F$  tests if the algebraic function field  $F$  has a singularity at infinity.

**sinh** (*expression*)

**sinhIfCan** (*expression*)

Argument  $x$  can be a Complex, Float, DoubleFloat, or Expression value or a series.

**sinh** ( $x$ ) returns the hyperbolic sine of  $x$  if possible, and calls **error** otherwise.

**sinhIfCan** ( $x$ ) returns **sinh** ( $x$ ) if possible, and "failed" otherwise.

**sinh2csch** (*expression*)

**sinh2csch** ( $f$ ) converts every **sinh** ( $u$ ) appearing in  $f$  into  $1/\text{csch}(u)$ .

**size** ()

**size** () $\$F$  returns the number of elements in the domain of category Finite. By definition, each such domain must have a finite number of elements. See also FreeAbelianMonoidCategory.

**size?** (*aggregate*, *nonNegativeInteger*)

**size?** ( $a$ ,  $n$ ) tests if aggregate  $a$  has exactly  $n$  elements.

**sizeLess?** (*element*, *element*)

**sizeLess?** ( $x$ ,  $y$ ) tests whether  $x$  is strictly smaller than  $y$  with respect to the **euclideanSize**.

**sizeMultiplication** ()

**sizeMultiplication** () $\$F$  returns the number of entries in the multiplication table of the field. Note: The time of multiplication of field elements depends on this size.

**skewSFunction** (*listOfIntegers*, *listOfIntegers*)

**skewSFunction** ( $l_{i1}$ ,  $l_{i2}$ ) is the S-function of the partition difference  $l_{i1} - l_{i2}$ , expressed in terms of power sum symmetric functions. See CycleIndicators for details.

**solve** ( $u$ ,  $v$  [,  $w$ ])

$eq$  denotes an equation to be solved; alternatively, an expression  $u$  may be given for  $eq$  in which case the equation  $eq$  is defined as  $u = 0$ .

$leq$  denotes a list [ $eq_1 \dots eq_n$ ] of equations; alternatively, a list of expressions [ $u_1 \dots u_n$ ] may be given for  $leq$  in which case the equations  $eq_i$  are defined by  $u_i = 0$ .

$epsilon$  is either a rational number or a float.

**complexSolve** ( $eq$ ,  $epsilon$ ) finds all the real solutions to precision  $epsilon$  of the univariate equation  $eq$  of rational functions with respect to the unique variable appearing in

$eq$ . The complex solutions are either expressed as rational numbers or floats depending on the type of  $epsilon$ .

**complexSolve** ( $[eq_1 \dots eq_n]$ ,  $epsilon$ ) computes the real solutions to precision  $epsilon$  of a system of equations  $eq_i$  involving rational functions. The complex solutions are either expressed as rational numbers or floats depending on the type of  $epsilon$ .

**radicalSolve** ( $eq$  [,  $x$ ]) finds solutions expressed in terms of radicals of the equation  $eq$  involving rational functions. Solutions will be found with respect to a Symbol given as a second argument to the operation. This second argument may be omitted when  $eq$  contains a unique symbol.

**radicalSolve** ( $leq$ ,  $lv$ ) finds solutions expressed in terms of radicals of the system of equations  $leq$  involving rational functions. Solutions are found with respect to a list  $lv$  of Symbols, or with respect to all variables appearing in the equations, if no second argument is given.

**solve** ( $eq$  [,  $x$ ]) finds exact symbolic solutions to equation  $eq$  involving either rational functions or expressions of type Expression(R). Solutions will be found with respect to a Symbol given as a second argument to the operation. The second argument may be omitted when  $eq$  contains a unique symbol.

**solve** ( $leq$ ,  $lv$ ) finds exact solutions to a system of equations  $leq$  involving rational functions or expressions of type Expression(R). Solutions are found with respect to a list of  $lv$  of Symbols, or with respect to all variables appearing in the equations if no second argument is given.

**solve** ( $eq$ ,  $epsilon$ ) finds all the real solutions to precision  $epsilon$  of the univariate equation  $eq$  of rational functions with respect to the unique variable appearing in  $eq$ . The real solutions are either expressed as rational numbers or floats depending on the type of  $epsilon$ .

**solve** ( $[eq_1 \dots eq_n]$ ,  $epsilon$ ) computes the real solutions to precision  $epsilon$  of a system of equations  $eq_i$  involving rational functions. The real solutions are either expressed as rational numbers or floats depending on the type of  $epsilon$ .

**solve** ( $M$ ,  $v$ ), where  $M$  is a matrix and  $v$  is a Vector of coefficients, finds a particular solution of the system  $Mx = v$  and a basis of the associated homogeneous system  $MX = 0$ .

**solve** ( $eq$ ,  $y$ ,  $x = a$ , [ $y_0 \dots y_m$ ]) returns either the solution of the initial value problem  $eq$ ,  $y(a) = y_0$ ,  $y'(a) = a_1, \dots$  or "failed" if no solution can be found. Note: an error occurs if the equation  $eq$  is not a linear ordinary equation or of the form  $dy/dx = f(x, y)$ .

**solve** ( $eq$ ,  $y$ ,  $x$ ) returns either a solution of the ordinary differential equation  $eq$  or "failed" if no non-trivial solution can be found. If  $eq$  is a linear ordinary differential equation, a solution is of the form  $[h, [b_1, \dots, ]]$  where  $h$  is a



particular solution and  $[b_1, \dots, b_m]$  are linearly independent solutions of the associated homogeneous equation  $f(x, y) = 0$ . The value returned is a basis for the solution of the homogeneous equation which are found (note: this is not always a full basis).

See also **dioSolve**, **contractSolve**, **polSolve**, **seriesSolve**, **linSolve**.

**solveLinearlyOverQ** (*vector*)

**solveLinearlyOverQ** ( $[v_1, \dots, v_n], u$ ) returns  $[c_1, \dots, c_n]$  such that  $c_1 v_1 + \dots + c_n v_n = u$ , or "failed" if no such rational numbers  $c_i$  exist. The elements of the  $v_i$  and  $u$  can be from any extension ring with an explicit linear dependence test, for example, expressions, complex values, polynomials, rational functions, or exact numbers. See **LinearExplicitRingOver**.

**solveLinearPolynomialEquation** (*listOfPolys*, *poly*)

**solveLinearPolynomialEquation** ( $[f_1, \dots, f_n], g$ ), where  $g$  is a polynomial and the  $f_i$  are polynomials relatively prime to one another, returns a list of polynomials  $a_i$  such that  $g/\prod_i f_i = \sum_i a_i/f_i$ , or "failed" if no such list of  $a_i$ 's exists.

**sort** ( $[predicate, ]aggregate$ )

**sort!** ( $[predicate, ]aggregate$ )

**sort** ( $[p, ]a$ ) returns a copy of  $a$  sorted using total ordering predicate  $p$ .

**sort!** ( $[p, ]u$ ) returns  $u$  destructively changed with its elements ordered by comparison function  $p$ .

By default,  $p$  is the operation  $\leq$ . Thus both **sort** ( $u$ ) and **sort!** ( $u$ ) returns  $u$  with its elements in ascending order.

Also: **sort** ( $lp$ ) sorts a list of permutations  $lp$  according to cycle structure, first according to the length of cycles, second, if  $S$  has Finite or  $S$  has OrderedSet, according to lexicographical order of entries in cycles of equal length.

**spherical** (*point*)

**spherical** ( $pt$ ) transforms point  $pt$  from spherical coordinates to Cartesian coordinates, mapping  $(r, \theta, \phi)$  to  $x = r \sin(\phi) \cos(\theta)$ ,  $y = r \sin(\phi) \sin(\theta)$ ,  $z = r \cos(\phi)$ .

**split** (*element*, *binarySearchTree*)

**split** ( $x, t$ ) splits binary search tree  $t$  into two components, returning a record of two components: *less*, a binary search tree whose components are all less than  $x$ ; and, *greater*, a binary search tree with all the rest of the components of  $t$ .

**split!** (*aggregate*, *integer*)

**split!** ( $u, n$ ) splits  $u$  into two aggregates: the first consisting of  $v$ , the first  $n$  elements of  $u$ , and  $w$  consisting of all the rest. The value of  $w$  is returned. Thus  $v = \mathbf{first}(u, n)$  and  $w := \mathbf{rest}(u, n)$ . Note: afterwards **rest** ( $u, n$ ) returns **empty** ().

**splitDenominator** (*listOfFractions*)

**splitDenominator** ( $u$ ), where  $u$  is a list of fractions  $[q_1, \dots, q_n]$ , returns  $[[p_1, \dots, p_n], d]$  such that  $q_i = p_i/d$  and  $d$  is a common denominator for the  $q_i$ 's. Similarly, the function is defined for a matrix (respectively, a polynomial)  $u$  in which case the  $q_i$  are the elements of (respectively, the coefficients of)  $u$ .

**sqfrFactor** (*element*, *integer*)

**sqfrFactor** ( $base, exponent$ ) creates a factored object with a single factor whose *base* is asserted to be square-free (flag = "sqfr").

**sqrt** (*expression* [, *option*])

**sqrt** ( $x$ ) returns the square root of  $x$ .

**sqrt** ( $x, y$ ), where  $x$  and  $y$  are  $p$ -adic integers, returns a square root of  $x$  where argument  $y$  is a square root of  $x \bmod p$ . See also **PAdicIntegerCategory**.

**square?** (*matrix*)

**square?** ( $m$ ) tests if  $m$  is a square matrix.

**squareFree** (*element*)

**squareFree** ( $x$ ) returns the square-free factorization of  $x$ , that is, such that the factors are pairwise relatively prime and each has multiple prime factors. Argument  $x$  can be a member of any domain of category **UniqueFactorizationDomain** such as a polynomial or integer.

**squareFreePart** (*element*)

**squareFreePart** ( $p$ ) returns product of all the prime factors of  $p$  each taken with multiplicity one. Argument  $p$  can be a member of any domain of category **UniqueFactorizationDomain** such as a polynomial or integer.

**squareFreePolynomial** (*polynomial*)

**squareFreePolynomial** ( $p$ ) returns the square-free factorization of the univariate polynomial  $p$ .

**squareTop** (*matrix*)

**squareTop** ( $A$ ) returns an  $n$ -by- $n$  matrix consisting of the first  $n$  rows of the  $m$ -by- $n$  matrix  $A$ . The operation calls **error** if  $m < n$ .

**stack** (*list*)

**stack** ( $[x, y, \dots, z]$ ) creates a stack with first (top) element  $x$ , second element  $y$ ,  $\dots$ , and last element  $z$ .

**standardBasisOfCyclicSubmodule** (*listOfMatrices*, *vector*)

**standardBasisOfCyclicSubmodule** ( $lm, v$ ) returns a matrix representation of cyclic submodule over a ring  $R$ , where  $lm$  is a list of matrices and  $v$  is a vector, such that the non-zero column vectors are an  $R$ -basis for  $Av$ . See **RepresentationPackage2** using **Browse**.

**stirling1** (*integer, integer*)  
**stirling2** (*integer, integer*)  
**stirling1** ( $n, m$ ) returns the Stirling number of the first kind.  
**stirling2** ( $n, m$ ) returns the Stirling number of the second kind.

**string?** (*various*)  
**string** (*sExpression*)  
**string?** ( $s$ ) tests if SExpression object  $s$  is a string.  
**string** ( $s$ ) converts the symbol  $s$  to a string. An **error** is called if the symbol is subscripted.  
**string** ( $s$ ) returns SExpression object  $s$  as an element of String if possible, and otherwise calls **error**.

**strongGenerators** (*listOfPermutations*)  
**strongGenerators** ( $gp$ ) returns strong generators for the permutation group  $gp$ .

**structuralConstants** (*basis*)  
**structuralConstants** (*basis*) calculates the structural constants  $[(\gamma_{i,j,k}) \text{ for } k \text{ in } 1..rank(\$R)]$  of a domain  $R$  of category FramedNonAssociativeAlgebra over a ring  $R$ , defined by:  $v_i v_j = \gamma_{i,j,1} v_1 + \dots + \gamma_{i,j,n} v_n$ , where  $v_1, \dots, v_n$  is the fixed  $R$ -module basis.

**style** (*string*)  
**style** ( $s$ ) specifies the drawing style in which the graph will be plotted by the indicated string  $s$ . This option is expressed in the form **style == s**.

**sub** (*outputForm, outputForm*)  
**sub** ( $o_1, o_2$ ), where  $o_1$  and  $o_2$  are objects of type OutputForm (normally unexposed), creates an output form for  $o_1$  subscripted by  $o_2$ .

**subMatrix** (*matrix, integer, integer, integer, integer*)  
**subMatrix** ( $m, i_1, i_2, j_1, j_2$ ) extracts the submatrix  $[m(i, j)]$  where the index  $i$  ranges from  $i_1$  to  $i_2$  and the index  $j$  ranges from  $j_1$  to  $j_2$ .

**submod** (*integerNumber, integerNumber, integerNumber*)  
**submod** ( $a, b, p$ ), where  $0 \leq a < b < p > 1$ , returns  $a - b \bmod p$ , for integer numbers  $a, b$  and  $p$ .

**subResultantGcd** (*polynomial, polynomial*)  
**subResultantGcd** ( $p, q$ ) computes the *gcd* of the polynomials  $p$  and  $q$  using the SubResultant *GCD* algorithm.

**subscript** (*symbol, listOfOutputForms*)  
**subscript** ( $s, [a_1, \dots, a_n]$ ) returns symbol  $s$  subscripted by output forms  $a_1, \dots, a_n$  as a symbol.

**subset** (*integer, integer, integer*)  
**subSet** ( $n, m, k$ ) calculates the  $k^{\text{th}}$   $m$ -subset of the set

$0, 1, \dots, (n-1)$  in the lexicographic order considered as a decreasing map from  $0, \dots, (m-1)$  into  $0, \dots, (n-1)$ . See SymmetricGroupCombinatoricFunctions.

**subset?** (*set, set*)  
**subset?** ( $u, v$ ) tests if set  $u$  is a subset of set  $v$ .

**subspace** (*threeSpace*)  
**subspace** ( $s$ ) returns the space component which holds all the point information in the ThreeSpace object  $s$ .

**substring?** (*string, string, integer*)  
**substring?** ( $s, t, i$ ) tests if  $s$  is a substring of  $t$  beginning at index  $i$ . Note: **substring?(s, t, 0) = prefix?(s, t)**.

**subst** (*expression, equations*)  
**subst** ( $f, k = g$ ) formally replaces the kernel  $k$  by  $g$  in  $f$ .  
**subst** ( $f, [k_1 = g_1, \dots, k_n = g_n]$ ) formally replaces the kernels  $k_1, \dots, k_n$  by  $g_1, \dots, g_n$  in  $f$ .  
**subst** ( $f, [k_1, \dots, k_n], [g_1, \dots, g_n]$ ) formally replaces kernels  $k_i$  by  $g_i$  in  $f$ .

**suchThat** (*symbol, predicates*)  
**suchThat** ( $sy, pred$ ) attaches the predicate *pred* to symbol *sy*. Argument *pred* may also be a list  $[p_1, \dots, p_n]$  of predicates  $p_i$ . In this case, the predicate *pred* attached to *sy* is  $p_1$  **and**  $\dots$  **and**  $p_n$ .  
**suchThat** ( $r, [a_1, \dots, a_n], f$ ) returns the rewrite rule  $r$  with the predicate  $f(a_1, \dots, a_n)$  attached to it.

**suffix?** (*string, string*)  
**suffix?** ( $s, t$ ) tests if the string  $s$  is the final substring of  $t$ .

**sum** (*rationalFunction, symbolOrSegmentBinding*)  
**sum** ( $a(n), n$ ), where  $a(n)$  is an rational function or expression involving the symbol  $n$ , returns the indefinite sum  $A$  of  $a$  with respect to upward difference on  $n$ , that is,  $A(n+1) - A(n) = a(n)$ .  
**sum** ( $f(n), n = a..b$ ), where  $f(n)$ ,  $a$ , and  $b$  are rational functions (or polynomials), computes and returns the sum  $f(a) + f(a+1) + \dots + f(b)$  as a rational function (or polynomial).

**summation** (*expression, segmentBinding*)  
**summation** ( $f, n = a..b$ ) returns the formal sum  $\sum_{n=a}^b f(n)$ .

**sumOfDivisors** (*integer*)  
**sumOfDivisors** ( $n$ ) returns the sum of the integers between 1 and integer  $n$  (inclusive) which divide  $n$ . This sum is often denoted in the literature by  $\sigma(n)$ .

**sumOfKthPowerDivisors** (*integer, nonNegativeInteger*)  
**sumOfKthPowerDivisors** ( $n, k$ ) returns the sum of the  $k^{\text{th}}$  powers of the integers between 1 and  $n$  (inclusive)

which divide  $n$ . This sum is often denoted in the literature by  $\sigma_k(n)$ .

**sumSquares** (*integer*)

**sumSquares** ( $p$ ) returns the list  $[a, b]$  such that  $a^2 + b^2$  is equal to the integer prime  $p$ , and calls **error** if this is not possible. It will succeed if  $p$  is 2 or congruent to 1 **mod** 4.

**sup** (*element, element*)

**sup** ( $x, y$ ) returns the least element from which both  $x$  and  $y$  can be subtracted. The purpose of **sup** is to act as a supremum with respect to the partial order imposed by the  $-$  operation on the domain. See `OrderedAbelianMonoidSup` for details.

**super** (*outputForm, outputForm*)

**super** ( $o_1, o_2$ ), where  $o_1$  and  $o_2$  are objects of type `OutputForm` (normally unexposed), creates an output form for  $o_1$  superscripted by  $o_2$ .

**superscript** (*symbol, listOfOutputForms*)

**superscript** ( $s, [a_1, \dots, a_n]$ ) returns symbol  $s$  superscripted by output forms  $[a_1, \dots, a_n]$ .

**supersub** (*outputForm, listOfOutputForms*)

**supersub** ( $o, lo$ ), where  $o$  is an object of type `OutputForm` (normally unexposed) and  $lo$  is a list of output forms of the form  $[sub_1, super_1, sub_2, super_2, \dots, sub_n, super_n]$  creates an output form with each subscript aligned under each superscript.

**surface** (*function, function, function*)

**surface** ( $c_1, c_2, c_3$ ) creates a surface from three parametric component functions  $c_1$ ,  $c_2$ , and  $c_3$ .

**swap!** (*aggregate, index, index*)

**swap!** ( $u, i, j$ ) interchanges elements  $i$  and  $j$  of aggregate  $u$ . No meaningful value is returned.

**swapColumns!** (*matrix, integer, integer*)

**swapColumns!** ( $m, i, j$ ) interchanges the  $i^{\text{th}}$  and  $j^{\text{th}}$  columns of  $m$  returning  $m$  which is destructively altered.

**swapRows!** (*matrix, integer, integer*)

**swapRows!** ( $m, i, j$ ) interchanges the  $i^{\text{th}}$  and  $j^{\text{th}}$  rows of  $m$ , returning  $m$  which is destructively altered.

**symbol?** (*sExpression*)

**symbol?** ( $s$ ) tests if `SExpression` object  $s$  is a symbol.

**symbol** (*sExpression*)

**symbol** ( $s$ ) returns  $s$  as an element of type `Symbol`, or calls **error** if this is not possible.

**symmetric?** (*matrix*)

**symmetric?** ( $m$ ) tests if the matrix  $m$  is square and symmetric, that is, if each  $m(i, j) = m(j, i)$ .

**symmetricDifference** (*set, set*)

**symmetricDifference** ( $u, v$ ) returns the set aggregate of elements  $x$  which are members of set aggregate  $u$  or set aggregate  $v$  but not both. If  $u$  and  $v$  have no elements in common, **symmetricDifference** ( $u, v$ ) returns a copy of  $u$ . Note: **symmetricDifference** ( $u, v$ ) = **union**(**difference** ( $u, v$ ), **difference** ( $v, u$ ))

**symmetricGroup** (*integers*)

**symmetricGroup** ( $n$ ) constructs the symmetric group  $S_n$  acting on the integers  $1, \dots, n$ . The generators are the  $n$ -cycle  $(1, \dots, n)$  and the 2-cycle  $(1, 2)$ .

**symmetricGroup** ( $li$ ), where  $li$  is a list of integers, constructs the symmetric group acting on the integers in the list  $li$ . The generators are the cycle given by  $li$  and the 2-cycle  $(li(1), li(2))$ . Duplicates in the list will be removed.

**symmetricRemainder** (*integer, integer*)

**symmetricRemainder** ( $a, b$ ), where  $b > 1$ , yields  $r$  where  $-b/2 \leq r < b/2$ .

**symmetricTensors** (*matrices, positiveInteger*)

**symmetricTensors** ( $la, n$ ), where  $la$  is a list  $[a_1, \dots, a_k]$  of  $m$ -by- $m$  square matrices, applies to each matrix  $a_i$ , the irreducible, polynomial representation of the general linear group  $GL_m$  corresponding to the partition  $(n, 0, \dots, 0)$  of  $n$ .

**systemCommand** (*string*)

**systemCommand** ( $cmd$ ) takes the string  $cmd$  and passes it to the runtime environment for execution as a system command. Although various things may be printed, no usable value is returned.

**tableau** (*listOfListOfElements*)

**tableau** ( $ll$ ) converts a list of lists  $ll$  to an object of type `Tableau`.

**tableForDiscreteLogarithm** (*integer*)

**tableForDiscreteLogarithm** ( $n$ ) returns a table of the discrete logarithms of  $a^0$  up to  $a^{n-1}$  which, when called with the key **lookup** ( $a^i$ ), returns  $i$  for  $i$  in  $0..n-1$  for a finite field. This operation calls **error** if not called for prime divisors of order of multiplicative group.

**table** (*[listOfRecords]*)

**table** ( $[p_1, p_2, \dots, p_n]$ ) creates a table with keys of type `Key` and entries of type `Entry`. Each pair  $p_i$  is a record with selectors `key` and `entry` with values from the corresponding domains `Key` and `Entry`.

**table** ()\$ $T$  creates a empty table of domain  $T$  of category `TableAggregate`.

**tail** (*aggregate*)

**tail** (*a*) returns the last node of recursive aggregate *a*.

**tan** (*expression*)

**tanIfCan** (*expression*)

Argument *x* can be a Complex, Float, DoubleFloat, or Expression value or a series.

**tan** (*x*) returns the tangent of *x*.

**tanIfCan** (*x*) returns **tan** (*x*) if possible, and "failed" otherwise.

**tan2cot** (*expression*)

**tan2cot** (*f*) converts every **tan** (*u*) appearing in *f* into  $1/\cot(u)$ .

**tan2trig** (*expression*)

**tan2trig** (*f*) converts every **tan** (*u*) appearing in *f* into  $\sin(u)/\cos(u)$ .

**tanh** (*expression*)

**tanhIfCan** (*expression*)

Argument *x* can be a Complex, Float, DoubleFloat, or Expression value or a series.

**tanh** (*x*) returns the hyperbolic tangent of *x*.

**tanhIfCan** (*x*) returns **tanh** (*x*) if possible, and "failed" otherwise.

**tanh2coth** (*expression*)

**tanh2coth** (*f*) converts every **tanh** (*u*) appearing in *f* into  $1/\coth(u)$ .

**tanh2trigh** (*expression*)

**tanh2trigh** (*f*) converts every **tanh** (*u*) appearing in *f* into  $\sinh(u)/\cosh(u)$ .

**taylor** (*various, ..*)

**taylor** (*u*) converts the Laurent series *u*(*x*) to a Taylor series if possible, and if not, calls **error**.

**taylor** (*f*) converts the expression *f* into a Taylor expansion of the expression *f*. Note: *f* must have only one variable.

**taylor** (*sy*), where *sy* is a symbol, returns *sy* as a Taylor series.

**taylor** ( $n + - > a(n), x = a$ ) returns  $\sum_{n=0...} a(n)(x-a)^n$ .

**taylor** (*f*,  $x = a, [n]$ ) expands the expression *f* as a series in powers of  $(x-a)$  with *n* terms. If *n* is missing, the number of terms is governed by the value set by the system command **set** streams calculate.

**taylor** ( $i + - > a(i), x = a, m..[n, k]$ ) creates the Taylor series  $\sum_{i=m..n \text{ by } k} a(i)(x-a)^i$ . Here *m*, *n* and *k* are integers. Upper-limit *n* and stepsize *k* are optional and have default values *n* = ∞ and *k* = 1.

**taylor** (*a*(*i*), *i*,  $x = a, m..[n, k]$ ) returns  $\sum_{i=m..n \text{ by } k} a(n)(x-a)^n$ .

**taylorIfCan** (*laurentSeries*)

**taylorIfCan** (*f*(*x*)) converts the Laurent series *f*(*x*) to a Taylor series if possible, and returns "failed" if this is not possible.

**taylorRep** (*laurentSeries*)

**taylorRep** (*f*(*x*)) returns *g*(*x*), where  $f = x^n g(x)$  is represented by  $[n, g(x)]$ .

**tensorProduct** (*listOfMatrices* [, *listOfMatrices*])

**tensorProduct** ( $[a_1, \dots, a_k][, [b_1, \dots, b_k]]$ ) calculates the list of Kronecker products of the matrices *a<sub>i</sub>* and *b<sub>i</sub>* for  $1 \leq i \leq k$ . If a second argument is missing, the *b<sub>i</sub>* is defined as the corresponding *a<sub>i</sub>*. Also, **tensorProduct** (*m*), where *m* is a matrix, is defined as **tensorProduct** ( $[m], [m]$ ).

Note: If each list of matrices corresponds to a group representation (representation of generators) of one group, then these matrices correspond to the tensor product of the two representations.

**terms** (*various*)

**terms** (*s*) returns a stream of the non-zero terms of series *s*. Each term is returned as a record with selectors *k* and *c*, which correspond to the exponent and coefficient, respectively. The terms are ordered by increasing order of exponents.

**terms** (*m*), where *m* is a free abelian monoid of the form  $e_1 a_1 + \dots + e_n a_n$ , returns  $[[a_1, e_1], \dots, [a_n, e_n]]$ . See FreeAbelianMonoidCategory.

**tex** (*formattedObject*)

**tex** (*t*) extracts the TeX section of a TeX formatted object *t*.

**third** (*aggregate*)

**third** (*u*) returns the third element of a recursive aggregate *u*. Note: **third** (*u*) = *first*(*rest*(*rest*(*u*))).

**title** (*string*)

**title** (*s*) specifies string *s* as the title for a plot. This option is expressed as an option to the **draw** command in the form **title == s**.

**top** (*stack*)

**top!** (*dequeue*)

**top** (*s*) returns the top element *x* from *s*.

**top!** (*d*) returns the element at the top (front) of the dequeue.

**toroidal** (*value*)

**toroidal** (*element*) transforms from toroidal coordinates to Cartesian coordinates: **toroidal** (*a*) is a function that maps the point (*u*, *v*, *φ*) to  $x = a \sinh(v) \cos(\phi) / (\cosh(v) - \cos(u))$ ,  $y = a \sinh(v) \sin(\phi) / (\cosh(v) - \cos(u))$ ,  $z = a \sin(u) / (\cosh(v) - \cos(u))$ .

**toScale** (*boolean*)

**toScale** (*b*) specifies whether or not a plot is to be drawn to scale. This command may be expressed as an option to the **draw** command in the form *toScale* == *b*.

**totalDegree** (*polynomial*, *listOfVariables*)

**totalDegree** (*p*, *lv*) returns the maximum sum (over all monomials of polynomial *p*) of the variables in the list *lv*. If a second argument is missing, *lv* is defined to be all the variables appearing in *p*.

**totalfract** (*polynomial*)

**totalfract** (*prf*) takes a polynomial whose coefficients are themselves fractions of polynomials and returns a record containing the numerator and denominator resulting from putting *prf* over a common denominator.

**totalGroebner** (*listOfPolynomials*, *listOfVariables*)

**totalGroebner** (*lp*, *lv*) computes the Gröbner basis for the list of polynomials *lp* with the terms ordered first by total degree and then refined by reverse lexicographic ordering. The variables are ordered by their position in the list *lv*.

**tower** (*expression*)

**tower** (*f*) returns all the kernels appearing in *f*, regardless of level.

**trace** (*various*, ..)

**trace** (*m*) returns the trace of the matrix *m*, that is, the sum of its diagonal elements.

**trace** (*a*) returns the trace of the regular representation of *a*, an element of an algebra of finite rank. See **FiniteRankAlgebra**.

**trace** (*a*, *d*), where *a* is an element of a finite algebraic extension field, computes the trace of *a* with respect to the field of extension degree *d* over the ground field of size *q*. This operation calls **error** if *d* does not divide the extension degree of *a*. The default value of *d* is 1. Note:

$$\text{trace}(a, d) = \sum_{i=0}^{n/d} a^{q^{di}}.$$

**traceMatrix** ( [*basis*] )

**traceMatrix** (*[v1, ..., vn]*) is the *n*-by-*n* matrix whose *i*, *j* element is  $\text{Tr}(v_i v_j)$ . If no argument is given, the *v<sub>i</sub>* are assumed to be elements of the fixed basis.

**tracePowMod** (*poly*, *nonNegativeInteger*, *poly*)

**tracePowMod** (*u*, *k*, *v*) returns  $\sum_{i=0}^k u^{2^i}$ , all computed modulo the polynomial *v*.

**transcendenceDegree** ()

**transcendenceDegree** ()\$*F* returns the transcendence degree of the field extension *F*, or 0 if the extension is algebraic.

**transcendent?** (*element*)

**transcendent?** (*a*) tests whether an element *a* of a domain that is an extension field over a ground field *F* is transcendent with respect to *F*.

**transpose** (*matrix* [, *options*])

**transpose** (*m*) returns the transpose of the matrix *m*.

**transpose** (*t*, *i*, *j*) exchanges the *i*<sup>th</sup> and *j*<sup>th</sup> indices of *t*. For example, if *r* = **transpose**(*t*, 2, 3) for a rank four tensor *t*, then *r* is the rank four tensor given by  $r(i, j, k, l) = t(i, k, j, l)$ . If *i* and *j* are not given, they are assumed the first and last index of *t*.

**tree** (*value* [, *listOfChildren*])

**tree** (*x*, *ls*) creates an element of **Tree** with value *x* at the root node, and immediate children *ls* in left-to-right order.

**tree** (*x*) is equivalent to **tree** (*x*, []\$*List*(*S*)) where *x* has type *S*.

**trapezoidal** (*floatFunction*, *fourFloats*, *threeIntegers*)

**trapezoidalClosed** (*floatFunction*, *fourFloats*, *twoIntegers*)

**trapezoidalOpen** (*floatFunction*, *fourFloats*, *twoIntegers*)

**trapezoidal** (*fn*, *a*, *b*, *epsrel*, *epsabs*, *nmin*, *nmax*, *nint*) uses the adaptive trapezoidal method to numerically integrate function *fn* over the closed interval from *a* to *b*, with relative accuracy *epsrel* and absolute accuracy *epsabs*, where the refinement levels for the checking of convergence vary from *nmin* to *nmax*. The method is called “adaptive” since it requires an additional parameter *nint* giving the number of subintervals over which the integrator independently applies the convergence criteria using *nmin* and *nmax*; this is useful when a large number of points are needed only in a small fraction of the entire interval. Parameter *fn* is a function of type **Float** → **Float**; *a*, *b*, *epsrel*, and *epsabs* are floats; *nmin*, *nmax*, and *nint* are integers. The operation returns a record containing: **value**, an estimate of the integral; **error**, an estimate of the error in the computation; **totalpts**, the total integral number of function evaluations, and **success**, a boolean value that is *true* if the integral was computed within the user specified error criterion. See **NumericalQuadrature** for details.

**trapezoidalClosed** (*fn*, *a*, *b*, *epsrel*, *epsabs*, *nmin*, *nmax*)

similarly uses the trapezoidal method to numerically integrate function *fn* over the closed interval *a* to *b*, but is not adaptive.

**trapezoidalOpen** (*fn*, *a*, *b*, *epsrel*, *epsabs*, *nmin*, *nmax*)

is similar to **trapezoidalClosed**, except that it integrates function *fn* over the open interval from *a* to *b*.

**triangularSystems** (*listOfFractions*, *listOfSymbols*)

**triangularSystems** (*lf*, *lv*) solves the system of equations defined by *lf* with respect to the list of symbols *lv*; the system of equations is obtaining by equating to zero the

list of rational functions  $lf$ . The result is a list of solutions where each solution is expressed as a “reduced” triangular system of polynomials.

**trigs** (*expression*)

**trigs** ( $f$ ) rewrites all the complex logs and exponentials appearing in  $f$  in terms of trigonometric functions.

**trim** (*string*, *characterOrCharacterClass*)

**trim** ( $s, c$ ) returns  $s$  with all characters  $c$  deleted from right and left ends. For example, **trim**(" abc ", char " ") returns "abc". Argument  $c$  may also be a character class, in which case  $s$  is returned with all characters in  $cc$  deleted from right and left ends. For example, **trim**("abc", charClass "()") returns "abc".

**truncate** (*various* [, *options*])

**truncate** ( $x$ ) returns the integer between  $x$  and 0 closest to  $x$ .

**truncate** ( $f, m$  [,  $n$ ]) returns a (finite) power series consisting of the sum of all terms of  $f$  of degree  $d$  with  $n \leq d \leq m$ . Upper bound  $m$  is  $\infty$  by default.

**tubePoints** (*positiveInteger*)

**tubePoints** ( $n$ ) specifies the number of points,  $n$ , defining the circle that creates the tube around a three-dimensional curve. The default is 6. This option is expressed in the form **tubePoints** ==  $n$ .

**tubePointsDefault** ([*positiveInteger*])

**tubePointsDefault** ( $i$ ) sets the number of points to use when creating the circle to be used in creating a three-dimensional tube plot to  $i$ .

**tubePointsDefault** () returns the number of points to be used when creating the circle to be used in creating a three-dimensional tube plot.

**tubeRadius** (*float*)

**tubeRadius** ( $r$ ) specifies a radius  $r$  for a tube plot around a three-dimensional curve. This operation may be expressed as an option to the **draw** command in the form **tubeRadius** ==  $r$ .

**tubeRadiusDefault** ([*float*])

**tubeRadiusDefault** ( $r$ ) sets the default radius for a three-dimensional tube plot to  $r$ .

**tubeRadiusDefault** () returns the radius used for a three-dimensional tube plot.

**twist** ()

**twist** ( $f$ ), where  $f$  is a function of type  $(A, B)C$ , is the function  $g$  such that  $g(a, b) = f(b, a)$ . See MappingPackage for related functions.

**unary?** (*basicOperator*)

**unary?** ( $op$ ) tests if basic operator  $op$  is unary, that is,

takes exactly one argument.

**union** (*set*, *elementOrSet*)

**union** ( $u, x$ ) returns the set aggregate  $u$  with the element  $x$  added. If  $u$  already contains  $x$ , **union** ( $u, x$ ) returns a copy of  $x$ .

**union** ( $u, v$ ) returns the set aggregate of elements that are members of either set aggregate  $u$  or  $v$ . See also Multiset.

**unit** ([*various*])

**unit** () returns a unit of the algebra (necessarily unique), or "failed" if there is none.

**unit** ( $u$ ) extracts the unit part of the factored object  $u$ .

**unit** ( $l$ ) marks off the units on a viewport according to the indicated list  $l$ . This option is expressed in the draw command in the form **unit** == [ $f_1, f_2$ ].

**unit?** (*element*)

**unit?** ( $x$ ) tests whether  $x$  is a unit, that is, if  $x$  is invertible.

**unitCanonical** (*element*)

**unitCanonical** ( $x$ ) returns **unitNormal** ( $x$ ).canonical.

**unitNormalize** (*factored*)

**unitNormalize** ( $u$ ) normalizes the unit part of the factorization. For example, when working with factored integers, this operation ensures that the bases are all positive integers.

**unitNormal** (*element*)

**unitNormal** ( $x$ ) tries to choose a canonical element from the associate class of  $x$ . If successful, it returns a record with three components "unit", "canonical" and "associate". The attribute **canonicalUnitNormal**, if asserted, means that the "canonical" element is the same across all associates of  $x$ . If **unitNormal** ( $x$ ) = [ $u, c, a$ ] then  $ux = c$ ,  $au = 1$ .

**unitsColorDefault** ([*palette*])

**unitsColorDefault** ( $p$ ) sets the default color of the unit ticks in a two-dimensional viewport to the palette  $p$ .

**unitsColorDefault** () returns the default color of the unit ticks in a two-dimensional viewport.

**unitVector** (*positiveInteger*)

**unitVector** ( $n$ ) produces a vector with 1 in position  $n$  and zero elsewhere.

**univariate** (*polynomial* [, *variable*])

**univariate** ( $p$  [,  $v$ ]) converts the multivariate polynomial  $p$  into a univariate polynomial in  $v$  whose coefficients are multivariate polynomials in all the other variables. If  $v$  is omitted, then  $p$  must involve exactly one variable.

**universe** ()

**universe** ()\$ $R$  returns the universal set for finite set

aggregate *R*.

**unparse** (*inputForm*)

**unparse** (*f*) returns a string *s* such that the parser would transform *s* to *f*, or calls **error** if *f* is not the parsed form of a string.

**unrankImproperPartitions0** (*integer, integer, integer*)

**unrankImproperPartitions0** (*n, m, k*) computes the *k*<sup>th</sup> improper partition of nonnegative *n* in *m* nonnegative parts in reverse lexicographical order. Example:  $[0, 0, 3] < [0, 1, 2] < [0, 2, 1] < [0, 3, 0] < [1, 0, 2] < [1, 1, 1] < [1, 2, 0] < [2, 0, 1] < [2, 1, 0] < [3, 0, 0]$ . The operation calls **error** if *k* is negative or too big. Note: counting of subtrees is done by **numberOfImproperPartitions**.

**unrankImproperPartitions1** (*integer, integer, integer*)

**unrankImproperPartitions1** (*n, m, k*) computes the *k*<sup>th</sup> improper partition of nonnegative *n* in at most *m* nonnegative parts ordered as follows: first, in reverse lexicographical order according to their non-zero parts, then according to their positions (i.e. lexicographical order using *subSet*:  $[3, 0, 0] < [0, 3, 0] < [0, 0, 3] < [2, 1, 0] < [2, 0, 1] < [0, 2, 1] < [1, 2, 0] < [1, 0, 2] < [0, 1, 2] < [1, 1, 1]$ ). Note: counting of subtrees is done by **numberOfImproperPartitionsInternal**.

**unravel** (*listOfElement*)

**unravel** (*t*) produces a tensor from a list of components such that **unravel** (**ravel**(*t*)) = *t*.

**upperCase** (*string*)

**upperCase?** (*string*)

**upperCase!** (*string*)

**upperCase!** (*s*) destructively replaces the alphabetic characters in *s* by upper case characters.

**upperCase** () returns the class of all characters for which **upperCase?** is *true*.

**upperCase** (*c*) converts a lower case letter *c* to the corresponding upper case letter. If *c* is not a lower case letter, then it is returned unchanged.

**upperCase** (*s*) returns the string with all characters in upper case.

**upperCase?** (*c*) tests if *c* is an upper case letter, that is, one of A...Z.

**validExponential** (*listOfKernels, expression, symbol*)

**validExponential** ( $[k_1, \dots, k_n], f, x$ ) returns *g* if **exp** (*f*) = *g* and *g* involves only  $k_1 \dots k_n$ , and "failed" otherwise.

**value** (*recursiveAggregate*)

**value** (*a*) returns the "value" part of a recursive aggregate *a*, typically the root of tree. See, for example, **BinaryTree**.

**var1Steps** (*positiveInteger*)

**var1Steps** (*n*) indicates the number of subdivisions *n* of the first range variable. This command may be expressed as an option to the **draw** command in the form **var1Steps** == *n*.

**var1StepsDefault** ([*positiveInteger*])

**var1StepsDefault** () returns the current setting for the number of steps to take when creating a three-dimensional mesh in the direction of the first defined free variable (a free variable is considered defined when its range is specified (that is,  $x = 0..10$ )).

**var1StepsDefault** (*i*) sets the number of steps to take when creating a three-dimensional mesh in the direction of the first defined free variable to *i* (a free variable is considered defined when its range is specified (that is,  $x = 0..10$ )).

**var2Steps** (*positiveInteger*)

**var2Steps** (*n*) indicates the number of subdivisions, *n*, of the second range variable. This option is expressed in the form **var2Steps** == *n*.

**var2StepsDefault** ([*positiveInteger*])

**variable** (*various*)

**variable** (*f*) returns the (unique) power series variable of the power series *f*.

**variable** (*segb*) returns the variable from the left hand side of the SegmentBinding *segb*. For example, if *segb* is  $v = a..b$ , then **variable** (*segb*) returns *v*.

**variable** (*v*) returns *s* if *v* is any derivative of the differential indeterminate *s*.

**variables** (*expression*)

**variables** (*f*) returns the list of all the variables of expression, polynomial, rational function, or power series *f*.

**vconcat** (*outputForms* [, *OutputForm*] (*normally unexposed*))

**vconcat** ( $o_1, o_2$ ), where  $o_1$  and  $o_2$  are objects of type *OutputForm* (normally unexposed), returns an output form for the vertical concatenation of forms  $o_1$  and  $o_2$ .

**vconcat** (*lo*), where *lo* is a list of objects of type *OutputForm* (normally unexposed), returns an output form for the vertical concatenation of the elements of *lo*.

**vector** (*listOfElements*)

**vector** (*l*) converts the list *l* to a vector.

**vectorise** (*polynomial, nonNegativeInteger*)

**vectorise** (*p, n*) returns  $[a_0, \dots, a_{n-1}]$  where  $p = a_0 + a_1x + \dots + a_{n-1}x^{n-1} + \text{higher order terms}$ . The degree of polynomial *p* can be different from *n* - 1.

**vertConcat** (*matrix*, *matrix*)

**vertConcat** ( $x, y$ ) vertically concatenates two matrices with an equal number of columns. The entries of  $y$  appear below the entries of  $x$ .

**viewDefaults** ()

**viewDefaults** () resets all the default graphics settings.

**viewPosDefault** ([*listOfNonNegativeIntegers*])

**viewPosDefault** ( $[x, y]$ ) sets the default  $X$  and  $Y$  position of a viewport window. Unless overridden explicitly, newly created viewports will have the  $X$  and  $Y$  coordinates  $x, y$ .

**viewPosDefault** () returns the default  $X$  and  $Y$  position of a viewport window unless overridden explicitly, newly created viewports will have these  $X$  and  $Y$  coordinate.

**viewSizeDefault** ([*listOfPositiveIntegers*])

**viewSizeDefault** ( $[w, h]$ ) sets the default viewport width to  $w$  and height to  $h$ .

**viewWriteAvailable** ()

**viewWriteAvailable** () returns a list of available methods for writing, such as BITMAP, POSTSCRIPT, etc.

**viewWriteDefault** (*listOfStrings*)

**viewWriteDefault** () returns the list of things to write in a viewport data file; a viewAlone file is always generated.

**viewWriteDefault** ( $l$ ) sets the default list of things to write in a viewport data file to the strings in  $l$ ; a viewAlone file is always generated.

**void** ()

**void** () produces a void object.

**weakBiRank** (*element*)

**weakBiRank** ( $x$ ) determines the number of linearly independent elements in the  $b_i x b_j$ ,  $i, j = 1, \dots, n$ , where  $b = [b_1, \dots, b_n]$  is the fixed basis of a domain of category FramedNonAssociativeAlgebra.

**weight** ( $u$ )

**weight** ( $u$ ) returns

if  $u$  is a differential polynomial: the maximum weight of all differential monomials appearing in the differential polynomial  $u$ .

if  $u$  is a derivative: the weight of the derivative  $u$ .

if  $u$  is a basic operator: the weight attached to  $u$ .

**weight** ( $p, s$ ) returns the maximum weight of all differential monomials appearing in the differential polynomial  $p$  when  $p$  is viewed as a differential polynomial in the differential indeterminate  $s$  alone.

**weight** ( $op, n$ ) attaches the weight  $n$  to  $op$ .

**weights** (*differentialPolynomial*,

*differentialIndeterminated*)

**weights** ( $p, s$ ) returns a list of weights of differential monomials appearing in the differential polynomial  $p$  when  $p$  is viewed as a differential polynomial in the differential indeterminate  $s$  alone. If  $s$  is missing, a list of weights of differential monomials appearing in differential polynomial  $p$ .

**whatInfinity** (*orderedCompletion*)

**whatInfinity** ( $x$ ) returns 0 if  $x$  is finite, 1 if  $x$  is  $\infty$ , and  $-1$  if  $x$  is  $-\infty$ .

**wholePart** (*various*)

**wholePart** ( $x$ ) returns the whole part of the fraction  $x$ , that is, the truncated quotient of the numerator by the denominator.

**wholePart** ( $x$ ) extracts the whole part of  $x$ . That is, if  $x = \text{continuedFraction}(b_0, [a_1, a_2, \dots], [b_1, b_2, \dots])$ , then **wholePart** ( $x$ ) =  $b_0$ .

**wholePart** ( $p$ ) extracts the whole part of the partial fraction  $p$ .

**wholeRadix** (*listOfIntegers*)

**wholeRadix** ( $l$ ) creates an integral radix expansion from a list of ragits. For example, **wholeRadix** ( $[1, 3, 4]$ ) returns 134.

**wholeRagits** (*listOfIntegers*)

**wholeRagits** ( $rx$ ) returns the ragits of the integer part of a radix expansion.

**wordInGenerators** (*permutation*, *permutationGroup*)

**wordInGenerators** ( $p, gp$ ) returns the word for the permutation  $p$  in the original generators of the permutation group  $gp$ , represented by the indices of the list, given by **generators**.

**wordInStrongGenerators** (*permutation*, *permutationGroup*)

**wordInStrongGenerators** ( $p, gp$ ) returns the word for the permutation  $p$  in the strong generators of the permutation group  $gp$ , represented by the indices of the list, given by **strongGenerators**.

**wordsForStrongGenerators** (*listOfListsOfIntegers*)

**wordsForStrongGenerators** ( $gp$ ) returns the words for the strong generators of the permutation group  $gp$  in the original generators of  $gp$ , represented by their indices in the list of nonnegative integers, given by **generators**.

**wreath** (*symmetricPolynomial*, *symmetricPolynomial*)

**wreath** ( $s_1, s_2$ ) is the cycle index of the wreath product of the two groups whose cycle indices are  $s_1$  and  $s_2$ , symmetric polynomials with rational number coefficients.



**writable?** (*file*)

**writable?** (*f*) tests if the named file can be opened for writing. The named file need not already exist.

**write!** (*file*, *value*)

**write!** (*f*, *s*) puts the value *s* into the file *f*. The state of *f* is modified so that subsequent calls to **write!** will append values to the end of the file.

**writeLine!** (*textfile* [, *string*])

**writeLine!** (*f*) finishes the current line in the file *f*. An empty string is returned. The call **writeLine!** (*f*) is equivalent to **writeLine!** (*f*, "").

**writeLine!** (*f*, *s*) writes the contents of the string *s* and finishes the current line in the file *f*. The value of *s* is returned.

**xor** (*boolean*, *boolean*)

**xor** (*a*, *b*) returns the logical *exclusive-or* of booleans or bit aggregates *a* and *b*.

**xor** (*n*, *m*) returns the bit-by-bit logical *xor* of the small integers *n* and *m*.

**xRange** (*curve*)

**xRange** (*c*) returns the range of the *x*-coordinates of the points on the curve *c*.

**yCoordinates** (*function*)

**yCoordinates** (*f*), where *f* is a function defined over a curve, returns the coordinates of *f* with respect to the natural basis for the curve. Specifically, the operation returns  $[[a_1, \dots, a_n], d]$  such that  $f = (a_1 + \dots + a_n y^{n-1})/d$ .

**yellow** ()

**yellow** () returns the position of the yellow hue from total hues.

**youngGroup** (*various*)

**youngGroup** ( $[n_1, \dots, n_k]$ ) constructs the direct product of the symmetric groups  $S_{n_1}, \dots, S_{n_k}$ .

**youngGroup** (*lambda*) constructs the direct product of the symmetric groups given by the parts of the partition *lambda*.

**yRange** (*curve*)

**yRange** (*c*) returns the range of the *y*-coordinates of the points on the curve *c*.

**zag** (*outputForm*, *outputForm*)

**zag** ( $o_1, o_2$ ), where  $o_1$  and  $o_2$  are objects of type *OutputForm* (normally *unexposed*), return an output form displaying the continued fraction form for  $o_2$  over  $o_1$ .

**zero** (*nonNegativeInteger* [, *nonNegativeInteger*])

**zero** (*n*) creates a zero vector of length *n*.

**zero** (*m*, *n*) returns an *m*-by-*n* zero matrix.

**zero?** (*element*)

**zero?** (*x*) tests if *x* is equal to 0.

**zeroDim?** (*ideal*)

**zeroDim?** (*I*) tests if the ideal *I* is zero dimensional, that is, all its associated primes are maximal.

**zeroDimPrimary?** (*ideal*)

**zeroDimPrimary?** (*I*) tests if the ideal *I* is 0-dimensional primary.

**zeroDimPrime?** (*ideal*)

**zeroDimPrime?** (*I*) tests if the ideal *I* is a 0-dimensional prime.

**zeroOf** (*polynomial* [, *symbol*])

**zeroOf** (*p* [, *y*]) returns *y* such that  $p(y) = 0$ . If possible, *y* is expressed in terms of radicals. Otherwise it is an implicit algebraic quantity that displays as '*y*'. If no second argument is given, then *p* must have a unique variable *y*.

**zerosOf** (*polynomial* [, *symbol*])

**zerosOf** (*p*, *y*) returns  $[y_1, \dots, y_n]$  such that  $p(y_i) = 0$ . The  $y_i$ 's are expressed in radicals if possible. Otherwise they are implicit algebraic quantities that display as  $y_i$ . The returned symbols  $y_1, \dots, y_n$  are bound in the interpreter to respective root values. If no second argument is given, then *p* must have a unique variable *y*.

**zRange** (*curve*)

**zRange** (*c*) returns the range of the *z*-coordinates of the points on the curve *c*.



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## APPENDIX F

---

# Programs for AXIOM Images

This appendix contains the AXIOM programs used to generate the images in the AXIOM Images color insert of this book. All these input files are included with the AXIOM system. To produce the images on page 6 of the AXIOM Images insert, for example, issue the command:

```
)read images6
```

These images were produced on an IBM RS/6000 model 530 with a standard color graphics adapter. The smooth shaded images were made from X Window System screen dumps. The remaining images were produced with AXIOM-generated PostScript output. The images were reproduced from slides made on an Agfa ChromaScript PostScript interpreter with a Matrix Instruments QCR camera.

### F.1 images1.input

---

Read torus knot program.

```
)read tknot
```

1

A (15,17) torus knot.

```
torusKnot(15,17, 0.1, 6, 700)
```

2

3

## F.2 images2.input

Read the programs from  
Chapter 10.  
Create a Newton's iteration  
function for  $x^3 = 2$ .

These images illustrate how Newton's method converges when computing the complex cube roots of 2. Each point in the  $(x, y)$ -plane represents the complex number  $x + iy$ , which is given as a starting point for Newton's method. The poles in these images represent bad starting values. The flat areas are the regions of convergence to the three roots.

```
)read newton 1
)read vectors 2
f := newtonStep(x**3 - 2) 3
4
```

The function  $f^n$  computes  $n$  steps of Newton's method.

Clip values with magnitude  $\leq 4$ .  
The vector field for  $f^3$   
The surface for  $f^3$   
The surface for  $f^4$

```
clipValue := 4 5
drawComplexVectorField(f**3, -3..3, -3..3) 6
drawComplex(f**3, -3..3, -3..3) 7
drawComplex(f**4, -3..3, -3..3) 8
```

## F.3 images3.input

```
)r tknot 1
for i in 0..4 repeat torusKnot(2, 2 + i/4, 0.5, 25, 250) 2
```

## F.4 images5.input

The parameterization of the Etruscan Venus is due to George Frances.

```
venus(a,r,steps) == 1
surf := (u:DFLOAT, v:DFLOAT): Point DFLOAT +-> 2
cv := cos(v) 3
sv := sin(v) 4
cu := cos(u) 5
su := sin(u) 6
x := r * cos(2*u) * cv + sv * cu 7
y := r * sin(2*u) * cv - sv * su 8
z := a * cv 9
point [x,y,z] 10
draw(surf, 0..%pi, -%pi..%pi, var1Steps==steps, 11
var2Steps==steps, title == "Etruscan Venus") 12
13
14
The Etruscan Venus venus(5/2, 13/10, 50)
```

The Figure-8 Klein Bottle parameterization is from "Differential Geometry and Computer Graphics" by Thomas Banchoff, in *Perspectives in Mathematics*, Anniversary of Oberwolfach 1984, Birkhäuser-Verlag, Basel, pp. 43-60.

```
klein(x,y) == 15
cx := cos(x) 16
cy := cos(y) 17
sx := sin(x) 18
sy := sin(y) 19
sx2 := sin(x/2) 20
cx2 := cos(x/2) 21
sq2 := sqrt(2.0@DFLOAT) 22
point [cx * (cx2 * (sq2 + cy) + (sx2 * sy * cy)), - 23
```

	$\begin{aligned} & sx * (cx2 * (sq2 + cy) + (sx2 * sy * cy)), - \\ & -sx2 * (sq2 + cy) + cx2 * sy * cy \end{aligned}$	24 25 26
<b>Figure-8 Klein bottle</b>	<code>draw(klein, 0..4*%pi, 0..2*%pi, var1Steps==50, var2Steps==50,title=="Figure Eight Klein Bottle")</code>	27 28
	The next two images are examples of generalized tubes.	
Rotate a point $p$ by $\theta$ around the origin.	<code>)read ntube rotateBy(p, theta) == c := cos(theta) s := sin(theta) point [p.1*c - p.2*s, p.1*s + p.2*c]</code>	29 30 31 32 33 34
A circle in three-space.	<code>bcircle t == point [3*cos t, 3*sin t, 0]</code>	35 36 37
An ellipse that twists around four times as $t$ revolves once.	<code>twist(u, t) == theta := 4*t p := point [sin u, cos(u)/2] rotateBy(p, theta)</code>	38 39 40 41 42
<b>Twisted Torus</b>	<code>ntubeDrawOpt(bcicle, twist, 0..2*%pi, 0..2*%pi, var1Steps == 70, var2Steps == 250)</code>	43 44 45
Create a twisting circle.	<code>twist2(u, t) == theta := t p := point [sin u, cos(u)] rotateBy(p, theta)</code>	46 47 48 49 50
Color function with 21 stripes.	<code>cf(u,v) == sin(21*u)</code>	51 52
<b>Striped Torus</b>	<code>ntubeDrawOpt(bcicle, twist2, 0..2*%pi, 0..2*%pi, colorFunction == cf, var1Steps == 168, var2Steps == 126)</code>	53 54 55

## F.5 images6.input

The height and color are the real and argument parts of the Gamma function, respectively.	<code>gam(x,y) == g := Gamma complex(x,y) point [x,y,max(min(real g, 4), -4), argument g]</code>	1 2 3 4 5
<b>The Gamma Function</b>	<code>draw(gam, -%pi..%pi, -%pi..%pi, title == "Gamma(x + %i*y)", - var1Steps == 100, var2Steps == 100)</code>	6 7 8 9
	<code>b(x,y) == Beta(x,y)</code>	10
<b>The Beta Function</b>	<code>draw(b, -3.1..3, -3.1 .. 3, title == "Beta(x,y)")</code>	11 12
	<code>atf(x,y) == a := atan complex(x,y)</code>	13 14 15

	point [x,y,real a, argument a]	16
		17
The Arctangent function	draw(atf, -3.0..%pi, -3.0..%pi)	18

## F.6 images7.input

Read program for drawing conformal maps.	)read conformal	1
		2
		3
The coordinate grid for the complex plane.	f z == z	4
Mapping 1: Source	conformalDraw(f, -2..2, -2..2, 9, 9, "cartesian")	5
		6
The map $z \mapsto z + 1/z$	f z == z + 1/z	7
		8
Mapping 1: Target	conformalDraw(f, -2..2, -2..2, 9, 9, "cartesian")	9
		10
	The map $z \mapsto -(z+1)/(z-1)$ maps the unit disk to the right half-plane, as shown on the Riemann sphere.	
The unit disk.	f z == z	11
		12
Mapping 2: Source	riemannConformalDraw(f,0.1..0.99,0..2*%pi,7,11,"polar")	13
		14
The map $x \mapsto -(z+1)/(z-1)$ .	f z == -(z+1)/(z-1)	15
		16
Mapping 2: Target	riemannConformalDraw(f,0.1..0.99,0..2*%pi,7,11,"polar")	17
		18
Riemann Sphere Mapping	riemannSphereDraw(-4..4, -4..4, 7, 7, "cartesian")	19

## F.7 images8.input

Sierpinsky's Tetrahedron	)read dhtri	1
	)read tetra	2
	drawPyramid 4	3
		4
	)read antoine	5
Antoine's Necklace	drawRings 2	6
		7
	)read scherk	8
Scherk's Minimal Surface	drawScherk(3,3)	9
		10
	)read ribbonsNew	11
Ribbon Plot	drawRibbons([x**i for i in 1..5], x=-1..1, y=0..2)	12

## F.8 conformal.input

Complex Numbers  
Draw ranges  
Points in 3-space

The functions in this section draw conformal maps both on the plane and on the Riemann sphere.

```
C := Complex DoubleFloat 1
S := Segment DoubleFloat 2
R3 := Point DFLOAT 3
4
```

**conformalDraw**(*f*, *rRange*, *tRange*, *rSteps*, *tSteps*, *coord*) draws the image of the coordinate grid under *f* in the complex plane. The grid may be given in either polar or Cartesian coordinates. Argument *f* is the function to draw; *rRange* is the range of the radius (in polar) or real (in Cartesian); *tRange* is the range of  $\theta$  (in polar) or imaginary (in Cartesian); *tSteps*, *rSteps*, are the number of intervals in the *r* and  $\theta$  directions; and *coord* is the coordinate system to use (either "polar" or "cartesian").

```
conformalDraw: (C -> C, S, S, PI, PI, String) -> VIEW3D 5
conformalDraw(f,rRange,tRange,rSteps,tSteps,coord) == 6
transformC := 7
coord = "polar" => polar2Complex 8
cartesian2Complex 9
cm := makeConformalMap(f, transformC) 10
sp := createThreeSpace() 11
adaptGrid(sp, cm, rRange, tRange, rSteps, tSteps) 12
makeViewport3D(sp, "Conformal Map") 13
```

Function for changing an (*x*, *y*)  
pair into a complex number.

Create a fresh space.  
Plot the coordinate lines.  
Draw the image.

**riemannConformalDraw**(*f*, *rRange*, *tRange*, *rSteps*, *tSteps*, *coord*) draws the image of the coordinate grid under *f* on the Riemann sphere. The grid may be given in either polar or Cartesian coordinates. Its arguments are the same as those for **conformalDraw**.

```
riemannConformalDraw: (C->C,S,S,PI,PI,String)->VIEW3D 14
riemannConformalDraw(f, rRange, tRange, 15
rSteps, tSteps, coord) == 16
transformC := 17
coord = "polar" => polar2Complex 18
cartesian2Complex 19
sp := createThreeSpace() 20
cm := makeRiemannConformalMap(f, transformC) 21
adaptGrid(sp, cm, rRange, tRange, rSteps, tSteps) 22
curve(sp, [point [0,0,2.0@DFLOAT,0],point [0,0,2.0@DFLOAT,0]]) 23
makeViewport3D(sp,"Map on the Riemann Sphere") 24
25
adaptGrid(sp, f, uRange, vRange, uSteps, vSteps) == 26
delU := (hi(uRange) - lo(uRange))/uSteps 27
delV := (hi(vRange) - lo(vRange))/vSteps 28
uSteps := uSteps + 1; vSteps := vSteps + 1 29
u := lo uRange 30
for i in 1..uSteps repeat 31
c := curryLeft(f,u) 32
cf := (t:DFLOAT):DFLOAT +-> 0 33
makeObject(c,vRange::SEG Float,colorFunction==cf, 34
space == sp, tubeRadius == .02, tubePoints == 6) 35
```

Function for changing an (*x*, *y*)  
pair into a complex number.

Create a fresh space.

Plot the coordinate lines.  
Add an invisible point at  
the north pole for scaling.

Plot the coordinate grid  
using adaptive plotting for  
coordinate lines, and draw  
tubes around the lines.

Draw coordinate lines in the *v*  
direction; curve *c* fixes the  
current value of *u*.  
Draw the *v* coordinate line.

	u := u + delU	36
	v := lo vRange	37
Draw coordinate lines in the u	for i in 1..vSteps repeat	38
direction; curve c fixes the	c := curryRight(f,v)	39
current value of v.	cf := (t:DFLOAT):DFLOAT +-> 1	40
Draw the u coordinate line.	makeObject(c,uRange::SEG Float,colorFunction==cf,	41
	space == sp, tubeRadius == .02, tubePoints == 6)	42
	v := v + delV	43
	void()	44
		45
Map a point in the complex	riemannTransform(z) ==	46
plane to the Riemann sphere.	r := sqrt norm z	47
	cosTheta := (real z)/r	48
	sinTheta := (imag z)/r	49
	cp := 4*r/(4+r**2)	50
	sp := sqrt(1-cp*cp)	51
	if r>2 then sp := -sp	52
	point [cosTheta*cp, sinTheta*cp, -sp + 1]	53
		54
Convert Cartesian coordinates	cartesian2Complex(r:DFLOAT, i:DFLOAT):C ==	55
to		
complex Cartesian form.	complex(r, i)	56
		57
Convert polar coordinates to	polar2Complex(r:DFLOAT, th:DFLOAT):C ==	58
complex Cartesian form.	complex(r*cos(th), r*sin(th))	59
		60
Convert complex function f	makeConformalMap(f, transformC) ==	61
to a mapping:	(u:DFLOAT,v:DFLOAT):R3 +->	62
(DFLOAT,DFLOAT) $\mapsto$ R3		
in the complex plane.	z := f transformC(u, v)	63
	point [real z, imag z, 0.0@DFLOAT]	64
		65
Convert a complex function f	makeRiemannConformalMap(f, transformC) ==	66
to a mapping:	(u:DFLOAT, v:DFLOAT):R3 +->	67
(DFLOAT,DFLOAT) $\mapsto$ R3		
on the Riemann sphere.	riemannTransform f transformC(u, v)	68
		69
Draw a picture of the mapping	riemannSphereDraw: (S, S, PI, PI, String) -> VIEW3D	70
of the complex plane to	riemannSphereDraw(rRange,tRange,rSteps,tSteps,coord) ==	71
the Riemann sphere.	transformC :=	72
	coord = "polar" => polar2Complex	73
	cartesian2Complex	74
Coordinate grid function.	grid := (u:DFLOAT, v:DFLOAT): R3 +->	75
	z1 := transformC(u, v)	76
	point [real z1, imag z1, 0]	77
Create a fresh space.	sp := createThreeSpace()	78
Draw the flat grid.	adaptGrid(sp, grid, rRange, tRange, rSteps, tSteps)	79
	connectingLines(sp,grid,rRange,tRange,rSteps,tSteps)	80
Draw the sphere.	makeObject(riemannSphere,0..2*%pi,0..%pi,space==sp)	81
	f := (z:C):C +-> z	82
	cm := makeRiemannConformalMap(f, transformC)	83
Draw the sphere grid.	adaptGrid(sp, cm, rRange, tRange, rSteps, tSteps)	84
	makeViewPort3D(sp, "Riemann Sphere")	85
		86
Draw the lines that connect	connectingLines(sp,f,uRange,vRange,uSteps,vSteps) ==	87
the points in the complex	delU := (hi(uRange) - lo(uRange))/uSteps	88



plane to the north pole of the Riemann sphere.	delV := (hi(vRange) - lo(vRange))/vSteps	89
	uSteps := uSteps + 1; vSteps := vSteps + 1	90
	u := lo uRange	91
For each u.	for i in 1..uSteps repeat	92
	v := lo vRange	93
For each v.	for j in 1..vSteps repeat	94
	p1 := f(u,v)	95
Project p1 onto the sphere.	p2 := riemannTransform complex(p1.1, p1.2)	96
Create a line function.	fun := lineFromTo(p1,p2)	97
	cf := (t:DFLOAT):DFLOAT +-> 3	98
Draw the connecting line.	makeObject(fun, 0..1,space==sp,tubePoints==4,	99
	tubeRadius==0.01,colorFunction==cf)	100
	v := v + delV	101
	u := u + delU	102
	void()	103
		104
A sphere sitting on the complex plane, with radius 1.	riemannSphere(u,v) ==	105
	sv := sin(v)	106
	0.99@DFLOAT*(point [cos(u)*sv,sin(u)*sv,cos(v),0.0@DFLOAT])+	107
	point [0.0@DFLOAT, 0.0@DFLOAT, 1.0@DFLOAT, 4.0@DFLOAT]	108
		109
Create a line function that goes from p1 to p2	lineFromTo(p1, p2) ==	110
	d := p2 - p1	111
	(t:DFLOAT):Point DFLOAT +->	112
	p1 + t*d	113

## F.9 tknot.input

	Create a $(p, q)$ torus-knot with radius $r$ around the curve. The formula was derived by Larry Lambe.	
	)read ntube	1
	torusKnot: (DFLOAT, DFLOAT, DFLOAT, PI, PI) -> VIEW3D	2
	torusKnot(p, q ,r, uSteps, tSteps) ==	3
Function for the torus knot.	knot := (t:DFLOAT):Point DFLOAT +->	4
	fac := 4/(2.2@DFLOAT-sin(q*t))	5
	fac * point [cos(p*t), sin(p*t), cos(q*t)]	6
The cross section.	circle := (u:DFLOAT, t:DFLOAT): Point DFLOAT +->	7
	r * point [cos u, sin u]	8
Draw the circle around the knot.	ntubeDrawOpt(knot, circle, 0..2*%pi, 0..2*%pi,	9
	var1Steps == uSteps, var2Steps == tSteps)	10
		11

## F.10 ntube.input

	The functions in this file create generalized tubes (also known as gener- alized cylinders). These functions draw a 2-d curve in the normal planes around a 3-d curve.	
Points in 3-Space	R3 := Point DFLOAT	1
Points in 2-Space	R2 := Point DFLOAT	2
Draw ranges	S := Segment Float	3
Introduce types for functions for:		4

—the space curve function	ThreeCurve := DFLOAT -> R3	5
—the plane curve function	TwoCurve := (DFLOAT, DFLOAT) -> R2	6
—the surface function	Surface := (DFLOAT, DFLOAT) -> R3	7
Frenet frames define a		8
coordinate system around a	FrenetFrame :=	9
point on a space curve.	Record(value:R3,tangent:R3,normal:R3,binormal:R3)	10
The current Frenet frame	frame: FrenetFrame	11
for a point on a curve.		12

**ntubeDraw**(*spaceCurve*, *planeCurve*,  $u_0..u_1$ ,  $t_0..t_1$ ) draws *planeCurve* in the normal planes of *spaceCurve*. The parameter  $u_0..u_1$  specifies the parameter range for *planeCurve* and  $t_0..t_1$  specifies the parameter range for *spaceCurve*. Additionally, the plane curve function takes a second parameter: the current parameter of *spaceCurve*. This allows the plane curve to change shape as it goes around the space curve. See Section F.4 on page 862 for an example of this.

	ntubeDraw: (ThreeCurve,TwoCurve,S,S) -> VIEW3D	13
	ntubeDraw(spaceCurve,planeCurve,uRange,tRange) ==	14
	ntubeDrawOpt(spaceCurve, planeCurve, uRange, -	15
	tRange, []\$List DROPT)	16
		17
	ntubeDrawOpt: (ThreeCurve,TwoCurve,S,S,List DROPT)	18
	-> VIEW3D	19
	ntubeDrawOpt(spaceCurve,planeCurve,uRange,tRange,1) ==	20
		21
	delT:DFLOAT := (hi(tRange) - lo(tRange))/10000	22
	oldT:DFLOAT := lo(tRange) - 1	23
	fun := ngeneralTube(spaceCurve,planeCurve,delT,oldT)	24
	draw(fun, uRange, tRange, 1)	25
		26

This function is similar to **ntubeDraw**, but takes optional parameters that it passes to the axiomFundraw command.

**nfrenetFrame**(*c*, *t*, *delT*) numerically computes the Frenet frame about the curve *c* at *t*. Parameter *delT* is a small number used to compute derivatives.

	nfrenetFrame(c, t, delT) ==	27
	f0 := c(t)	28
	f1 := c(t+delT)	29
The tangent.	t0 := f1 - f0	30
	n0 := f1 + f0	31
The binormal.	b := cross(t0, n0)	32
The normal.	n := cross(b,t0)	33
	ln := length n	34
	lb := length b	35
	ln = 0 or lb = 0 =>	36
	error "Frenet Frame not well defined"	37
	n := (1/ln)*n	38
	b := (1/lb)*b	39
Make into unit length vectors.	[f0, t0, n, b]\$FrenetFrame	40

**ngeneralTube**(*spaceCurve*, *planeCurve*, *delT*, *oltT*) creates a function that can be passed to the system axiomFundraw command. The function

is a parameterized surface for the general tube around *spaceCurve*. *delT* is a small number used to compute derivatives. *oldT* is used to hold the current value of the *t* parameter for *spaceCurve*. This is an efficiency measure to ensure that frames are only computed once for each value of *t*.

	<code>ngeneralTube: (ThreeCurve, TwoCurve, DFLOAT, DFLOAT) -&gt; Surface</code>	41
	<code>ngeneralTube(spaceCurve, planeCurve, delT, oldT) ==</code>	42
Indicate that <b>frame</b> is global.	<code>  free frame</code>	43
	<code>  (v:DFLOAT, t: DFLOAT): R3 +-&gt;</code>	44
If not already computed,	<code>    if (t ~= oldT) then</code>	45
compute new frame.	<code>      frame := nfrenetFrame(spaceCurve, t, delT)</code>	46
	<code>      oldT := t</code>	47
	<code>  p := planeCurve(v, t)</code>	48
Project <b>p</b> into the normal plane.	<code>  frame.value + p.1*frame.normal + p.2*frame.binormal</code>	49

## F.11

### dhtri.input

Create affine transformations (DH matrices) that transform a given triangle into another.

Compute a DHMATRIX that	<code>tri2tri: (List Point DFLOAT, List Point DFLOAT) -&gt; DHMATRIX(DFLOAT)</code>	1
	<code>tri2tri(t1, t2) ==</code>	2
transforms <b>t1</b> to <b>t2</b> , where	<code>  n1 := triangleNormal(t1)</code>	3
<b>t1</b> and <b>t2</b> are the vertices	<code>  n2 := triangleNormal(t2)</code>	4
of two triangles in 3-space.	<code>  tet2tet(concat(t1, n1), concat(t2, n2))</code>	5
		6
Compute a DHMATRIX that	<code>tet2tet: (List Point DFLOAT, List Point DFLOAT) -&gt; DHMATRIX(DFLOAT)</code>	7
	<code>tet2tet(t1, t2) ==</code>	8
transforms <b>t1</b> to <b>t2</b> ,	<code>  m1 := makeColumnMatrix t1</code>	9
where <b>t1</b> and <b>t2</b> are the	<code>  m2 := makeColumnMatrix t2</code>	10
vertices of two tetrahedrons	<code>  m2 * inverse(m1)</code>	11
in 3-space.		12
Put the vertices of a tetra-	<code>makeColumnMatrix(t) ==</code>	13
hedron into matrix form.	<code>  m := new(4,4,0)\$DHMATRIX(DFLOAT)</code>	14
	<code>  for x in t for i in 1..repeat</code>	15
	<code>    for j in 1..3 repeat</code>	16
	<code>      m(j,i) := x.j</code>	17
	<code>  m(4,i) := 1</code>	18
	<code>  m</code>	19
		20
Compute a vector normal to	<code>triangleNormal(t) ==</code>	21
the given triangle, whose	<code>  a := triangleArea t</code>	22
length is the square root	<code>  p1 := t.2 - t.1</code>	23
of the area of the triangle.	<code>  p2 := t.3 - t.1</code>	24
	<code>  c := cross(p1, p2)</code>	25
	<code>  len := length(c)</code>	26
	<code>  len = 0 =&gt; error "degenerate triangle!"</code>	27
	<code>  c := (1/len)*c</code>	28
	<code>  t.1 + sqrt(a) * c</code>	29
		30
Compute the area of a	<code>triangleArea t ==</code>	31

triangle using Heron's	a := length(t.2 - t.1)	32
formula.	b := length(t.3 - t.2)	33
	c := length(t.1 - t.3)	34
	s := (a+b+c)/2	35
	sqrt(s*(s-a)*(s-b)*(s-c))	36

## F.12

### tetra.input

Bring DH matrices into the environment.	)set expose add con DenavitHartenbergMatrix	1
Set up the coordinates of the corners of the tetrahedron.	x1:DFLOAT := sqrt(2.0@DFLOAT/3.0@DFLOAT)	2
	x2:DFLOAT := sqrt(3.0@DFLOAT)/6	3
		4
Some needed points.	p1 := point [-0.5@DFLOAT, -x2, 0.0@DFLOAT]	5
	p2 := point [0.5@DFLOAT, -x2, 0.0@DFLOAT]	6
	p3 := point [0.0@DFLOAT, 2*x2, 0.0@DFLOAT]	7
	p4 := point [0.0@DFLOAT, 0.0@DFLOAT, x1]	8
		9
The base of the tetrahedron.	baseTriangle := [p2, p1, p3]	10
		11
The "middle triangle" inscribed in the base of the tetrahedron.	mt := [0.5@DFLOAT*(p2+p1), 0.5@DFLOAT*(p1+p3), 0.5@DFLOAT*(p3+p2)]	12
		13
The bases of the triangles of the subdivided tetrahedron.	bt1 := [mt.1, p1, mt.2]	14
	bt2 := [p2, mt.1, mt.3]	15
	bt3 := [mt.2, p3, mt.3]	16
	bt4 := [0.5@DFLOAT*(p2+p4), 0.5@DFLOAT*(p1+p4), 0.5@DFLOAT*(p3+p4)]	17
		18
		19
Create the transformations that bring the base of the tetrahedron to the bases of the subdivided tetrahedron.	tt1 := tri2tri(baseTriangle, bt1)	20
	tt2 := tri2tri(baseTriangle, bt2)	21
	tt3 := tri2tri(baseTriangle, bt3)	22
	tt4 := tri2tri(baseTriangle, bt4)	23
		24
Draw a Sierpinsky tetrahedron with n levels of recursive subdivision.	drawPyramid(n) ==	25
	s := createThreeSpace()	26
	dh := rotatex(0.0@DFLOAT)	27
	drawPyramidInner(s, n, dh)	28
	makeViewport3D(s, "Sierpinsky Tetrahedron")	29
		30
Recursively draw a Sierpinsky tetrahedron.	drawPyramidInner(s, n, dh) ==	31
	n = 0 => makeTetrahedron(s, dh, n)	32
Draw the 4 recursive pyramids.	drawPyramidInner(s, n-1, dh * tt1)	33
	drawPyramidInner(s, n-1, dh * tt2)	34
	drawPyramidInner(s, n-1, dh * tt3)	35
	drawPyramidInner(s, n-1, dh * tt4)	36
		37
Draw a tetrahedron into the given space with the given color, transforming it by the given DH matrix.	makeTetrahedron(sp, dh, color) ==	38
	w1 := dh*p1	39
	w2 := dh*p2	40
	w3 := dh*p3	41
	w4 := dh*p4	42
	polygon(sp, [w1, w2, w4])	43

```

polygon(sp, [w1, w3, w4]) 44
polygon(sp, [w2, w3, w4]) 45
void() 46

```

## F.13 antoine.input

Bring DH matrices into  
the environment.  
The current transformation for  
drawing a sub ring.

Draw Antoine's Necklace with  $n$   
levels of recursive subdivision.  
The number of subrings is  
 $10^n$ .  
Do the real work.

Draw Antoine's Necklace. Thank you to Matthew Grayson at IBM's T.J  
Watson Research Center for the idea.

```

)set expose add con DenavitHartenbergMatrix 1
torusRot: DHMATRIX(DFLOAT) 2
drawRings(n) == 3
s := createThreeSpace() 4
dh:DHMATRIX(DFLOAT) := identity() 5
drawRingsInner(s, n, dh) 6
makeViewport3D(s, "Antoine's Necklace") 7
8
9
10
11

```

In order to draw Antoine rings, we take one ring, scale it down to a  
smaller size, rotate it around its central axis, translate it to the edge of  
the larger ring and rotate it around the edge to a point corresponding to  
its count (there are 10 positions around the edge of the larger ring). For  
each of these new rings we recursively perform the operations, each ring  
becoming 10 smaller rings. Notice how the DHMATRIX operations are used  
to build up the proper matrix composing all these transformations.

Recursively draw Antoine's  
Necklace.

Angle around ring.  
Angle of subring from plane.  
Amount to translate subring.  
The translation increment.  
Subdivide into 10 linked rings.

```

drawRingsInner(s, n, dh) == 12
n = 0 => 13
drawRing(s, dh) 14
void() 15
t := 0.0@DFLOAT 16
p := 0.0@DFLOAT 17
tr := 1.0@DFLOAT 18
inc := 0.1@DFLOAT 19
for i in 1..10 repeat 20
tr := tr + inc 21
inc := -inc 22
dh' := dh*rotatez(t)*translate(tr,0.0@DFLOAT,0.0@DFLOAT)*
23
rotatey(p)*scale(0.35@DFLOAT, 0.48@DFLOAT, 0.4@DFLOAT)
24
drawRingsInner(s, n-1, dh') 25
t := t + 36.0@DFLOAT 26
p := p + 90.0@DFLOAT 27
void() 28
29

```

Transform ring in center  
to a link.

Draw a single ring into  
the given subspace,  
transformed by the given  
DHMATRIX.

```

drawRing(s, dh) == 30
free torusRot 31
torusRot := dh 32
makeObject(torus, 0..2*%pi, 0..2*%pi, var1Steps == 6, 33
space == s, var2Steps == 15) 34

```

Parameterization of a torus,	torus(u ,v) ==	35
transformed by the	cu := cos(u)/6	36
DHMATRIX in torusRot.	torusRot*point [(1+cu)*cos(v),(1+cu)*sin(v),(sin u)/6]	37
		38

## F.14 scherk.input

Scherk's minimal surface, defined by:  $e^z \cos(x) = \cos(y)$ . See: *A Comprehensive Introduction to Differential Geometry*, Vol. 3, by Michael Spivak, Publish Or Perish, Berkeley, 1979, pp. 249-252.

Offsets for a single piece of Scherk's minimal surface.	(xOffset, yOffset):DFLOAT	1
		2
		3
Draw Scherk's minimal surface on an m by n patch.	drawScherk(m,n) ==	4
	free xOffset, yOffset	5
	space := createThreeSpace()	6
	for i in 0..m-1 repeat	7
	xOffset := i*pi	8
	for j in 0 .. n-1 repeat	9
Draw only odd patches.	rem(i+j, 2) = 0 => 'iter	10
	yOffset := j*pi	11
Draw a patch.	drawOneScherk(space)	12
	makeViewport3D(space, "Scherk's Minimal Surface")	13
		14
The first patch that makes up a single piece of Scherk's minimal surface.	scherk1(u,v) ==	15
	x := cos(u)/exp(v)	16
	point [xOffset + acos(x), yOffset + u, v, abs(v)]	17
		18
The second patch.	scherk2(u,v) ==	19
	x := cos(u)/exp(v)	20
	point [xOffset - acos(x), yOffset + u, v, abs(v)]	21
		22
The third patch.	scherk3(u,v) ==	23
	x := exp(v) * cos(u)	24
	point [xOffset + u, yOffset + acos(x), v, abs(v)]	25
		26
The fourth patch.	scherk4(u,v) ==	27
	x := exp(v) * cos(u)	28
	point [xOffset + u, yOffset - acos(x), v, abs(v)]	29
		30
Draw the surface by breaking it into four patches and then drawing the patches.	drawOneScherk(s) ==	31
	makeObject(scherk1,-%pi/2..%pi/2,0..%pi/2,space==s,	32
	var1Steps == 28, var2Steps == 28)	33
	makeObject(scherk2,-%pi/2..%pi/2,0..%pi/2,space==s,	34
	var1Steps == 28, var2Steps == 28)	35
	makeObject(scherk3,-%pi/2..%pi/2,-%pi/2..0,space==s,	36
	var1Steps == 28, var2Steps == 28)	37
	makeObject(scherk4,-%pi/2..%pi/2,-%pi/2..0,space==s,	38
	var1Steps == 28, var2Steps == 28)	39
	void()	40

[





---

# Glossary

] !  
(*syntax*) Suffix character for *destructive operations*.

,  
(*syntax*) a separator for items in a *tuple*, for example, to separate arguments of a function  $f(x, y)$ .

=>  
(*syntax*) the expression  $a \Rightarrow b$  is equivalent to *if a then exit b*.

?  
1. (*syntax*) a suffix character for Boolean-valued **function** names, for example, **odd?**. 2. Prefix character for “optional” pattern variables. For example, the pattern  $f(x + y)$  does not match the expression  $f(7)$ , but  $f(?x + y)$  does, with  $x$  matching 0 and  $y$  matching 7. 3. The special type  $?$  means *don't care*. For example, the declaration:  $x : \text{Polynomial } ?$  means that values assigned to  $x$  must be polynomials over an arbitrary *underlying domain*.

### abstract datatype

a programming language principle used in AXIOM where a datatype definition has defined in two parts: (1) a *public* part describing a set of *exports*, principally operations that apply to objects of that type, and (2) a *private* part describing the implementation of the datatype usually in terms of a *representation* for objects of the type. Programs that create and otherwise manipulate objects of the type may only do so through its exports. The representation and other implementation information is specifically hidden.

### abstraction

described functionally or conceptually without regard to implementation.

### accuracy

the degree of exactness of an approximation or measurement. In computer algebra systems, computations are typically carried out with complete accuracy using integers or rational numbers of indefinite size. Domain Float provides a function **precision** to change the precision for floating-point computations. Computations using Double-Float have a fixed precision but uncertain accuracy.

### add-chain

a hierarchy formed by *domain extensions*. If domain A extends domain B and domain B extends domain C, then A has *add-chain* B-C.

### aggregate

a data structure designed to hold multiple values. Examples of aggregates are List, Set, Matrix and Bits.

### AKCL

Austin Kyoto Common LISP, a version of *KCL* produced by William Schelter, Austin, Texas.

### algorithm

a step-by-step procedure for a solution of a problem; a program

### ancestor

(of a domain or category) a category that is a *parent*, or a *parent* of a *parent*, and so on. See a **Cross Reference** page of a constructor in Browse.

### application

(*syntax*) an expression denoting “application” of a function to a set of *argument* parameters. Appli-

cations are written as a *parameterized form*. For example, the form  $f(x,y)$  indicates the “application of the function  $f$  to the tuple of arguments  $x$  and  $y$ .” See also *evaluation* and *invocation*.

### **apply**

See *application*.

### **argument**

1. (actual argument) a value passed to a function at the time of a *function call*; also called an *actual parameter*. 2. (formal argument) a variable used in the definition of a function to denote the actual argument passed when the function is called.

### **arity**

1. (function) the number of arguments. 2. (operator or operation) corresponds to the arity of a function implementing the operator or operation.

### **assignment**

(*syntax*) an expression of the form  $x := e$ , meaning “assign the value of  $e$  to  $x$ .” After *evaluation*, the *variable*  $x$  *points* to an object obtained by evaluating the expression  $e$ . If  $x$  has a *type* as a result of a previous *declaration*, the object assigned to  $x$  must have that type. The interpreter must often *coerce* the value of  $e$  to make that happen. For example, the expression  $x : \text{Float} := 11$  first *declares*  $x$  to be a float, then forces the interpreter to coerce the integer 11 to 11.0 in order to assign a floating-point value to  $x$ .

### **attribute**

a name or functional form denoting *any* useful computational or mathematical property. For example, `commutative("*")` asserts that “ $*$ ” is commutative. Also, `finiteAggregate` is used to assert that an aggregate has a finite number of immediate components.

### **basis**

(*algebra*)  $S$  is a basis of a module  $M$  over a *ring* if  $S$  generates  $M$ , and  $S$  is linearly independent.

### **benefactor**

(of a given domain) a domain or package that the given domain explicitly references (for example, calls functions from) in its implementation. See a **Cross Reference** page of a constructor in Browse.

### **binary**

operation or function with *arity* 2.

### **binding**

the association of a variable with properties such as *value* and *type*. The top-level *environment* in the

interpreter consists of bindings for all user variables and functions. When a *function* is applied to arguments, a local environment of bindings is created, one for each formal *argument* and *local variable*.

### **block**

(*syntax*) a control structure where expressions are sequentially *evaluated*.

### **body**

a *function body* or *loop body*.

### **boolean**

objects denoted by the *literals* `true` and `false`; elements of domain Boolean. See also Bits.

### **built-in function**

a *function* in the standard AXIOM library. Contrast *user function*.

### **cache**

1. (noun) a mechanism for immediate retrieval of previously computed data. For example, a function that does a lengthy computation might store its values in a *hash table* using the function argument as the key. The hash table then serves as a cache for the function (see also `)set function cache`). Also, when *recurrence relations* that depend upon  $n$  previous values are compiled, the previous  $n$  values are normally cached (use `)set functions recurrence` to change this). 2. (verb) to save values in a cache.

### **capsule**

the part of the *body* of a *domain constructor* that defines the functions implemented by the constructor.

### **case**

(*syntax*) an operator used to evaluate code conditionally based on the branch of a Union. For example, if value  $u$  is `Union(Integer, "failed")`, the conditional expression `if u case Integer then A else B` evaluates  $A$  if  $u$  is an integer and  $B$  otherwise.

### **Category**

the distinguished object denoting the type of a category; the class of all categories.

### **category**

(*basic concept*) types denoting classes of domains. Examples of categories are Ring (“the class of all rings”) and Aggregate (“the class of all aggregates”). Categories form a hierarchy (formally, a directed acyclic graph) with the distinguished category Type at the top. Each category inherits the properties of all its ancestors. Categories op-

tionally provide “default definitions” for operations they export. Categories are defined in AXIOM by functions called *category constructors*. Technically, a category designates a class of domains with common *operations* and *attributes* but usually with different *functions* and *representations* for its constituent *objects*. Categories are always defined using the AXIOM library language (see also *category extension*). See also file **catdef.spad** for definitions of basic algebraic categories in AXIOM, **agggcat.spad** for data structure

### category constructor

a function that creates categories, described by an abstract datatype in the AXIOM programming language. For example, the category constructor Module is a function that takes a domain parameter R and creates the category “modules over R.”

### category extension

A category A *directly extends* a category B if its definition has the form `A == B with ...` or `A == Join(...,B,...)`. In this case, we also say that B is the *parent* of A. We say that a category A extends B if B is an *ancestor* of A. A category A may also directly extend B if B appears in a conditional expression within the **Exports** part of the definition to the right of a **with**. See, for example, file **catdef.spad** for definitions of the algebra categories in AXIOM, **agggcat.spad** for data structure categories.

### category hierarchy

hierarchy formed by category extensions. The root category is Type. A category can be defined as a *Join* of two or more categories so as to have multiple *parents*. Categories may also be parameterized so as to allow conditional inheritance.

### character

1. an element of a character set, as represented by a keyboard key. 2. a component of a string. For example, the 1st element of the string “hello there” is the character *h*.

### client

(of a given domain) any domain or package that explicitly calls functions from the given domain. See a **Cross Reference** page of a constructor in Browse.

### coercion

an automatic transformation of an object of one *type* to an object of a similar or desired target type. In the interpreter, coercions and *retractions* are done automatically by the interpreter when a type mismatch occurs. Compare *conversion*.

### comment

textual remarks imbedded in code. Comments are preceded by a double dash (“--”). For AXIOM library code, stylized comments for on-line documentation are preceded by two plus signs (“++”).

### Common LISP

A version of *LISP* adopted as an informal standard by major users and suppliers of LISP.

### compile-time

the time when category or domain constructors are compiled. Contrast *run-time*.

### compiler

a program that generates low-level code from a higher-level source language. AXIOM has three compilers. A *graphics compiler* converts graphical formulas to a compiled subroutine so that points can be rapidly produced for graphics commands. An *interpreter compiler* optionally compiles *user functions* when first *invoked* (use `)set functions compile` to turn this feature on). A *library compiler* compiles all constructors (available on an “as-is” basis for Release 1).

### computational object

In AXIOM, domains are objects. This term is used to distinguish the objects that are members of domains rather than the domains themselves.

### conditional

a *control structure* of the form `if A then B else C`. The *evaluation* of A produces **true** or **false**. If **true**, B evaluates to produce a value; otherwise C evaluates to produce a value. When the value is not required, the `else C` part can be omitted.

### constant

(*syntax*) a reserved word used in *signatures* in AXIOM programming language to signify that an operation always returns the same value. For example, the signature `0: constant -> $` in the source code of AbelianMonoid tells the AXIOM compiler that 0 is a constant so that suitable optimizations might be performed.

### constructor

a *function* that creates a *category*, *domain*, or *package*.

### continuation

when a line of a program is so long that it must be broken into several lines, then all but the first line are called *continuation lines*. If such a line is given interactively, then each incomplete line must end with an underscore.

**control structure**

program structures that can specify a departure from normal sequential execution. AXIOM has four kinds of control structures: *blocks*, *case* statements, *conditionals*, and *loops*.

**conversion**

the transformation of an object of one *type* to one of another type. Conversions that can be performed automatically by the interpreter are called *coercions*. These happen when the interpreter encounters a type mismatch and a similar or declared target type is needed. In general, the user must use the infix operation “:” to cause this transformation.

**copying semantics**

the programming language semantics used in PASCAL but *not* in AXIOM. See also *pointer semantics* for details.

**data structure**

a structure for storing data in the computer. Examples are *lists* and *hash tables*.

**datatype**

equivalent to *domain* in AXIOM.

**declaration**

(*syntax*) an expression of the form  $x : T$  where  $T$  is some *type*. A declaration forces all values *assigned* to  $x$  to be of that type. If a value is of a different type, the interpreter will try to *coerce* the value to type  $T$ . Declarations are necessary in case of ambiguity or when a user wants to introduce an *unexposed* domain.

**default definition**

a function defined by a *category*. Such definitions appear in category definitions of the form

```
C: Category == T add I
```

in an optional implementation part  $I$  to the right of the keyword *add*.

**default package**

an optional *package* of *functions* associated with a category. Such functions are necessarily defined in terms of other operations exported by the category.

**definition**

(*syntax*) 1. An expression of the form  $f(a) == b$  defining function  $f$  with *formal arguments*  $a$  and *body*  $b$ ; equivalent to the statement  $f == (a) +-> b$ . 2. An expression of the form  $a == b$  where  $a$  is a *symbol*, equivalent to  $a() == b$ . See also *macro* where a similar substitution is done at *parse* time.

**delimiter**

a *character* that marks the beginning or end of

some syntactically correct unit in the language, for example, “” for strings, blanks for identifiers.

**dependent**

(of a given constructor) another constructor that mentions the given constructor as an argument or among the types of an exported operation. See a **Cross Reference** page of a constructor in Browse.

**destructive operation**

An operation that changes a component or structure of a value. In AXIOM, destructive operations have names ending with an exclamation mark (“!”). For example, domain List has two operations to reverse the elements of a list, one named **reverse** that returns a copy of the original list with the elements reversed, another named **reverse!** that reverses the elements *in place*, thus destructively changing the original list.

**documentation**

1. on-line or hard-copy descriptions of AXIOM; 2. text in library code preceded by “++” comments as opposed to general comments preceded by “--”.

**domain**

(*basic concept*) a domain corresponds to the usual notion of datatypes. Examples of domains are List Float (“lists of floats”), Fraction Polynomial Integer (“fractions of polynomials of integers”), and Matrix Stream CardinalNumber (“matrices of infinite *streams* of cardinal numbers”). The term *domain* actually abbreviates *domain of computation*. Technically, a domain denotes a class of objects, a class of *operations* for creating and otherwise manipulating these objects, and a class of *attributes* describing computationally useful properties. Domains may also define *functions* for its exported operations, often in terms of some *representation* for the objects. A domain itself is an *object* created by a *function* called a *domain constructor*. The types of the exported operations of a domain are arbitrary; this gives rise to a special class of domains called *packages*.

**domain constructor**

a function that creates domains, described by an abstract datatype in the AXIOM programming language. Simple domains like Integer and Boolean are created by domain constructors with no arguments. Most domain constructors take one or more parameters, one usually denoting an *underlying domain*. For example, the domain Matrix(R) denotes “matrices over R.” Domains Mapping, Record, and Union are primitive domains. All other domains are written in the AXIOM programming language and

can be modified by users with access to the library source code and the library compiler.

### domain extension

a domain constructor *A* is said to *extend* a domain constructor *B* if *A*'s definition has the form *A* == *B add* .... This intuitively means “functions not defined by *A* are assumed to come from *B*.” Successive domain extensions form *add-chains* affecting the *search order* for functions not implemented directly by the domain during *dynamic lookup*.

### dot notation

using an infix dot (“.”) for the operation *elt*. If *u* is the list [7,4,-11] then both *u*(2) and *u*.2 return 4. Dot notation nests to the left: *f.g.h* is equivalent to (*f.g*).*h*.

### dynamic

that which is done at *run-time* as opposed to *compile-time*. For example, the interpreter may build a domain “matrices over integers” dynamically in response to user input. However, the compilation of all functions for matrices and integers is done during *compile-time*. Contrast *static*.

### dynamic lookup

In AXIOM, a *domain* may or may not explicitly provide *function* definitions for all its exported *operations*. These definitions may instead come from domains in the *add-chain* or from *default packages*. When a *function call* is made for an operation in the domain, up to five steps are carried out.

1. If the domain itself implements a function for the operation, that function is returned.
2. Each of the domains in the *add-chain* are searched; if one of these domains implements the function, that function is returned.
3. Each of the *default packages* for the domain are searched in order of the *lineage*. If any of the default packages implements the function, the first one found is returned.
4. Each of the *default packages* for each of the domains in the *add-chain* are searched in the order of their *lineage*. If any of the default packages implements the function, the first one found is returned.
5. If all of the above steps fail, an error message is reported.

### empty

the unique value of objects with type Void.

### environment

a set of *bindings*.

### evaluation

a systematic process that transforms an *expression*

into an object called the *value* of the expression. Evaluation may produce *side effects*.

### exit

(*reserved word*) an *operator* that forces an exit from the current *block*. For example, the block (*a* := 1; if *i* > 0 then exit *a*; *a* := 2) will prematurely exit at the second statement with value 1 if the value of *i* is greater than zero. See “=>” for an alternate syntax.

### explicit export

1. (of a domain *D*) any *attribute*, *operation*, or *category* explicitly mentioned in the *type* exports part *E* for the domain constructor definition *D*: *E* == *I*
2. (of a category *C*) any *attribute*, *operation*, or *category* explicitly mentioned in the *type* specification part *E* for the category constructor definition *C*: *Category* == *E*

### export

*explicit export* or *implicit export* of a domain or category

### expose

some constructors are *exposed*, others *unexposed*. Exposed domains and packages are recognized by the interpreter. Use *)set expose* to control what is exposed. Unexposed constructors will appear in Browse prefixed by a star (“\*”).

### expression

1. any syntactically correct program fragment.
2. an element of domain Expression.

### extend

see *category extension* or *domain extension*.

### field

(*algebra*) a *domain* that is a *ring* where every non-zero element is invertible and where *xy=yx*; a member of category Field. For a complete list of fields, click on **Domains** under **Cross Reference** for Field in Browse.

### file

1. a program or collection of data stored on disk, tape or other medium.
2. an object of a File domain.

### float

a floating-point number with user-specified precision; an element of domain Float. Floats are *literals* written either without an exponent (for example, 3.1416), or with an exponent (for example, 3.12E-12). Use function *precision* to change the precision of the mantissa (20 digits by default). See also *small float*.

**formal parameter**

(of a function) an identifier *bound* to the value of an actual *argument* on *invocation*. In the function definition  $f(x,y) == u$ , for example,  $x$  and  $y$  are the formal parameters.

**frame**

the basic unit of an interactive session; each frame has its own *step number*, *environment*, and *history*. In one interactive session, users can create and drop frames, and have several active frames simultaneously.

**free**

(*syntax*) A keyword used in user-defined functions to declare that a variable is a *free variable* of that function. For example, the statement **free**  $x$  declares the variable  $x$  within the body of a function  $f$  to be a free variable in  $f$ . Without such a declaration, any variable  $x$  that appears on the left-hand side of an assignment before it is referenced is regarded as a *local variable* of that function. If the intention of the assignment is to give a value to a *global variable*  $x$ , the body of that function must contain the statement **free**  $x$ . A variable that is a parameter to the function is always local.

**free variable**

(of a function) a variable that appears in a body of a function but is not *bound* by that function. Contrast with *local variable*.

**function**

implementation of *operation*. A function takes zero or more *argument* parameters and produces a single return value. Functions are objects that can be passed as parameters to functions and can be returned as values of functions. Functions can also create other functions (see also `InputForm`). See also *application* and *invocation*. The terms *operation* and *function* are distinct notions in AXIOM. An operation is an abstraction of a function, described by a *name* and a *signature*. A function is created by providing an implementation of that operation by AXIOM code. Consider the example of defining a user-function **fact** to compute the **factorial** of a nonnegative integer. The AXIOM statement **fact: Integer -> Integer** describes the operation, whereas the statement **fact(n) = reduce(\*,[1..n])** defines the function. See also *generic function*.

**function body**

the part of a *function's* definition that is evaluated when the function is called at *run-time*; the part of the function definition to the right of the `"=="`.

**garbage collection**

a system function that automatically recycles memory cells from the *heap*. AXIOM is built upon *Common LISP* that provides this facility.

**garbage collector**

a mechanism for reclaiming storage in the *heap*.

**Gaussian**

a complex-valued expression, for example, one with both a real and imaginary part; a member of a Complex domain.

**generic function**

the use of one function to operate on objects of different types. One might regard AXIOM as supporting generic *operations* but not generic functions. One operation  $+: (D, D) \rightarrow D$  exists for adding elements in a ring; each ring however provides its own type-specific function for implementing this operation.

**global variable**

A variable that can be referenced freely by functions. In AXIOM, all top-level user-defined variables defined during an interactive user session are global variables. AXIOM does not allow *fluid variables*, that is, variables *bound* by a function  $f$  that can be referenced by functions that  $f$  calls.

**Gröbner basis**

(*algebra*) a special basis for a polynomial ideal that allows a simple test for membership. It is useful in solving systems of polynomial equations.

**group**

(*algebra*) a monoid where every element has a multiplicative inverse.

**hash table**

a data structure designed for fast lookup of information stored under "keys". A hash table consists of a set of *entries*, each of which associates a *key* with a *value*. Finding the object stored under a key can be fast for a large number of entries since keys are *hashed* into numerical codes for fast lookup.

**heap**

1. an area of storage used by data in programs. For example, AXIOM will use the heap to hold the partial results of symbolic computations. When cancellations occur, these results remain in the heap until *garbage collected*. 2. an object of a Heap domain.

**history**

a mechanism that records input and output data for an interactive session. Using the history facility, users can save computations, review previous steps of a computation, and restore a previous interac-

tive session at some later time. For details, issue the system command *)history ?* to the interpreter. See also *frame*.

**ideal**

(*algebra*) a subset of a ring that is closed under addition and multiplication by arbitrary ring elements; thus an ideal is a module over the ring.

**identifier**

(*syntax*) an AXIOM name; a *literal* of type Symbol. An identifier begins with an alphabetical character, %, ?, or !, and may be followed by any of these or digits. Certain distinguished *reserved words* are not allowed as identifiers but have special meaning in AXIOM.

**immutable**

an object is immutable if it cannot be changed by an *operation*; it is not a *mutable object*. Algebraic objects are generally immutable: changing an algebraic expression involves copying parts of the original object. One exception is an object of type Matrix. Examples of mutable objects are data structures such as those of type List. See also *pointer semantics*.

**implicit export**

(of a domain or category) any exported *attribute* or *operation* or *category* that is not an *explicit export*. For example, Monoid and \* are implicit exports of Ring.

**index**

1. a variable that counts the number of times a *loop* is repeated. 2. the “address” of an element in a data structure (see also category LinearAggregate).

**infix**

(*syntax*) an *operator* placed between two *operands*; also called a *binary operator*. For example, in the expression *a + b*, “+” is the infix operator. An infix operator may also be used as a *prefix*. Thus *+(a, b)* is also permissible in the AXIOM language. Infix operators have a *precedence* relative to one another.

**input area**

a rectangular area on a HyperDoc screen into which users can enter text.

**instantiate**

to build a *category*, *domain*, or *package* at run-time.

**integer**

a *literal* object of domain Integer, the class of integers with an unbounded number of digits. Integer literals consist of one or more consecutive digits (0-

9) with no embedded blanks. Underscores can be used to separate digits in long integers if desirable.

**interactive**

a system where the user interacts with the computer step-by-step.

**interpreter**

the part of AXIOM responsible for handling user input during an interactive session. The interpreter parses the user’s input expression to create an expression tree, then does a bottom-up traversal of the tree. Each subtree encountered that is not a value consists of a root node denoting an operation name and one or more leaf nodes denoting *operands*. The interpreter resolves type mismatches and uses type-inferencing and a library database to determine appropriate types for the operands and the result, and an operation to be performed. The interpreter next builds a domain to perform the indicated operation, and invokes a function from the domain to compute a value. The subtree is then replaced by that value and the process continues. Once the entire tree has been processed, the value replacing the top node of the tree is displayed back to the user as the value of the expression.

**invocation**

(of a function) the run-time process involved in *evaluating a function application*. This process has two steps. First, a local *environment* is created where *formal arguments* are locally *bound* by *assignment* to their respective actual *argument*. Second, the *function body* is evaluated in that local environment. The evaluation of a function is terminated either by completely evaluating the function body or by the evaluation of a **return** expression.

**iteration**

repeated evaluation of an expression or a sequence of expressions. Iterations use the reserved words **for**, **while**, and **repeat**.

**Join**

a primitive AXIOM function taking two or more categories as arguments and producing a category containing all of the operations and attributes from the respective categories.

**KCL**

Kyoto Common LISP, a version of *Common LISP* that features compilation of LISP into the C Programming Language.

**library**

In AXIOM, a collection of compiled modules representing *category* or *domain* constructors.

**lineage**

the sequence of *default packages* for a given domain to be searched during *dynamic lookup*. This sequence is computed first by ordering the category *ancestors* of the domain according to their *level number*, an integer equal to the minimum distance of the domain from the category. Parents have level 1, parents of parents have level 2, and so on. Among categories with equal level numbers, ones that appear in the left-most branches of Joins in the source code come first. See a **Cross Reference** page of a constructor in Browse. See also *dynamic lookup*.

**LISP**

acronym for List Processing Language, a language designed for the manipulation of non-numerical data. The AXIOM library is translated into LISP then compiled into machine code by an underlying LISP system.

**list**

an object of a List domain.

**literal**

an object with a special syntax in the language. In AXIOM, there are five types of literals: *booleans*, *integers*, *floats*, *strings*, and *symbols*.

**local**

(*syntax*) A keyword used in user-defined functions to declare that a variable is a *local variable* of that function. Because of default assumptions on variables, such a declaration is often not necessary but is available to the user for clarity when appropriate.

**local variable**

(of a function) a variable *bound* by that function and such that its binding is invisible to any function that function calls. Also called a *lexical variable*. By default in the interpreter:

1. any variable *x* that appears on the left-hand side of an assignment is normally regarded a local variable of that function. If the intention of an assignment is to change the value of a *global variable x*, the body of the function must then contain the statement **free x**.
2. any other variable is regarded as a *free variable*.

An optional declaration **local x** is available to declare explicitly a variable to be a local variable. All *formal parameters* are local variables to the function.

**loop**

1. an expression containing a **repeat**. 2. a collec-

tion expression having a **for** or a **while**, for example, `[f(i) for i in S]`.

**loop body**

the part of a loop following the **repeat** that tells what to do each iteration. For example, the body of the loop **for x in S repeat B** is *B*. For a collection expression, the body of the loop precedes the initial **for** or **while**.

**macro**

1. (*interactive syntax*) An expression of the form **macro a == b** where *a* is a *symbol* causes *a* to be textually replaced by the expression *b* at *parse time*. 2. An expression of the form **macro f(a) == b** defines a parameterized macro expansion for a parameterized form *f*. This macro causes a form *f(x)* to be textually replaced by the expression *c* at parse time, where *c* is the expression obtained by replacing *a* by *x* everywhere in *b*. See also *definition* where a similar substitution is done during *evaluation*. 3. (*programming language syntax*) An expression of the form **a ==> b** where *a* is a *symbol*.

**mode**

a type expression containing a question-mark (“?”). For example, the mode **POLY ?** designates *the class of all polynomials over an arbitrary ring*.

**mutable**

objects that contain *pointers* to other objects and that have operations defined on them that alter these pointers. Contrast *immutable*. AXIOM uses *pointer semantics* as does *LISP* in contrast with many other languages such as PASCAL that use *copying semantics*. See *pointer semantics* for details.

**name**

1. a *symbol* denoting a *variable*, such as the variable *x*. 2. a *symbol* denoting an *operation*, that is, the operation **divide**: `(Integer, Integer) -> Integer`.

**nullary**

a function with no arguments, for example, **characteristic**; operation or function with *arity* zero.

**object**

a data entity created or manipulated by programs. Elements of domains, functions, and domains themselves are objects. The most basic objects are *literals*; all other objects must be created by *functions*. Objects can refer to other objects using *pointers* and can be *mutable*.

**object code**

code that can be directly executed by hardware;



also known as *machine language*.

### **operand**

an argument of an *operator* (regarding an operator as a *function*).

### **operation**

an abstraction of a *function*, described by a *signature*. For example, `fact: NonNegativeInteger -> NonNegativeInteger` describes an operation for “the factorial of a (non-negative) integer.”

### **operator**

special reserved words in the language such as “+” and “\*”; operators can be either *prefix* or *infix* and have a relative *precedence*.

### **overloading**

the use of the same name to denote distinct operations; an operation is identified by a *signature* identifying its name, the number and types of its arguments, and its return types. If two functions can have identical signatures, a *package call* must be made to distinguish the two.

### **package**

a special case of a domain, one for which the exported operations depend solely on the parameters and other explicit domains (contain no \$). Intuitively, packages are collections of (*polymorphic*) functions. Facilities for integration, differential equations, solution of linear or polynomial equations, and group theory are provided by packages.

### **package call**

(*syntax*) an expression of the form `e $ P` where `e` is an *application* and `P` denotes some *package* (or *domain*).

### **package constructor**

same as *domain constructor*.

### **parameter**

see *argument*.

### **parameterized datatype**

a domain that is built on another, for example, polynomials with integer coefficients.

### **parameterized form**

a expression of the form `f(x,y)`, an *application* of a function.

### **parent**

(of a domain or category) a category which is explicitly declared in the source code definition for the domain either to the left of the `with` or as an *export* of the domain. See *category extension*. See

also a **Cross Reference** page of a constructor in Browse.

### **parse**

1. (verb) to transform a user input string representing a valid AXIOM expression into an internal representation as a tree-structure; the resulting internal representation is then “interpreted” by AXIOM to perform some indicated action.

### **partially ordered set**

a set with a reflexive, transitive and antisymmetric *binary* operation.

### **pattern matching**

1. (on expressions) Given an expression called the “subject” `u`, the attempt to rewrite `u` using a set of “rewrite rules.” Each rule has the form `A == B` where `A` indicates an expression called a “pattern” and `B` denotes a “replacement.” The meaning of this rule is “replace `A` by `B`.” If a given pattern `A` matches a subexpression of `u`, that subexpression is replaced by `B`. Once rewritten, pattern matching continues until no further changes occur. 2. (on strings) the attempt to match a string indicating a “pattern” to another string called a “subject”, for example, for the purpose of identifying a list of names. In Browse, users may enter *search strings* for the purpose of identifying constructors, operations, and attributes.

### **pile**

alternate syntax for a block, using indentation and column alignment (see also *block*).

### **pointer**

a reference implemented by a link directed from one object to another in the computer memory. An object is said to *refer* to another if it has a pointer to that other object. Objects can also refer to themselves (cyclic references are legal). Also more than one object can refer to the same object. See also *pointer semantics*.

### **pointer semantics**

the programming language semantics used in languages such as LISP that allow objects to be *mutable*. Consider the following sequence of AXIOM statements:

```
x : Vector Integer := [1,4,7]
y := x
swap!(x,2,3)
```

The function `swap!` is used to interchange the second and third value in the list `x`, producing the value `[1,7,4]`. What value does `y` have after evaluation of the third statement? The answer is different in AXIOM than it is in a language with *copying*

*semantics.* In AXIOM, first the vector [1,2,3] is created and the variable *x* set to *point* to this object. Let's call this object *V*. Next, the variable *y* is made to point to *V* just as *x* does. Now the third statement interchanges the last 2 elements of *V* (the “!” at the end of the name **swap!** tells you that this operation is destructive, that is, it changes the elements *in place*). Both *x* and *y* perceive this change to *V*. Thus both *x* and *y* then have the value [1,7,4]. In PASCAL, the second statement causes a copy of *V* to be stored under *y*. Thus the change to *V* made by the third statement does not affect *y*.

### polymorphic

a *function* (for example, one implementing an *algorithm*) defined with categorical types so as to be applicable over a variety of domains (the domains which are members of the categorical types). Every AXIOM function defined in a domain or package constructor with a domain-valued parameter is polymorphic. For example, the same matrix “+” function is used to add “matrices over integers” as “matrices over matrices over integers.”

### postfix

an *operator* that follows its single *operand*. Postfix operators are not available in AXIOM.

### precedence

(*syntax*) refers to the so-called *binding power* of an operator. For example, “\*” has higher binding power than “+” so that the expression *a + b \* c* is equivalent to *a + (b \* c)*.

### precision

the number of digits in the specification of a number. The operation **digits** sets this for objects of Float.

### predicate

1. a Boolean-valued function, for example, **odd: Integer -> Boolean**. 2. a Boolean-valued expression.

### prefix

(*syntax*) an *operator* such as “-” that is written *before* its single *operand*. Every function of one argument can be used as a prefix operator. For example, all of the following have equivalent meaning in AXIOM: *f(x)*, *f x*, and *f.x*. See also *dot notation*.

### quote

the prefix operator “'” meaning *do not evaluate*.

### Record

(basic domain constructor) a domain constructor used to create an inhomogeneous aggregate composed of pairs of *selectors* and *values*. A Record domain is written in the form **Record(a1: D1, ..., an: Dn)** (*n* > 0) where *a1*, ..., *an* are identifiers called the *selectors* of the record, and *D1*, ..., *Dn* are domains indicating the type of the component stored under selector *an*.

### recurrence relation

A relation that can be expressed as a function *f* with some argument *n* which depends on the value of *f* at *k* previous values. In most cases, AXIOM will rewrite a recurrence relation on compilation so as to *cache* its previous *k* values and therefore

make the computation significantly more efficient.

### **recursion**

use of a self-reference within the body of a function. Indirect recursion is when a function uses a function below it in the call chain.

### **recursive**

1. A function that calls itself, either directly or indirectly through another function. 2. self-referential. See also *recursive*.

### **reference**

see *pointer*

### **relative**

(of a domain) A package that exports operations relating to the domain, in addition to those exported by the domain. See a **Cross Reference** page of a constructor in Browse.

### **representation**

a *domain* providing a data structure for elements of a domain, generally denoted by the special identifier *Rep* in the AXIOM programming language. As domains are *abstract datatypes*, this representation is not available to users of the domain, only to functions defined in the *function body* for a domain constructor. Any domain can be used as a representation.

### **reserved word**

a special sequence of non-blank characters with special meaning in the AXIOM language. Examples of reserved words are names such as **for**, **if**, and **free**, operator names such as “+” and **mod**, special character strings such as “==” and “:=”.

### **retraction**

to move an object in a parameterized domain back to the underlying domain, for example to move the object 7 from a “fraction of integers” (domain Fraction Integer) to “the integers” (domain Integer).

### **return**

when leaving a function, the value of the expression following **return** becomes the value of the function.

### **ring**

a set with a commutative addition, associative multiplication, a unit element, where multiplication is distributive over addition and subtraction.

### **rule**

(*syntax*) 1. An expression of the form **rule A == B** indicating a “rewrite rule.” 2. An expression of the form **rule (R1;...;Rn)** indicating a set of “rewrite rules” R1,...,Rn. See *pattern matching* for details.

### **run-time**

the time when computation is done. Contrast with *compile-time*, and *dynamic* as opposed to *static*. For example, the decision of the interpreter to build a structure such as “matrices with power series entries” in response to user input is made at run-time.

### **run-time check**

an error-checking that can be done only when the program receives user input; for example, confirming that a value is in the proper range for a computation.

### **search string**

a string entered into an *input area* on a HyperDoc screen.

### **selector**

an identifier used to address a component value of a Record datatype.

### **semantics**

the relationships between symbols and their meanings. The rules for obtaining the *meaning* of any syntactically valid expression.

### **semigroup**

(*algebra*) a *monoid* which need not have an identity; it is closed and associative.

### **side effect**

action that changes a component or structure of a value. See *destructive operation* for details.

### **signature**

(*syntax*) an expression describing the type of an *operation*. A signature has the form **name : source -> target**, where **source** is the type of the arguments of the operation, and **target** is the type of the result.

### **small float**

an object of the domain DoubleFloat for floating-point arithmetic as provided by the computer hardware.

### **small integer**

an object of the domain SingleInteger for integer arithmetic as provided by the computer hardware.

### **source**

the *type* of the argument of a *function*; the type expression before the -> in a *signature*. For example, the source of **f : (Integer, Integer) -> Integer** is **(Integer, Integer)**.

### **sparse**

data structure whose elements are mostly identical (a sparse matrix is one filled mostly with zeroes).

**static**

that computation done before run-time, such as compilation. Contrast *dynamic*.

**step number**

the number that precedes user input lines in an interactive session; the output of user results is also labeled by this number.

**stream**

an object of `Stream(R)`, a generalization of a *list* to allow an infinite number of elements. Elements of a stream are computed “on demand.” Streams are used to implement various forms of power series.

**string**

an object of domain `String`. Strings are *literals* consisting of an arbitrary sequence of *characters* surrounded by double-quotes (“”), for example, “Look here!”.

**subdomain**

(*basic concept*) a *domain* together with a *predicate* characterizing the members of the domain that belong to the subdomain. The exports of a subdomain are usually distinct from the domain itself. A fundamental assumption however is that values in the subdomain are automatically *coerceable* to values in the domain. For example, if `n` and `m` are declared to be members of a subdomain of the integers, then *any binary* operation from `Integer` is available on `n` and `m`. On the other hand, if the result of that operation is to be assigned to, say, `k`, also declared to be of that subdomain, a *run-time* check is generally necessary to ensure that the result belongs to the subdomain.

**such that clause**

(*syntax*) the use of “|” followed by an expression to filter an iteration.

**suffix**

(*syntax*) an *operator* that is placed after its operand. Suffix operators are not allowed in the AXIOM language.

**symbol**

objects denoted by *identifier literals*; an element of domain `Symbol`. The interpreter, by default, converts the symbol `x` into `Variable(x)`.

**syntax**

rules of grammar and punctuation for forming correct expressions.

**system commands**

top-level AXIOM statements that begin with “)”. System commands allow users to query the database, read files, trace functions, and so on.

**tag**

an identifier used to discriminate a branch of a Union type.

**target**

the *type* of the result of a *function*; the type expression following the “->” in a *signature*.

**top-level**

refers to direct user interactions with the AXIOM interpreter.

**totally ordered set**

(*algebra*) a partially ordered set where any two elements are comparable.

**trace**

use of system function `)trace` to track the arguments passed to a function and the values returned.

**tuple**

an expression of two or more other expressions separated by commas, for example, `4,7,11`. Tuples are also used for multiple arguments both for *applications* (for example, `f(x,y)`) and in *signatures* (for example, `(Integer, Integer) -> Integer`). A tuple is not a data structure, rather a syntax mechanism for grouping expressions.

**type**

The type of any *category* is the unique symbol *Category*. The type of a *domain* is any *category* to which the domain belongs. The type of any other object is either the (unique) domain to which the object belongs or a *subdomain* of that domain. The type of objects is in general not unique.

**Type**

a category with no operations or attributes, of which all other categories in AXIOM are *extensions*.

**type checking**

a system function that determines whether the datatype of an object is appropriate for a given operation.

**type constructor**

a *domain constructor* or *category constructor*.

**type inference**

when the interpreter chooses the type for an object based on context. For example, if the user interactively issues the definition `f(x) == (x + %i)**2` then issues `f(2)`, the interpreter will infer the type of `f` to be `Integer -> Complex Integer`.

**unary**

operation or function with *arity* 1.

**underlying domain**

for a *domain* that has a single domain-valued parameter, the *underlying domain* refers to that parameter. For example, the domain “matrices of integers” (Matrix Integer) has underlying domain Integer.

**Union**

(basic domain constructor) a domain constructor used to combine any set of domains into a single domain. A Union domain is written in the form `Union(a1: D1, ..., an: Dn)` ( $n > 0$ ) where `a1`, ..., `an` are identifiers called the *tags* of the union, and `D1`, ..., `Dn` are domains called the *branches* of the union. The tags `ai` are optional, but required when two of the `Di` are equal, for example, `Union(inches: Integer, centimeters: Integer)`. In the interpreter, values of union domains are automatically coerced to values in the branches and vice-versa as appropriate. See also *case*.

**unit**

(*algebra*) an invertible element.

**user function**

a function defined by a user during an interactive session. Contrast *built-in function*.

**user variable**

a variable created by the user at top-level during an interactive session.

**value**

1. the result of *evaluating* an expression. 2. a property associated with a *variable* in a *binding* in an *environment*.

**variable**

a means of referring to an object, but not an object itself. A variable has a name and an associated *binding* created by *evaluation* of AXIOM expressions such as *declarations*, *assignments*, and *definitions*. In the top-level *environment* of the interpreter, variables are *global variables*. Such variables can be freely referenced in user-defined functions although a **free** declaration is needed to assign values to them. See *local variable* for details.

**Void**

the type given when the *value* and *type* of an expression are not needed. Also used when there is no guarantee at run-time that a value and predictable mode will result.

**wild card**

a symbol that matches any substring including the empty string; for example, the search string “**\*an\***” matches any word containing the consecutive let-

ters “a” and “n”.

**workspace**

an interactive record of the user input and output held in an interactive history file. Each user input and corresponding output expression in the workspace has a corresponding *step number*. The current output expression in the workspace is referred to as `%`. The output expression associated with step number `n` is referred to by `%(n)`. The `kth` previous output expression relative to the current step number `n` is referred to by `%(- k)`. Each interactive *frame* has its own workspace.



[





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## APPENDIX H

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